COMPOSITION OF QUANTUM OPERATIONS AND PRODUCTS OF RANDOM MATRICES

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Dedicated to the memory of Ryszard Zygadło

Spectral properties of evolution operators corresponding to random maps and quantized chaotic systems strongly interacting with an environment can be described by the ensemble of non-Hermitian random matrices from the real Ginibre ensemble. We analyze evolution operators $\Psi = \Psi_s \dots \Psi_1$ representing the composition of s random maps and demonstrate that their complex eigenvalues are asymptotically described by the law of Burda *et al.* obtained for a product of s independent random complex Ginibre matrices. Numerical data support the conjecture that the same results are applicable to characterize the distribution of eigenvalues of the s-th power of a random Ginibre matrix. Squared singular values of Ψ are shown to be described by the Fuss–Catalan distribution of the order of s. Results obtained for products of random Ginibre matrices are also capable to describe the s-step evolution operator for a model deterministic dynamical system — a generalized quantum baker map subjected to strong interaction with an environment.

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1. Introduction

Under the assumption of classical chaos the corresponding unitary quantum evolution, representing dynamics of an isolated quantum system, can be described [1, 2] by random unitary matrices of the circular ensembles of random matrices [3]. If the quantum system is not isolated, but it is coupled to an environment, its time evolution is not unitary. In such a case one describes the quantum state by a density operator ρ, which is Hermitian,

$\rho = \rho^\dagger$, positive, $\rho \geq 0$, and normalized, $\mathrm{Tr}\,\rho = 1$. The time evolution of such a system can be described by master equations [4], or by quantum operations [5], which correspond to a stroboscopic picture and discrete dynamics.

A quantum operation is described by a *superoperator* Ψ, which acts on the space of density operators. Let N denotes the size of a density matrix ρ. Then the superoperator is represented by a matrix Ψ of size N^2. Such a matrix is in general not unitary, but it obeys a quantum analogue of the Frobenius–Perron theorem [6], so its spectrum is confined to the unit disk. Spectral properties of superoperators representing interacting quantum systems were investigated in [7, 8, 9] and also analyzed in an NMR experiment [10]. Similar properties exhibit also non-unitary evolution operators analyzed earlier in context of quantum dissipative dynamics [11, 12].

Under the condition of classical chaos and strong decoherence the spectral properties of one-step evolution operators of deterministic systems do coincide with these of random operations [6] and can be described [13] by the ensemble of non-Hermitian Ginibre matrices. All entries of such a random matrix are independent Gaussian variables. Since a superoperator describing one-step evolution operator can be represented as a real matrix [14], we are going to apply random matrices of *the real Ginibre ensemble* [15, 16, 17].

The main aim of this work is to study spectra of evolution operators describing compositions of random quantum operations. Furthermore, we analyze s-step evolution operators representing quantum systems periodically interacting with the environment and compare statistical properties of complex spectra with predictions of the theory of non-Hermitian random matrices.

This work is organized as follows. In Section 2 we recall necessary definitions of relevant ensembles of random matrices and briefly review recent results concerning statistical properties of their products. Properties of random maps and their compositions are analyzed in Section 3. The model deterministic dynamical system — a variant of the baker map interacting with an environment is studied in Section 4.

2. Non-Hermitian random matrices and their products

Consider a random square matrix G of size N of the *complex Ginibre ensemble* [18], generated according to the probability density

$$P(G) \propto \exp\left(-\mathrm{Tr}\,GG^\dagger\right) . \tag{1}$$

This assumption implies that each entry G_{mn} of the random matrix is an independent complex Gaussian variable of a fixed variance $\sigma^2 = \xi^2/N$, where

ξ is a free parameter which sets the scale. Eigenvectors of a random matrix G from such an ensemble are distributed according to the Haar measure on the unitary group, while complex eigenvalues z_i are described by the joint probability distribution

$$P(z_1,\ldots,z_N) \propto \exp\left(-\sum_i |z_i|^2\right)\prod_{i<j}|z_i-z_j|^2 \,. \tag{2}$$

From this result one can evaluate the density of eigenvalues in the complex plane. The density is rotationally symmetric and is a function of the moduls $r=|z|$ of an eigenvalue [18, 17],

$$P(z) = \frac{1}{\pi}\frac{\Gamma(N,|z|^2)}{\Gamma(N)}\,, \tag{3}$$

where $\Gamma(s,x)$ denotes the incomplete Gamma function,

$$\Gamma(s,x) = \int\limits_x^\infty t^{s-1}\exp(-t)dt\,.$$

In the asymptotic limit of large matrix size N the level density becomes constant inside the disk of radius $R=\xi$, and decays exponentially outside the disk. This fact, known as the circular law of Girko [19], is conveniently formulated under the normalization $\sigma^2 = 1/N$ so that $\xi=1$, for which the spectrum of a random Ginibre matrix of a large dimension covers uniformly the unit disk.

Several recent applications including multiplicative diffusion processes [20], macroeconomic time series [21], lattice gauge field theories [22] and chiral ensembles of random matrices [23, 24] increased interest in statistical properties of products of non-Hermitian random matrices [25]. Let Y denote a product of s independent square random matrices of size N from the complex Ginibre ensemble, $Y=G_1G_2\cdots G_s$. The density of the spectrum of Y is rotationally invariant in the complex plane [26],

$$P(z) = \frac{1}{s\pi}\xi^{-2/s}|z|^{-2+(2/s)} \qquad \text{for} \qquad |z|\le\xi\,. \tag{4}$$

Here $\xi^2=\xi_1^2\xi_2^2\ldots\xi_s^2$ denotes the product of scale parameters of each of the random matrices. For simplicity we shall assume that all s random matrices are characterized by the same variance, so that $\xi=\xi_1^s$. The radial density of the eigenvalues reads

$$P(r) = \frac{2}{s}\xi^{-2/s}r^{-1+(2/s)} \qquad \text{for} \qquad r\le\xi\,. \tag{5}$$

Based on the exact results for the level density for the case $s = 1$ of a single random matrix [27, 17, 28] it was suggested by Burda *et al.* [29] to describe the finite size effects for the spectral density of a product of s random matrices by the dollowing ansatz, which involves the complementary error function $\mathrm{erfc}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-t^2/2)dt$,

$$P_N(r) \equiv P(r)\, \frac{1}{2}\mathrm{erfc}\Big(q(r-\xi)\sqrt{N}\Big)\,. \tag{6}$$

Here q is an adjustable parameter, which does not depend on the dimension N, and the above form was reported [29, 30] to describe well the data obtained numerically by diagonalization of products of random Ginibre matrices.

Another way to describe a non-Hermitian operator A is to study its singular values. Their squares are equal to the eigenvalues of the positive matrix $AA^\dagger$. To set the scale it is convenient to renormalize such a matrix and define

$$W = \frac{AA^\dagger}{\mathrm{Tr}\, AA^\dagger} \tag{7}$$

such that $\mathrm{Tr}\, W = 1$.

Let $\{\lambda_i\}$, $i = 1, \ldots, N$ denote the non-negative eigenvalues of W. The normalization implies that their sum is equal to unity, so if we use a rescaled variable, $x_i = N\lambda_i$, its mean value is equal to unity, $\langle x \rangle = 1$.

If the matrix A is a random square Gaussian matrix from the Ginibre ensemble the level density $P(x)$ describing the Wishart matrix W given in (7) is asymptotically (for a large matrix dimension N) described by the Marchenko–Pastur distribution [31],

$$\mathrm{FC}_1(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x} - 1} \qquad \text{for} \qquad 0 \leq x \leq 4\,. \tag{8}$$

If A is obtained as a product of s independent square random Ginibre matrices, $A = G_1 G_2 \cdots G_s$, the level density of the Wishart-like matrix W is given by the Fuss–Catalan distribution of order s [32, 33]. The name of this distribution is related to the fact that its moments are equal to the Fuss–Catalan numbers, often studied in combinatorics [34]. An explicit form of the FC distribution of order two,

$$\mathrm{FC}_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\, \frac{\left[\sqrt[3]{2}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}\right]}{x^{\frac{2}{3}}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{1}{3}}}\,, \tag{9}$$

valid for $x \in [0, 27/4]$, was derived first by Penson and Solomon [35] in context of construction of generalized coherent states. More recently this formula was used in [36, 37] to describe singular values of random quantum states, the construction of which involves the product of two random matrices. Treating the sequence of Fuss–Catalan numbers as given, one can solve the Hausdorff moment problem and find the corresponding probability distributions $\mathrm{FC}_s(x)$. They can be written down explicitly [38] and represented as a combination of the hypergeometric functions [39] of the type ${}_sF_{s-1}$ of the same argument. For instance, the Fuss–Catalan distribution of order three reads

$$\begin{aligned} \mathrm{FC}_3(x) \; = \; & \frac{1}{\sqrt{2}\pi x^{3/4}}\, {}_3F_2\left(-\frac{1}{12},\frac{1}{4},\frac{7}{12};\ \frac{1}{2},\frac{3}{4};\ \frac{27}{256}x\right) \\ & -\frac{1}{4\pi x^{1/2}}\, {}_3F_2\left(\frac{1}{6},\frac{1}{2},\frac{5}{6};\ \frac{3}{4},\frac{5}{4};\ \frac{27}{256}x\right) \\ & -\frac{1}{32\sqrt{2}\pi x^{1/4}}\, {}_3F_2\left(\frac{5}{12},\frac{3}{4},\frac{13}{12};\ \frac{5}{4},\frac{3}{2};\ \frac{27}{256}x\right). \end{aligned} \tag{10}$$

The support of the Fuss–Catalan distribution $\mathrm{FC}_s(x)$ of order s is formed by an interval $[0, (s+1)^{s+1}/s^s]$ [32, 36]. Although this distribution was shown first to describe asymptotic distribution of squared singular values of the product of s independent Ginibre matrices, $A = G_1 G_2 \cdots G_s$, the same law describes asymptotically the distribution of squared singular values of the s-th *power* of a square Ginibre matrix, $A = G^s$, [40].

Analyzing discrete time evolution of Hermitian density matrices in terms of quantum maps one copes with evolution operators which can be represented by a real matrix [5]. In particular, superoperators associated with random quantum maps [6] can be described [13] by real random matrices of the Ginibre ensemble. These matrices can be formally defined by the distribution (1) applied in the space of real matrices, so the argument of the exponent can be written as $\mathrm{Tr}\, AA^T$. To generate a random matrix pertaining to this ensemble one takes N^2 independent random Gaussian variables of the same variance and forms out of them a non-symmetric square matrix.

Statistical properties of the real Ginibre ensemble are more complicated to analyze [15, 16] than in the complex case. For instance, for the real Ginibre ensemble the joint probability distribution of eigenvalues depends explicitly on their imaginary parts, so the level density is not rotationally invariant. In fact, there exists an accumulation of eigenvalues along the real axis, which is compensated by the repulsion of complex eigenvalues in vicinity of the real axis [15, 17]. However, the number of real eigenvalues of a real random Ginibre matrix of size N scales as $\sqrt{N}$, so that the non-uniform features of the spectrum can be neglected in the asymptotic limit $N \to \infty$. Also the distribution of singular values of products of s real Ginibre matrices can be described [37] by Fuss–Catalan distribution of order s, originally applied [32] for products of complex Ginibre matrices.

3. Random operations and their compositions

A quantum operation is a linear map, $\rho \to \rho' = \Psi(\rho)$, which maps the set of the density matrices into itself, so it preserves positivity and trace of the input state ρ. Any quantum operation acting on a d-dimensional state can be described by a unitary evolution U applied to an extended system followed by the partial trace over the environment $\mathcal{E}$,

$$\Psi(\rho) = \mathrm{Tr}_{\mathcal{E}} \left(U(\rho \otimes |\nu\rangle\langle\nu|)U^{\dagger} \right) . \tag{11}$$

Here $|\nu\rangle \in \mathcal{H}_M$ denotes the initially pure state of the environment, which is assumed to be M dimensional, so the unitary matrix U of size Md acts on the composite Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_M$.

If we reshape a density matrix ρ of size N into a vector of length d^2, a quantum operation can be represented by a $d^2 \times d^2$ matrix called superoperator. It is convenient to represent a density operator ρ of size N by its Bloch vector,

$$\rho = \sum_{i=0}^{d^2-1} a_i \, \gamma^i \, . \tag{12}$$

Here γ^i denotes the set of $d^2 - 1$ generators of the group $\mathsf{SU}(d)$, which satisfy relations $\mathrm{Tr}\left(\gamma^i \gamma^j\right) = \delta^{ij}$, while $\gamma^0 = \mathbb{1}/\sqrt{N}$. As any density matrix is Hermitian, $\rho = \rho^{\dagger}$, the components of the Bloch vector are real, $a_i \in \mathbb{R}$ for $i = 0, \ldots, d^2 - 1$. Thus the action of a quantum operation can be described as an affine transformation on the Bloch vector $\vec{a}$ representing the quantum state, $\vec{a}' = C\vec{a} + \vec{\kappa}$, where C is a non-Hermitian distortion matrix of order $d^2 - 1$, while $\vec{\kappa}$ is a translation vector of length $d^2 - 1$. Using the Bloch vector representation one writes the superoperator Ψ as a real matrix,

$$\Psi = \begin{bmatrix} 1 & 0 \\ \vec{\kappa} & C \end{bmatrix} . \tag{13}$$

The spectrum of the non-symmetric matrix C belongs to the complex plane. Since the quantum operation Ψ preserves the trace of the density matrix, $\mathrm{Tr}\,\Psi(\rho) = \mathrm{Tr}(\rho) = 1$, the spectrum of the superoperator belongs to the unit disk.

Assuming that the matrix $U \in U(Md)$ in (11) is taken randomly with respect to the Haar measure one obtains a random quantum operation [6]. In such a case the spectrum of the associated one-step evolution operator Ψ was shown to consist of a single eigenvalue equal to unity, corresponding to the unique invariant state, and the remaining part localized in the disk of radius $R = 1/\sqrt{M}$ centered at the origin of the complex plane [13]. This characterization becomes exact for a large system size d. Since the matrix C

is real, the spectrum of Ψ is symmetric with respect to the real axis. As in the case of the real Ginibre ensemble [15, 16, 17] there exists a clustering of eigenvalues along the real axis, but this effect vanishes in the limit $d \to \infty$.

The number M, which determines the size of the disk of complex eigenvalues, equal to the dimension of the auxiliary subsystem $\mathcal{E}$, can be thus considered as a control parameter of the model describing the interaction of the principal system with the environment. Technically, M determines the rank of the Hermitian dynamical matrix [5], which describes the quantum operation.

3.1. Spectral density of a random superoperator

We analyzed spectra of evolution operators corresponding to compositions of random maps. Let $\Psi = \Psi_s \cdots \Psi_2 \circ \Psi_1$, where all s random maps Ψ_j are assumed to be independent. Figure 1 shows the spectra of such operators for $s = 1, 2$ and $s = 3$ for maps acting on a quantum system of size $d = 20$. To show the structure of the spectrum we magnified the scale accordingly, letting the leading eigenvalue $z_1 = 1$ to remain outside the plot. According to the prediction of the Ginibre ensemble for $s = 1$ the distribution of the spectrum is close to be uniform in the disk of radius $R = 1/\sqrt{M}$ (apart of the clustering of eigenvalues along the real axis), while its structure changes for larger s.

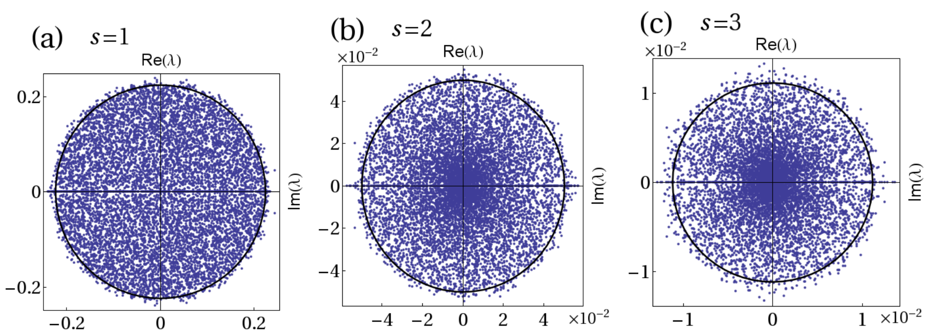

Fig. 1. Superimposed spectra of 25 superoperators C of dimension $d^2 - 1$ associated with random maps acting on density operators of dimension $d = 20$ and obtained by an interaction with an environment of size $M = 20$. The superoperators represent (a) single random maps, $s = 1$; the composition of (b) $s = 2$ and (c) $s = 3$ random maps. The disk of radius $R_s = 1/\sqrt{M^s}$ (note the rescaling of both axes) denotes the support predicted for the ensemble of random Ginibre matrices.

Consider first the simplest case, $s = 2$, in which two random operations (11) act successively. Assume that the first random operation Ψ_1 is due to the interaction with the environment $\mathcal{E}_1$ of dimension M_1, while the second operation Ψ_2 describes the interaction with the environment $\mathcal{E}_2$ of dimension M_2. The resulting dynamics takes place in a tri-partite system described in the Hilbert space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$. The first label p refers to the principal system of dimension d, while the other subsystems are labeled by the number of the environment $\mathcal{E}_1$ and $\mathcal{E}_2$.

After both operations the output state of the system, $\rho'' = \Psi_2(\rho') = \Psi_2[\Psi_1(\rho)]$ can be obtained by a three-subsystem unitary evolution,

$$U = V_2 V_1 = (U_{p2} \otimes_1 \mathbb{1}_1)(U_{p1} \otimes_2 \mathbb{1}_2)\,, \tag{14}$$

followed by the partial trace over the collective environment $\mathcal{E}_{12}$ of dimension $M = M_1 M_2$. Note that the resulting unitary U does not have a product structure, as the symbols $\otimes$ represent the tensor product with respect to two different splittings of the Hilbert space, $\mathcal{H} = \mathcal{H}_{p1} \otimes_2 \mathcal{H}_2 = \mathcal{H}_{p2} \otimes_1 \mathcal{H}_1$.

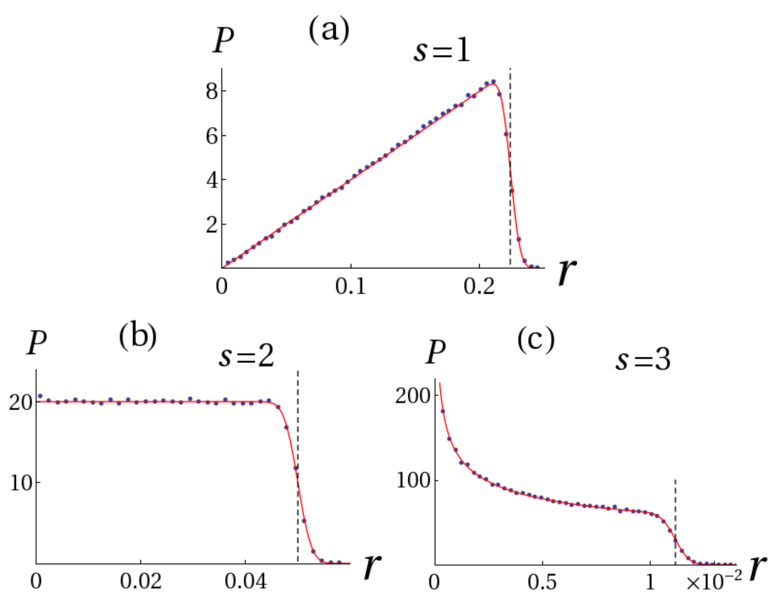

Fig. 2. Radial density of complex eigenvalues of superoperators associated with (a) a single random map, $s = 1$ and a composition of (b) $s = 2$ and (c) $s = 3$ random maps. Numerical data were obtained from a sample of 1000 superoperators of dimension $d^2 - 1$ with $d = 20$. The dimensions of all auxiliary subspaces are equal, $M = 20$. Solid lines represent predictions (5) for radial density for products of s Ginibre matrices with the correction (6) due to the finite size effects. Best fit gives the following values of the fitting parameter, (a) $q \approx 6$ for $s = 1$, (b) $q \approx 7$ for $s = 2$, (c) $q \approx 68$ for $s = 3$. Dashed vertical line represents the radius R_s of the disk, which determines the support of the essential spectrum in the limit $d \to \infty$.

It is natural to expect that in the case of two random operations, $\Psi = \Psi_2 \circ \Psi_1$, the resulting effect will be similar as the one caused by a single interaction with the combined environment $\mathcal{E}_{12}$ of dimension $M = M_1 M_2$. This statement is equivalent to an assumption that the effect of the action of the resulting unitary U of the structure (14) is statistically indistinguishable from the effect due to a global random unitary matrix $U_{p12} \in U(dM_1M_2)$ followed by the partial trace over $\mathcal{E}_{12}$. Our numerical results confirm that this approximation works fine, as it implies that the subleading eigenvalues of the superoperator Ψ live in the disk of radius $R = 1/\sqrt{M} = 1/\sqrt{M_1 M_2}$.

The same reasoning shows that in the case of a composition of s random operations, in which all dimensions of the environments are equal, $M_1 = M_2 = \cdots = M_s = M$, the essential spectrum of the resulting superoperator is asymptotically confined in the disk of radius $R_s = R_1^s = 1/\sqrt{M^s}$. These predictions describe well the spectra of the evolution operators associated with the compositions of two and three random maps and presented in Fig. 1 in panels (b) and (c), respectively.

The radial distribution of complex eigenvalues collected of 1000 superimposed spectra are presented in densities in Fig. 2. Numerical results can be described by the law (5) obtained in [26] for a product of s independent random complex Ginibre matrices. To take into account the finite size effects we used the ansatz (6) and fitted the parameter q.

3.2. Singular values of random superoperator

The distribution of squared singular values of a superoperator Ψ associated with a single random map or the composition of s of them was analyzed numerically. The data obtained presented in Fig. 3 show a fair agreement with the Fuss–Catalan distribution of order s, which describes properties of a product of s complex Ginibre matrices [32].

3.3. Average entropies

To characterize the eigenvalue distribution $P(\lambda)$ of the positive matrix $\Psi\Psi^\dagger / \operatorname{Tr}\Psi\Psi^\dagger$ one studies the Shannon entropy of the spectrum, $S = -\sum_{i=1}^{d^2-1} \lambda_i \ln \lambda_i$. We put aside the leading eigenvalue $\lambda_1 = 1$ and renormalize the remaining $d^2 - 1$ eigenvalues so that their sum is set to unity. The mean entropy computed numerically for a sample of 1000 superoperators Ψ representing random maps with parameters $d = M = 20$ reads $\langle S \rangle_\Psi \approx -0.505$. This value agrees with the asymptotic prediction for the Wishart matrices, $\langle S \rangle_1 = -\frac{1}{2}$ implied by the Marchenko–Pastur distribution (8). A similar agreement is obtained in the case $s = 2$, for which the average entropy implied by the Fuss–Catalan distribution of order two (9), $\langle S \rangle_2 = -\frac{5}{6} \approx -0.833$ [36], while the numerical data give the average

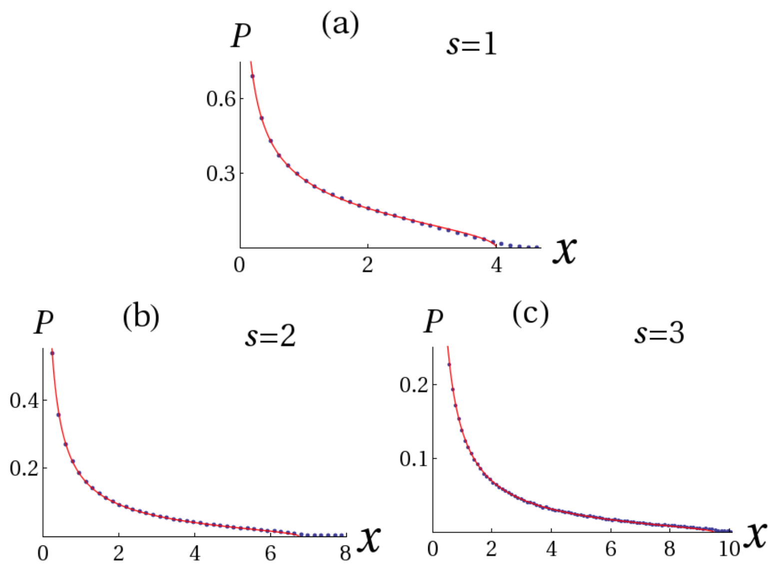

Fig. 3. Density of normalized squared singular values, $x_i = (d^2-1)\lambda_i$, of superoperators corresponding to (a) a single random map, $s = 1$; a composition of (b) $s = 2$ and (c) $s = 3$ random maps obtained for $d = 20$ and $M = 20$. Numerical data collected from a sample of 1000 random maps are compared with the corresponding Fuss–Catalan distributions ((8), (9), (10)) represented by solid curves.

entropy $\langle S \rangle_\Psi \approx -0.841$. Numerical results for $s = 3$ provide the value $\langle S \rangle_\Psi \approx -1.093$, whereas the Fuss–Catalan distribution $\mathrm{FC}_3(x)$ leads to $\langle S \rangle_3 = -\frac{13}{12} \approx -1.083$.

4. Generalized quantum baker map and s-step evolution operators

To investigate statistical properties of evolution operators associated with deterministic quantum systems interacting with an environment we shall concentrate on a model dynamical system, the classical analogues of which is known to be chaotic. Following the work of Balazs and Voros [41] we consider the unitary operator describing the one-step evolution of the quantum baker map,

$$B = F_d^\dagger \begin{bmatrix} F_{d/2} & 0 \\ 0 & F_{d/2} \end{bmatrix} . \tag{15}$$

Here F_d denotes the Fourier matrix of size d, namely $[F_d]_{jk} = \exp(\mathrm{i}jk/2\pi d)/\sqrt{d}$ and it is assumed that the dimension d of the Hilbert space $\mathcal{H}_d$ is even.

The standard quantum baker map B may be generalized, if in the definition of the unitary operator (15) the Fourier matrix F_d is replaced by a two-parameter matrix $F_d^{\phi_1,\phi_2}$,

$$\left[F_d^{\phi_1,\phi_2}\right]_{jk} = \frac{1}{\sqrt{d}} \exp\left[\mathrm{i}\frac{(j+\phi_1)(k+\phi_2)}{2\pi d}\right] , \tag{16}$$

see Appendix D in [42]. The choice of both phases in $[0, 2\pi)$ does not influence the classical limit, equal to the classical baker map. Thus combining Eq. (16) and (15) one obtains a two-parameter family of unitary quantum model dynamical systems, which we denote by B_{ϕ_1,ϕ_2}.

A certain variant of a non-unitary baker map introduced by Saraceno and Vallejos is capable to describe a dissipative quantum system [43]. Here we are going to investigate yet another model of non-unitary quantum baker map introduced in [7, 44], which is deterministic, conserves the probability, and is capable to describe projective measurements or a coupling with an external subsystem. Such a non-unitary dynamics can be represented as a quantum map and written in its Kraus form [5],

$$\rho' = \Phi(\rho) = \sum_{j=1}^{M} X_j \rho X_j^\dagger . \tag{17}$$

For any trace preserving operation the set of M Kraus operators satisfies the identity resolution, $\sum_{j=1}^{M} X_j^\dagger X_j = \mathbb{1}$.

The parameter M is equal to the size of the environment coupled to the principal system of the baker map. Alternatively, M can be interpreted as the number of different outcomes of a measurement process. It is a free parameter of the model, which describes the degree of the decoherence in the system introduced by the non-unitary map (17).

The model of the classical baker map is chaotic and its dynamics can be characterized by the dynamical entropy of Kolmogorov and Sinai H_{KS} equal to $\ln 2$ [45]. To increase the degree of chaos one can simply take L iterations of the classical system, for which the dynamical entropy reads $L \ln 2$. This corresponds, in the quantum model, to taking the L-th power of the unitary evolution B given by (15), or its generalized version B_{ϕ_1,ϕ_2} which involves (16). Increasing the parameter L one increases the degree of chaos in the classical model, and thus obtains unitary quantum operators the properties of which are well described [46] by the Haar random matrices of the circular unitary ensemble.

In our model system we take the L-step unitary dynamics of the baker map, B^L, followed by the non-unitary interaction with the environment of dimension M. In other words, the non-unitary map Φ given by (17) acts only every L steps of the unitary evolution. The structure of the complete evolution operator is presented schematically in Fig. 4.

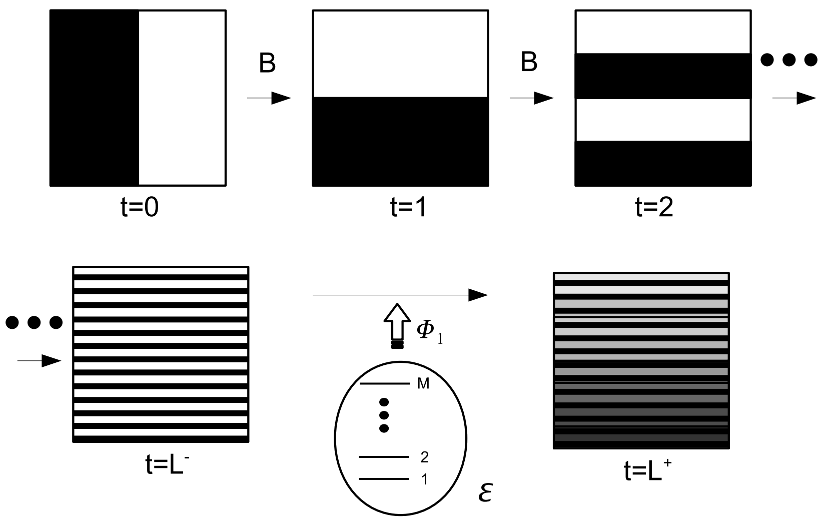

Fig. 4. Sketch of the deterministic dynamical system — the generalized quantum baker map $\Phi_{M,L}$ — analyzed in this work: L steps of the unitary dynamics followed by an interaction with an M-dimensional environment $\mathcal{E}$ described by the quantum operation Φ_1.

Thus the stochastic quantum baker map describing the non-unitary evolution of the generalized quantum baker map [7, 44] reads

$$\Phi_{M,L}(\rho) = \sum_{j=1}^{M} P_j \left[B^L \rho (B^\dagger)^L \right] P_j^\dagger \,. \tag{18}$$

It consists of M Kraus operators P_j, which act on the unitarily rotated state $B^L\rho(B^\dagger)^L$. It is assumed that the ratio $K = d/M$ is integer, so one can decompose the Hilbert space $\mathcal{H}_d$ into the direct sum of M mutually orthogonal subspaces $\mathcal{H}_{(j)}$, $j = 1, \ldots, M$, of dimension K each. Then the Kraus operator $X_j = P_j B^L$ is a projector operator onto a K-dimensional subspace $\mathcal{H}_{(j)}$, so the sum $\sum_{j=1}^{M} X_j^\dagger X_j = \sum_{j=1}^{M} X_j$ is equal to identity, as required. Thus the parameter M in the non-unitary quantum baker map studied in this section has a similar meaning that M parameterizing the random maps: it describes the degree of the interaction of the principal system with the environment. Additionally, for each choice of the system parameters (L, M) we may choose an pair of phases (ϕ_1, ϕ_2) which enter (16) and define the quantum model. To obtain a better statistics we shall superimpose spectra of superoperators obtained for fixed values of (L, M) and various phases (ϕ_1, ϕ_2).

Figure 5 presents exemplary spectra of superoperators Φ^s of the generalized quantum baker map for $s = 1, 2, 3$ obtained for fixed values of $L = 20$ and $M = 10$ and 60 pairs of phases (ϕ_1, ϕ_2). To display the structure of the bulk of the spectrum the scale increases with the power s, so the leading eigenvalue $z_1 = 1$ is located outside the figure. Apart of a few real eigenvalues, located for outside the circle of radius $R_s = M^{-s/2}$, the remaining eigenvalues are located close to the disk predicted for products of random matrices in the asymptotic limit $d \to \infty$. The spectra are symmetric with respect to the real axis and exhibit the clustering of eigenvalues along the real axis combined with the repulsion of eigenvalues in the vicinity of this axis. These effects, typical to the ensemble of real Ginibre matrices [15, 16], vanish in the asymptotic limit, in which properties of products of complex and real random matrices tend to coincide.

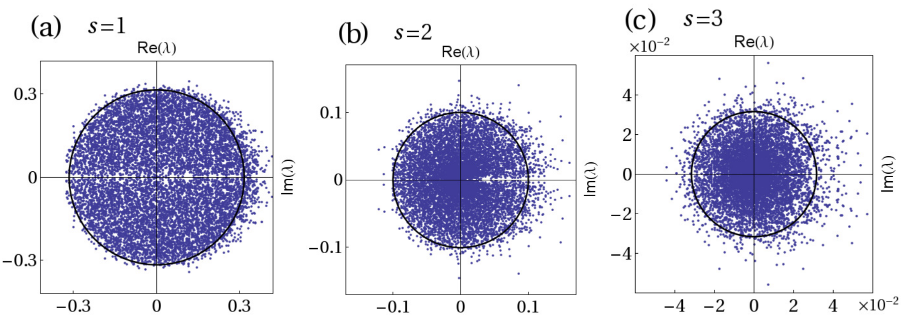

Fig. 5. Superimposed spectra of 60 superoperators of the generalized quantum baker map acting on density operators of dimension $d = 40$ and characterized by parameters $L = 20$ and $M = 10$. The superoperators represent (a) single baker map (18), $s = 1$; the s-step propagator $(\Phi_{M,L})^s$ for (b) $s = 2$ and (c) $s = 3$ time steps. The disk of radius $R_s = 1/\sqrt{M^s}$ (note the rescaling of both axes) denotes the support predicted for the ensemble of random Ginibre matrices.

Figure 6 shows the radial density distribution $P(r)$ for complex eigenvalues of superoperators of the generalized baker map Φ^s for $s = 1, 2, 3$. The dynamical parameters of the model are fixed, $d = 40$, $L = 20$ and $M = 10$, while to accumulate a necessary statistics we superimposed data of 100 superoperators obtained for different values of the phases (ϕ_1, ϕ_2). Note that already for $s = 1$ the spectral properties of the superoperator Φ can be described by the ensemble of random Ginibre matrices.

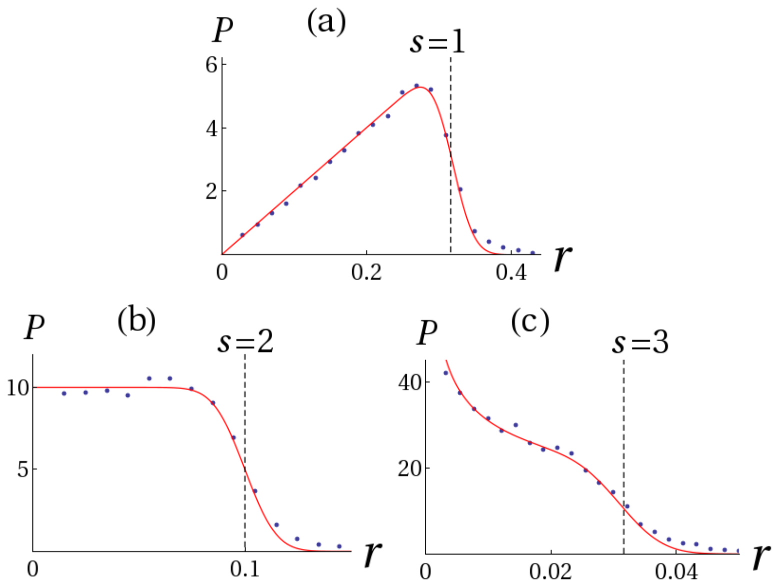

Fig. 6. Radial density of complex eigenvalues of superoperators associated with (a) one-step evolution operator for the baker map, $s = 1$ and the s-step propagators for (b) $s = 2$ and (c) $s = 3$. Numerical data were obtained from a sample of 100 superoperators of the generalized quantum baker map acting on density operators of dimension $d = 40$ and characterized by parameters $L = 20$ and $M = 10$. Solid lines represent predictions (5) for radial density for products of s Ginibre matrices with the correction (6) due to finite size effects. Best fit gives the following values of the fitting parameter, (a) $q \approx 1.5$ for $s = 1$, (b) $q \approx 1.5$ for $s = 2$, (c) $q \approx 3.5$ for $s = 3$. Dashed vertical line represent the radius R_s of the disk, which determines the support of the essential spectrum in the limit $d \to \infty$.

To analyze properties of superoperators associated with s-step propagator of the generalized baker map, we analyzed also statistical properties of squared singular values of Φ^s equal to eigenvalues of a positive operator $\Phi^s(\Phi^\dagger)^s$. Using the Kraus decomposition of a superoperator [5], $\Phi = \sum_{i=1}^{M} X_i \otimes \bar{X}_i$ we find that $\Phi\Phi^\dagger = \sum_{i,j=1}^{M} X_i X_j^\dagger \otimes \bar{X}_i X_j^T$. Since in our model each Kraus operator is a projector rotated by the same unitary matrix, $X_i = P_i U$, the unitaries cancel out. The projections operators are mutually orthogonal, $P_i P_j = \delta_{ij} P_j$, so the above expression reduces to a single sum, $\Phi\Phi^\dagger = \sum_{j=1}^{M} P_j \otimes P_j$. Any operator P_i projects onto the subspace of dimension $K = d/M$ so the operator $\Phi\Phi^\dagger = P$, where P is a projector on a space of dimension $N' = M(d/M)^2 = d^2/M$. Hence its spectrum consists of N' eigenvalues equal to unity and remaining $d^2(1 - 1/M)$ eigenvalues equal to zero — see Fig 7(a).

Consider now the case $s = 2$, in which we analyze the spectrum of $\Phi^2(\Phi^\dagger)^2$. This operator can be written as $\Phi P \Phi^\dagger = (\Phi P)(\Phi P)^\dagger$. Hence the singular values of Φ^2 are equal to the singular values of a truncated matrix ΦP. Thus the generic, non-zero eigenvalues of $W_2 = \Phi^2(\Phi^\dagger)^2/\mathrm{Tr}[\Phi^2(\Phi^\dagger)^2]$ will be described by the Marchenko–Pastur distribution — see Fig. 7(b). In a similar way, the singular values of Φ^s are equal to the singular values of a truncated matrix $\Phi^{s-1}P$. Therefore, its squared singular values are described by the Fuss–Catalan distribution of order $s' = s - 1$. Although the distributions $\mathrm{FC}_s(x)$ are known to describe the asymptotic distribution of squared singular values of a product of complex Ginibre matrices, they describe also statistical properties of $(s-1)$-step propagators of the generalized baker map.

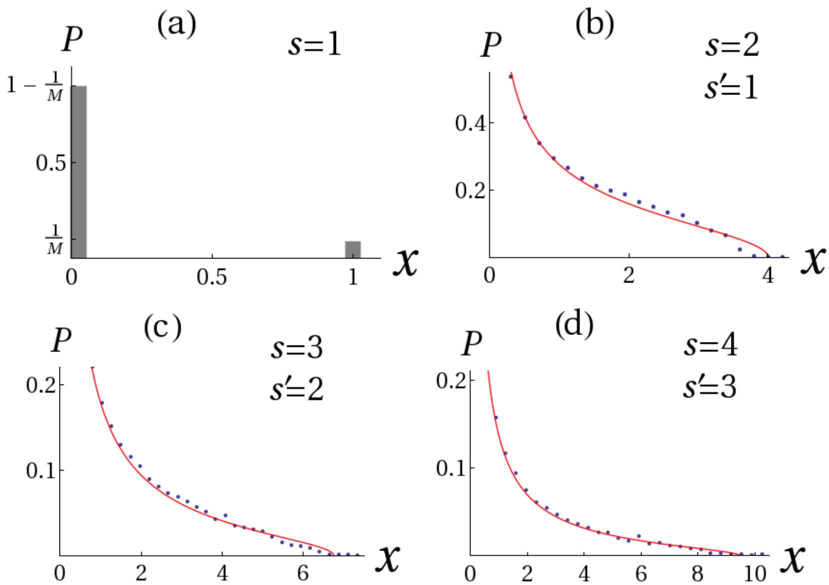

Fig. 7. Density of normalized positive squared singular values, $x_i = (d^2/M - 1)\lambda_i$, of superoperators corresponding to (a) one step evolution operator of the baker map, $s = 1$; the s-step propagators $(\Phi_{M,L})^s$ for (b) $s = 2$, (c) $s = 3$, and (d) $s = 4$ time steps (with the delta peak at $x = 0$ removed). Numerical data were obtained from a sample of 70 superoperators of the generalized quantum baker map acting on density operators of dimension $d = 40$ and characterized by parameters $L = 20$ and $M = 10$. Densities are compared with the corresponding Fuss–Catalan distributions of order $s' = s - 1$, Eqs. (8), (9), (10), represented by solid curves.

5. Concluding remarks

We analyzed complex spectra of superoperators associated to compositions of s random maps. Statistical properties of the eigenvalues can be described by products of random matrices from the Ginibre ensemble. Due to the fact that a quantum map Ψ preserves hermicity of a quantum state ρ, the superoperator Ψ can be described [13] by an ensemble of *real* Ginibre matrices, for which a clustering of the eigenvalues along the real axis occurs [15]. However, for large system sizes these finite-size effects can be neglected and the density of eigenvalues can be compared with predictions obtained for products of complex Ginibre matrices. In particular, the radial density $P(r)$ of complex eigenvalues of the superoperator associated with the composition s random maps can be described by the algebraic law of Burda *et al.* [26, 29], while the distribution of the squared singular values of the superoperator (*i.e.* the eigenvalues of the positive matrix $\Psi\Psi^\dagger$) are described by the Fuss–Catalan distributions of order s [32, 36, 37, 38].

Our numerical results support the conjecture that the distribution of eigenvalues of a product of s independent random Ginibre matrices, $G_1G_2 \ldots G_s$, obtained in [29, 30], describe also the spectrum of s-th *power* G^s of a given random Ginibre matrix G. This observation encouraged us to compare statistical properties of s-step propagators of non-unitary quantum dynamical systems with the predictions of random matrices. Under the condition of strong classical chaos and sufficiently large coupling with the environment the corresponding one-step evolution operators can be described by the ensemble of real Ginibre matrices [13].

Investigating a generalized version of a model dynamical system — the quantum baker map interacting with an environment — we demonstrate that statistical properties of complex eigenvalues of s-step evolution operators associated with such deterministic dynamical systems agree with predictions obtained for products of random Ginibre matrices. For the dynamical system investigated the operator $\Phi\Phi^\dagger$ is a projection operator with spectrum containing $\{0, 1\}$, so it cannot be described by random matrices. However, for a larger number s of the time steps, the squared singular values of Φ^s can be described by the Fuss–Catalan distribution of order $s' = s - 1$ characteristic to the s'-th power of random matrices.

Thus products of random non-Hermitian matrices, used to describe matrix valued diffusion [20] or random density operators [36, 37], can also be applied to characterize statistical properties of multi-step evolution operators corresponding to generic quantum dynamical systems strongly interacting with an environment.

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