THREE-CYCLE PROBLEM IN THE LOGISTIC MAP AND SHARKOVSKII'S THEOREM*

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In the logistic map a 3-cycle does not appear until after the end of stable 2^k cycles. An impetus for analytical studies of 3-cycles is provided by Sharkovskii's theorem, according to which the existence of a 3-cycle means the existence of all other cycles, hence chaos. It is a rigorous definition of chaos. We give a simple and direct proof of the existence of 3-cycles. The logistic map at fully developed chaos is shown to be isomorphic to the dynamics of a harmonic oscillator chain at the thermodynamic limit. Chaos in the logistic map is signified by a 3-cycle and in the harmonic oscillator chain by the thermodynamic limit.

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1. Introduction

The logistic map is to chaos nearly like the harmonic oscillator model is to many-body physics. It is basic to the study of chaos and no doubt best known as a means to understanding chaos [1, 2, 3]. Although it has been studied by many for over several decades, there seems to be still something new to be uncovered or at least to be understood more deeply. See [4, 5].

Let a function $f(x)$ be defined continuously in the interval $x = (0, 1)$, such that $x' = f(x)$, where $x' = (0, 1)$ too. Thus $f(x)$ maps onto itself. If the function is self similar, $x^{n+1} = f(x^n) = f^{n+1}(x)$, $n = 0, 1, 2, \ldots$, where $f^1 = f$ and x^0 (no prime) $= x$.

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A one-cycle is defined by $f(x) = x$, where x denotes the fixed points, the zeros of $P_1(x) = f(x) - x = x(x - x_0)$. A two-cycle is defined by $f^2(x) = x$. The fixed points are the zeros of $P_2(x) = f^2(x) - x$, excluding $x = 0$ and x_0, the zeros of $P_1(x)$. A three-cycle is defined by $f^3(x) = x$. The fixed points are the zeros of $P_3(x) = f^3(x) - x = 0$, excluding the zeros of $P_1(x)$.

It is now well known that if $f(x) = ax(1 - x)$, with the control parameter $a = (0, 4)$, which defines the logistic map, there develops a bifurcation sequence with increasing a, yielding 2^k cycles [1,2,3]. The process goes as: $1 \to 2 \to 4 \to 8 \to \ldots$ A 3-cycle never appears where 2^k cycles are stable up to $a = a_\infty = 3.56994\ldots$ [6]. If $a > a_\infty$, are there odd-numbered cycles such as a 3-cycle? What significance do they hold if they exist? If $a < a_\infty$, the domain is periodic, not chaotic. Is it chaotic everywhere $a > a_\infty$?

2. Sharkovskii's theorem

In the domain of a from $a = 1$ to a_∞, a 3-cycle (or any other odd-numbered cycles) cannot occur. If a 1-cycle bifurcates into two cycles (as if one branch begets two branches), the two begotten branches have exactly the same value of slope df^2/dx. Thus these two branches must themselves bifurcate at exactly the same value of $a = a_2 = 1 + \sqrt{6}$, each to two new branches, thus giving altogether 4 new branches or a 4-cycle. Thus a 3-cycle cannot occur while stable 2^k cycles are at work. Where $a > a_\infty$, one can show graphically that a 3-cycle can occur for some values of a. Exactly at what value of a can it occur?

There is a remarkable theorem due to Sharkovskii [7,8,9], which states that the existence of a 3-cycle implies the existence of a 5-cycle, a 7-cycle, *etc.* in the following sequence:

$3 \to 5 \to 7 \to \ldots$ (all odd numbers) $\to$
$2 \times 3 \to 2 \times 5 \to 2 \times 7 \to \ldots$, which we shall write as
$2\,\times$ (all odd numbers) $\to 2^2\times$ (all odd numbers) $\to \ldots \to$
$2^k\times$ (all odd numbers) $\to \ldots \to$
$\ldots \to 2^4 \to 2^3 \to 2^2 \to 2 \to 1\,.$

This sequence counts all natural numbers. We recognize at once that if the sequence is read in reverse, it begins as a 2^k cycles and ends in a 3-cycle. Thus the theorem says that if a 3-cycle exists, all other cycles must exist. The existence of all cycles defines chaos. There is a similar theorem due to Li and Yorke [10], which turns out to be a corollary of Sharkovskii's theorem. It states that a 3-cycle implies chaos. Thus to show that a 3-cycle exists is to show that chaos exists. It is of central importance thus to establish analytically that a 3-cycle does exist and to determine the value of a whence it comes into existence (the onset value of a denoted as b_3).

3. Three-cycles

A 3-cycle is defined by

$$f(x_1) = x_2\,, \tag{1}$$
$$f(x_2) = x_3\,, \tag{2}$$
$$f(x_3) = x_1\,. \tag{3}$$

That is, $f^3(x_i) = x_i$, $i = 1, 2, 3$, each x_i is a fixed point and a zero of $P_3(x) = f^3(x) - x$. Let us assume that a 3-cycle occurs at $a = b_3 > a_\infty = 3.569945\ldots$ If $a < b_3$, what would be the zeros of $P_3(x)$? This answer cannot be given until a general solution for P_3 is obtained. But one can gain an insight into it by looking at $P_2(x) = f^2(x) - x$ which may be put in the form

$$P_2(x) = x(x - x_0)Q_2(x)\,, \tag{4}$$

where $x_0 = 1 - 1/a$ for the logistic map and

$$Q_2(x) = (x - x_1)(x - x_2)\,. \tag{5}$$

If $a < a_1 = 3$, x_1 and x_2 are a pair of complex conjugates $x_2 = x_1*$. At $a = a_1$, they are real and $x_2 = x_1$. If $a > a_1$, they remain real but $x_2 \neq x_1$.

Thus it is reasonable to think that if $a < b_3$, there must be 3 pairs of complex conjugates: x_i, x_i*, $i = 1, 2, 3$. At $a = b_3$, they become real, hence there are 3 degenerate zeros. If $a > b_3$, there must be 6 unequal zeros (*not* 3 unequal zeros as we might have expected from our knowledge on the 2^k cycles.) We could put $P_3(x)$ in the form

$$P_3(x) = x(x - x_0)\ a^7\, Q_3(x)\,, \tag{6}$$

where

$$Q_3(x) = \prod_{i=1}^{3}(x - x_i)\left(x - x_i'\right)\,, \tag{7}$$

where x_i and x_i' would be a pair of complex conjugates if $a < b_3$, which become degenerate at $a = b_3$. If the above analysis is valid, there are two 3-cycles, not one, if $a > b_3$. The formation of an odd-numbered cycle seems fundamentally different from that of a 2^k cycle.

4. Three-cycles at special values

An explicit form for $Q_3(x)$ obtained by solving $f^3(x) = x$ using the logistic map has the following form

$$\begin{aligned} Q_3(x) \;=\; & x^6 - a^{-1}(3a+1)x^5 + a^{-2}\left(3a^2+4a+1\right)x^4 - a^{-3} \\ & \times \left(a^3+5a^2+3a+1\right)x^3 + a^{-4}\left(2a^3+3a^2+3a+1\right)x^2 \\ & - a^{-5}\left(a^3+2a^2+2a+1\right)x + a^{-6}\left(a^2+a+1\right)\,. \end{aligned} \tag{8}$$

The above is a sextic equation. If it were a general sextic equation, according to Abel one could not obtain zeros by radicals. As discussed above, there are two sets of three zeros which are degenerate at $a = b_3$, but independent if $a > b_3$. This behavior suggests that the sextic equation may be made up of two cubic equations tangled up in some manner. If it could be untangled, the sextic equation might still be solved by radicals. For studies of polynomials of high degrees in physical problems, see [4, 11, 12].

In this work we shall look for such solutions at special values of a: (1) $\bar{b}_3$ (superstability), (2) $a = 4$ (fully developed chaos), (3) $a = b_3$ (onset of 3-cycles). A general solution obtained by exploiting cubic character will be deferred to a later paper. It should be noted that this approach based on the cubic equation is in our view more natural and more direct than one due to Gordon based on a Fourier analysis [13].

4.1. At superstability

If one of the 3-cycles is superstable, one of its fixed point must be at the symmetry point, $x_1 = 1/2$, say. This value may be substituted in (8) to obtain a polynomial of degree 6 in a: $Q_3(1/2) = 2^{-6}a^{-6}Q(a)$,

$$Q(a) = a^6 - 6a^5 + 4a^4 + 24a^3 - 16a^2 - 32a - 64 = 0\,. \tag{9}$$

The above can also be deduced directly by (1), (2) and (3): If $x_1 = 1/2$, $x_2 = a/4$, and $x_3 = a \times a/4(1 - a/4)$. Hence $f(a \times a/4(1 - a/4)) = 1/2$. The result is a hexatic equation in a. If the superstability of the 1-cycle is factored out, the hexatic equation is reduced to the sextic equation given above.

To solve (8), we first note that $Q(-a) \neq 0$ if $Q(a) = 0$. Suppose there is $a' = F(a)$ such that $Q(-a') = Q(a') = 0$. Then $Q(a')$ must be a cubic equation in a'^2, solvable by radicals. This required transformation turns out to be a simple translation: If $a = a' + 1$,

$$Q\left(a'\right) = a'^6 - 11a'^4 + 35a'^2 - 89\,. \tag{10}$$

There are 6 values for a, of which only one is of interest to us: $a = \bar{b}_3$,

$$\begin{aligned} \bar{b}_3 &= 1 + \left\{11/3 + 2/3\left(100 + \sqrt{9936}\right)^{1/3} + 2/3\left(100 - \sqrt{9936}\right)^{1/3}\right\}^{1/2} \\ &= 3.83187402552\ldots\,. \end{aligned} \tag{11}$$

The structure of the above solution manifests the symmetry of a triangle of Galois group theory. For the detail, see [14].

4.2. At fully developed chaos [15]

Letting $a = 4$ and $x = t/4$ in (8), we obtain: $Q_3(x) = 4^{-6}Q(t)$,

$$\begin{aligned} Q(t) &= t^6 - 13t^5 + 65t^4 - 157t^3 + 189t^2 - 105t + 21 \\ &= \left(t^3 - 7t^2 + 14t - 7\right)\left(t^3 - 6t^2 + 9t - 3\right) . \end{aligned} \tag{12}$$

We have said that the sextic equation (8) might be made up of a pair of cubic equations. At fully developed chaos, the sextic equation indeed is a simple product of two cubic equations. There are two independent sets of zeros:

$$x = \sin^2 \pi/7 , \qquad \sin^2(2\pi/7) , \qquad \sin^2(3\pi/7) , \tag{13}$$

$$x' = \sin^2 \pi/9 , \qquad \sin^2(2\pi/9) , \qquad \sin^2(4\pi/9) . \tag{14}$$

We shall discuss the significance of these solutions in Sec. 5.

4.3. At the onset

We shall make one assumption with no other conditions introduced: At $a = b_3$, the two sets of zeros are real and degenerate. By this assumption, (7) becomes

$$Q_3(x) = q_3^2(x) , \tag{15}$$

where

$$q_3(x) = (x - x_1)(x - x_2)(x - x_3) = x^3 - \alpha x^2 + \beta x - \gamma , \tag{16}$$

$$\alpha = x_1 + x_2 + x_3 , \tag{17}$$

$$\beta = x_1x_2 + x_2x_3 + x_3x_1 , \tag{18}$$

$$\gamma = x_1x_2x_3 . \tag{19}$$

These coefficients α, β and γ represent symmetry properties of the cubic equation. That is, $q_3(x)$ is invariant under the permutation of the three zeros.

If (16) is squared, $Q_3(x)$ is now expressed in terms of the three symmetry coefficients only

$$Q_3(x) = x^6 - 2\alpha x^5 + \left(2\beta + \alpha^2\right) x^4 - 2\left(\gamma + \alpha\beta\right) x^3 + \left(2\alpha\gamma + \beta^2\right) x^2 - 2\beta\gamma x + \gamma^2 . \tag{20}$$

At $a = b_3$ and only at this value of a, Eq. (20) and Eq. (8) must be equal.

Let us compare them term by term:

(a) By the terms of x^5,

$$2\alpha = 3 + a^{-1} . \tag{21}$$

(b) By the term of x^4 and by (21),

$$2\beta = 3/4 + 5/2a^{-1} + 3/4a^{-2} \,. \tag{22}$$

(c) By the term of x^3 and by (21) and (22),

$$2\gamma = -1/8 + 7/8a^{-1} + 5/8a^{-2} + 5/8a^{-3} \,. \tag{23}$$

We have succeeded in expressing the three symmetry coefficients in terms of a. They may be used in the remaining three terms of x^2, x^1 and x^0 to obtain 3 new equations, resp., S_2, S_1 and S_0 now entirely in a:

$$S_2 = 3a^4 - 12a^3 - 141a^2 + 52a + 35 \,, \tag{24}$$
$$S_1 = 3a^5 - 11a^4 - 18a^3 + 42a^2 + 63a + 49 \,, \tag{25}$$
$$S_0 = a^6 - 14a^5 + 39a^4 + 60a^3 - 161a^2 - 206a - 231 \,. \tag{26}$$

The required solution $a = b_3$ must be one which makes $S_2 = S_1 = S_0 = 0$ simultaneously (to be referred to as the onset condition). Eq. (24) is solvable by radicals but (25) and (26) are not unless they can be reduced by some factoring. In fact, we find that the three equations are all factorable into the following form:

$$S_2 = B_2 A \,, \tag{27}$$
$$S_1 = B_1 A \,, \tag{28}$$
$$S_0 = B_0 A \,, \tag{29}$$

where

$$B_2 = 3a^2 - 6a - 5 \,, \tag{30}$$
$$B_1 = 3a^3 - 5a^2 - 7a - 7 \,, \tag{31}$$
$$B_0 = a^4 - 12a^3 + 22a^2 + 20a + 33 \,, \tag{32}$$
$$A = a^2 - 2a - 7 \,. \tag{33}$$

The three equations for B_2, B_1 and B_0 are now all solvable by radicals. Each has one solution in the interval (0,4): 2.6329 ..., 2.7984 ..., and 3.5822 ... for Eqs. (30), (31) and (32), resp. They are all distinct so that the onset condition cannot be met. It must mean that $A = 0$ is the only possibility. Eq. (33) yields at once

$$a = b_3 = 1 + \sqrt{8} \,. \tag{34}$$

This value has been long conjectured based on numerical studies [16]. It was first deduced analytically by Saha and Strogatz [17]. Their solution is rather complicated. Bachhoeffer has simplified it [18]. Ours is perhaps the simplest.

5. Implications by Sharkovskii's theorem

5.1. Stable 3-cycle

We have proved that at $a = b_3$ a pair of degenerate 3-cycles are formed, one of which becomes superstable at $a = \bar{b}_3$. At fully developed chaos they exist in pure cyclic form, see Sec. 4.2. Even though our proof of existence is only for 3 special values of a, it seems reasonable to conclude that 3-cycles exist from the onset value $a = b_3$ to fully developed chaos at $a = 4$. We can then assert by Sharkovskii's theorem (and by Li–Yorke's theorem) that in this interval $a = (b_3, 4)$, there is chaos.

This interval $a = (b_3, 4)$ includes a narrow window where one of the 3-cycles is stable. When a cycle is stable, its Lyaponov exponent is negative. It is commonly said that when the Lyapunov exponent is negative, it is periodic, not chaotic [19]. This is at odds with the strong statements due to Sharkovskii and Li and Yorke.

We are thus led to say that the Lyapunov exponent must not give a full picture for the 3-cycle window. There are infinitely many cycles in this window, which according to Sharkovskii and Li and Yorke denote chaos.

5.2. Unstable 3-cycles and others

We have deduced that, at fully developed chaos [15], there are two 3-cycles in pure cyclic form

$$x = \sin^2 \pi y/2\,, \qquad y/2 = 1/7, 2/7, 3/7; 1/9, 2/9, 4/9\,. \tag{35}$$

The values of y are discrete and bounded $0 < y < 1$, so that the values of x are irrational but discrete and bounded $0 < x < 1$. It can be further shown that cycles 1,2,4 and 5 have the same form as (35) with different sets of discrete values for y but still bounded the same way. Therefore without considering other higher cycles, we contend by Sharkovkii's theorem that at fully developed chaos y has all possible values in the interval $y = (0, 1)$, representing all possible cycles. That is,

$$x = \sin^2 \pi y/2\,, \qquad 0 < y < 1\,. \tag{36}$$

The interval $x = (0, 1)$ is now dense, replete with all possible irrational numbers, *i.e.* its invariant measure $\mu(x) = 1$, where $d\mu(x) = \rho(x)dx$, where $\rho(x)$ is an invariant distribution [20], satisfying

$$\int_0^1 \rho(x)dx = 1\,. \tag{37}$$

Eq. (36) implies that

$$\rho(x) = \frac{1}{\pi\sqrt{(x(1-x))}}\,, \qquad 0 < x < 1\,. \tag{38}$$

This invariant distribution was first derived by Ulam by a stochastic method [21].

In a chaotic region, trajectories are initial-value sensitive. In a periodic region such as where 2^k cycles exist, trajectories are not initial-value sensitive except when the initial values happen to take on fixed points. Then the iteration immediately goes into a cycle of that fixed point. The probability of such an occurrence is low since the x space of isolated fixed points is sparse. Thus almost any initial value upon iteration tends to converge toward an attractor. At fully developed chaos the x space is dense, in which every point in the interval $x = (0, 1)$ is a fixed point of some very high multi-cycle. Thus almost every initial value will give a unique trajectory.

6. Isomorphism and concluding remarks

We introduce a new function ω by

$$\omega = 2|1 - 2x|\,, \tag{39}$$

where x is given by (36). Since $0 < x < 1$, $0 < \omega < 2$. If we write $\rho(\omega)d\omega = \rho(x)dx$ and use (39) in (38), we obtain

$$\rho(\omega) = \frac{1}{\pi\sqrt{4-\omega^2}}\,, \qquad 0 < \omega < 2\,. \tag{40}$$

If $y = ka/2\pi + 1/2$ in (36),

$$\omega = 2|\sin ka/2|\,, \qquad -\pi/a < k < \pi/a\,. \tag{41}$$

We recognize that (41) is the thermodynamic limit of the dispersion relation for an nn coupled $1d$ harmonic oscillator chain in periodic boundary conditions where k is a wave vector if a is the nn distance [22]. (In our units, $\omega_o = \sqrt{\kappa/m} = 1$, κ the coupling constant and m the mass.). Similarly, (40) is the density of the frequencies in the harmonic oscillator chain in periodic boundary conditions at the thermodynamic limit [23, 24].

Since the x space and its invariant distribution $\rho(x)$ can be mapped on to the ω space and its frequency distribution $\rho(\omega)$, resp., the iterative behavior of the logistic map at fully developed chaos is isomorphic to the dynamics of the harmonic oscillator chain at the thermodynamic limit. There must be chaos in the dynamics of this *linear* many-body model. It comes into existence at the thermodynamic limit for there are infinitely many frequencies, the recipe for chaos according to Sharkovskii.

By the isomorphism the dynamic behavior of a harmonic oscillator chain must also be present in the iterative behavior of the logistic map at fully developed chaos. Among the most important may be the velocity autocorrelation function $r(t) = \langle v(t)v(0)\rangle$, where $v(t)$ is the time evolution of the velocity v of anyone of the oscillators in the chain and $\langle \ldots \rangle$ denotes an ensemble average. It is possible to show that (40) implies $r(t) = J_0(2t)$, in these units employed where J_0 is the Bessel function of order 0. If an equivalent quantity could be defined for the iterative trajectories, there would be found the Bessel function in it.

Irreversibility in Hermitian systems is given by $r(t \to \infty) = 0$ [25]. The irreversible behavior in the oscillator chain, a conservative system, stems from its density of frequencies $\rho(\omega)$ given by (40). Since $\rho(\omega)$ can be simply mapped on to $\rho(x)$ given by (38), irreversibility must also be present in the logistic map at fully developed chaos.

Perhaps more subtle is ergodicity. According to the ergometric theory of the ergodic hypothesis [26], a dynamical variable such as v in a Hermitian many-body system is ergodic if $W \equiv \rho(\omega = 0) \neq 0, \infty$. Irreversibility is determined by the low-frequency spectrum of the density of frequencies while ergodicity by the density at the origin. For a variable which is irreversible and possibly ergodic, its density of frequencies must be given by a multi-valued function. It means that the spectrum is dense as a consequence of the thermodynamic limit as shown in this work.

In the oscillator chain, v is ergodic by the ergodic condition W. The strange attractors such as those in the logistic map at fully developed chaos are often viewed as ergodic. In the end the ergometric theory may very well shed light on understanding the ergodic theory of chaos [27, 28] through the isomorphism established here in this work.

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