

CDT AS A SCALING LIMIT OF MATRIX MODELS*

JAN AMBJØRN

The Niels Bohr Institute, Copenhagen University
Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

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It is shown that generalized CDT, the two-dimensional theory of quantum gravity, constructed as a scaling limit from so-called causal dynamical triangulations, can be obtained from a cubic matrix model. It involves taking a new scaling limit of matrix models, which is more natural from a classical point of view.

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1. Introduction

The great versatility of matrix models or matrix integrals in theoretical physics is well illustrated by their particularly beautiful application in two-dimensional Euclidean quantum gravity (see [1, 2, 3, 4] for reviews). This theory can be defined as a suitable sum over triangulations, so-called “dynamical triangulations” (DT), whose continuum limit is obtained by taking the side lengths a of the triangles to zero. The method of DT was originally introduced as a non-perturbative worldsheet regularization of the Polyakov bosonic string [5, 6, 7]. There it was used with success (or to disappointment, depending on one's taste) to show rigorously that a tachyon-free version of Polyakov's bosonic string theory does not exist in target space dimensions $d > 1$ [8]. However, when viewed as a theory of 2d quantum gravity coupled to matter with central charge $c \leq 1$, the theory — non-critical string theory — is perfectly consistent, and matrix models have been used to solve in an elegant way the combinatorial aspects of the DT construction, where one sums over random surfaces glued together from equilateral triangles.

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The DT approach possesses a well-defined cut-off, the length a of the lattice links. As has been discussed in many reviews (for instance the ones mentioned above), a continuum limit can be defined when the lattice spacing is taken to zero while simultaneously renormalizing the bare cosmological constant and possibly other matter coupling constants. However, the continuum limit in question has some unconventional properties. We will show that there is another way of taking the scaling limit of the matrix models, which still relates them to a summation over triangulated random surfaces, the so-called *causal dynamical triangulations* (CDT) [9]. We will show that this limit is in a way more natural and it corresponds to starting out from a “classical” matrix-model theory. In accordance with this it does not lead to the somewhat unconventional renormalization encountered in the standard DT approach.

2. Hermitian matrix models

We can define the Hermitian matrix model for $N \times N$ matrices as a formal power series in \tilde{g}

$$\begin{aligned} Z(\tilde{g}) &= \int d\phi \, e^{-N\text{tr}(\frac{1}{2}\phi^2 - \frac{\tilde{g}}{3}\phi^3)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \, e^{-\frac{1}{2}N\text{tr}(\phi^2)} \left(\frac{N\tilde{g}}{3} \text{tr} \phi^3 \right)^k, \end{aligned} \quad (1)$$

$$d\phi = \prod_{\alpha \leq \beta} d\text{Re} \phi_{\alpha\beta} \prod_{\alpha < \beta} d\text{Im} \phi_{\alpha\beta}. \quad (2)$$

The integral can be evaluated in the standard way by performing all possible Wick contractions of $(\text{tr} \phi^3)^k$ and using

$$\langle \phi_{\alpha\beta} \phi_{\alpha'\beta'} \rangle = C \int d\phi \, e^{-\frac{1}{2} \sum_{\alpha\beta} |\phi_{\alpha\beta}|^2} \phi_{\alpha\beta} \phi_{\alpha'\beta'} = \delta_{\alpha\beta'} \delta_{\beta\alpha'}. \quad (3)$$

The geometric interpretation in the context of DT is illustrated in Fig. 1. An index is assigned with each vertex in a triangle and a matrix $\phi_{\alpha\beta}$ is assigned to the link (the side in the triangle) which contains the vertices labeled α and β . In that way we can associate $\text{tr} \phi^3$ with a triangle and the term $(\phi^3)^k$ in (2) with k triangles. Performing the Gaussian integrations indicated in Eq. (3) we are gluing together all triangles in all possible ways by identifying links as illustrated in the figure. This way of performing the integral will result in an asymptotic power series in \tilde{g} which can be Borel summed for $\tilde{g} < 0$.

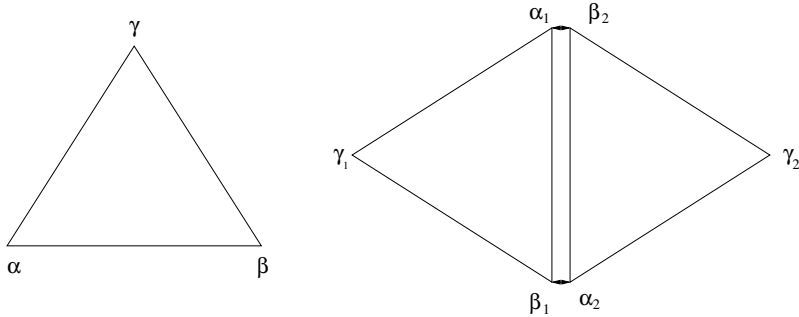


Fig. 1. The gluing of triangles via Gaussian integration.

If we want to perform the integral without power expanding the ϕ^3 part of the potential one takes advantage of the invariance of the action under $\phi \rightarrow U\phi U^\dagger$, where $U \in U(N)$. Thus the action depends only on the eigenvalues ℓ_i of ϕ and we can make the following decomposition of the measure of integration

$$d\phi e^{-\frac{N}{g_s} \text{tr} V(\phi)} \propto dU(N) \prod_{i=1}^N d\ell_i e^{-\frac{N}{g_s} V(\ell_i)} \prod_{i < j} |\ell_i - \ell_j|^2, \quad (4)$$

where $\prod_{i < j} |\ell_i - \ell_j|^2$ is the Jacobian, changing from ϕ to its eigenvalues and the unitary matrix U . The integral over the U matrices is now trivial since the action is independent of U .

The “classical” limit is obtained for $g_s \rightarrow 0$, where all eigenvalues are lumped together at ℓ_0 , where

$$V'(\ell_0) = 0. \quad (5)$$

However, for $g_s > 0$ the integration over the non-diagonal matrix elements produces the Vandermonde determinant $\prod_{i < j} |\ell_i - \ell_j|^2$, which acts as a “quantum” correction, a repulsion between different eigenvalues. The result is that eigenvalues are no longer concentrated at ℓ_0 . In the large N limit they occupy an interval around ℓ_0 .

As an example let us consider the matrix potential

$$\frac{1}{g_s} V(\phi) = \frac{1}{g_s} \left(-g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3 \right) \quad (6)$$

shown in Fig. 2. It is clear that this specific matrix integral only exists as the formal power series for finite N . For infinite N the eigenvalues will condense in an interval around ℓ_0 .

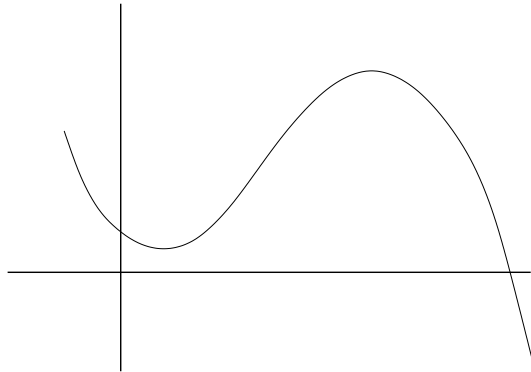


Fig. 2. The graph $-g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3$.

This interval is determined by a large N saddle-point equation. We will not discuss here the solution to that equation, but only mention that the so-called resolvent, which determines the distribution of eigenvalues is given by

$$w(z) := \left\langle \frac{1}{N} \text{tr} \frac{1}{z - \phi} \right\rangle = \frac{1}{Z} \int d\phi \frac{1}{N} \frac{1}{z - \phi} e^{-\frac{N}{g_s} \text{tr} V(\phi)}. \quad (7)$$

For $V(\phi) = -g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3$ from the saddlepoint equation or by other methods, (e.g. so-called loop equations) up to the leading order in N one gets

$$w(z) = \frac{1}{2g_s} \left(V'(z) + g(z - b) \sqrt{(z - c)(z - d)} \right), \quad (8)$$

where the constants b , c and d are determined by the requirement that $w(z) \rightarrow 1/z$ for $z \rightarrow \infty$. It should further be noticed that in the large N expansion each term has analyticity structure like $w(z)$ in the complex z -plane, *i.e.* a branch cut between c and d .

3. The conventional scaling limit

The usual scaling limit of the matrix model, relevant for non-critical strings and 2d Euclidean quantum gravity coupled to matter, is obtained (for fixed g_s) by fine-tuning g such that $b(g) = c(g)$. At this point the analytic structure of $w(z)$ changes from $(z - c(g))^{1/2} \rightarrow (z - c(g))^{3/2}$, and this change can only be accommodated by invoking arbitrary high k in the sum

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi e^{-\frac{N}{2g_s} \text{tr}(\phi^2)} \left(\frac{Ng}{3g_s} \text{tr} \phi^3 \right)^k. \quad (9)$$

This is why one geometrically can imagine a “continuum” limit where the size of each triangle shrinks to zero while the continuum size of the surface stays constant. More precisely, for our specific model one has

$$g = g_c (1 - \Lambda a^2) , \quad z = c(g_c) + aZ , \quad a \rightarrow 0 , \quad (10)$$

$$w(z) = \frac{1}{2} \left(V'(z) + g \sqrt{c(g_c) - d(g_c)} a^{3/2} W_E(Z, \Lambda) + O \left(a^{5/2} \right) \right) , \quad (11)$$

where the “continuum disk” amplitude is

$$W_E(Z, \Lambda) = \left(Z - \sqrt{2\Lambda/3} \right) \sqrt{Z + 2\sqrt{2\Lambda/3}} . \quad (12)$$

Here a has the interpretation as the length of the side of the triangles (polygons) which appear in $V(\phi)$.

Notice that actually the non-scaling part $V'(z)/2$ dominates when $a \rightarrow 0$ and renders the average number of polygons present in the ensemble with partition function $w(z)$ finite, even at the critical point. This somewhat embarrassing fact can be circumvented by differentiating $w(z)$ a sufficient number of times with respect to g and z , after which these “non-universal” contributions vanish since they are polynomials in g and z , but for the disk amplitude itself there is no such escape.

4. The new scaling limit

Until now we have considered two limits: the “classical limit”: $g_s = 0$ and “conventional scaling limit” of non-critical string theory: $g_s > 0$ and $g \rightarrow g_c(g_s)$. Is it possible to find a new, non-trivial scaling limit, closer to the classical limit when $g_s \rightarrow 0$. The answer is yes [10].

If one works out the details close to the conventional critical point $b(g_c) = c(g_c)$ one has

$$g_c(g_s) = \frac{1}{2} \left(1 - \frac{3}{2} g_s^{2/3} + O \left(g_s^{4/3} \right) \right) , \quad (13)$$

$$z_c(g_s) = c(g_c, g_s) = 1 + g_s^{1/3} + O \left(g_s^{2/3} \right) , \quad (14)$$

$$c(g_c) - d(g_c) = 4g_s^{1/3} + O \left(g_s^{2/3} \right) . \quad (15)$$

A non-trivial scaling can now be obtained if we fine-tune $g_s \rightarrow 0$ as

$$g_s = G_s a^3 . \quad (16)$$

Again, the scaling parameter a can be given the geometric interpretation as the link lengths of the polygons in $V(\phi)$. Note that the length of the cut goes to zero as $a \rightarrow 0$, thus we are closer to the “classical” limit. However, it will survive in the continuum limit

$$g = g_c(g_s) (1 - a^2 \Lambda) = \bar{g} (1 - a^2 \Lambda_{\text{cdt}} + O(a^4)) , \quad (17)$$

$$z = z_c + aZ = \bar{z} + aZ_{\text{cdt}} + O(a^2) , \quad (18)$$

$$\Lambda_{\text{cdt}} \equiv \Lambda + \frac{3}{2} G_s^{2/3} , \quad \bar{g} = \frac{1}{2} , \quad Z_{\text{cdt}} \equiv Z + G_s^{1/3} , \quad \bar{z} = 1 . \quad (19)$$

Using these definitions one computes in the limit $a \rightarrow 0$ that

$$w(z) = \frac{1}{a} \frac{\Lambda_{\text{cdt}} - \frac{1}{2} Z_{\text{cdt}}^2 + \frac{1}{2} (Z_{\text{cdt}} - H) \sqrt{(Z_{\text{cdt}} + H)^2 - \frac{4G_s}{H}}}{2G_s} , \quad (20)$$

$$h^3 - h + \frac{2G_s}{(2\Lambda_{\text{cdt}})^{3/2}} = 0 , \quad h = H / \sqrt{2\Lambda_{\text{cdt}}} , \quad (21)$$

$$w(z) = \frac{1}{a} W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s) . \quad (22)$$

Thus we have a situation where, contrary the situation we encountered taking the conventional scaling limit discussed above, the disk amplitude $w(z)$ really scales as one would expect from an ordinary correlation function.

We can now take the limit $G_s \rightarrow 0$ and we obtain

$$W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s) \rightarrow \frac{1}{Z_{\text{cdt}} + \sqrt{2\Lambda_{\text{cdt}}}} . \quad (23)$$

Thus we see that the cut where the eigenvalues are located shrinks to a point, indicating we have precisely the classical situation discussed above. This is indeed the case as we will discuss further shortly.

If we, on the other hand, take the limit $G_s \rightarrow \infty$ we obtain for the square root part of $W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)$

$$\frac{(Z_{\text{cdt}} - H) \sqrt{(Z_{\text{cdt}} + H)^2 - \frac{4G_s}{H}}}{2G_s} \rightarrow G_s^{-5/6} \left(Z - \sqrt{2\Lambda/3} \right) \sqrt{Z + 2\sqrt{2\Lambda/3}} .$$

Thus we recover the standard continuum disk function $W_E(Z, \Lambda)$ from (12) times a factor $G_s^{-5/6}$. If we write $g_s = a^3 G_s$ (in accordance with (16)) and keep g_s constant, while taking $a \rightarrow 0$, it means that $G_s \rightarrow \infty$ as a^{-3} . Thus $G_s^{-5/6} \sim a^{5/2}$ and we precisely recover the square root term in (11) if we remember that $w(z)$ and W_{cdt} according to (22) differ by a factor a . However, the part not related to the square root in W_{cdt} will not scale in the limit $G_s \rightarrow \infty$, in accordance with the previous discussion of standard scaling related to (11).

5. The matrix model

The new scaling can be obtained by a simple change of variables

$$\phi \rightarrow \bar{z} \hat{I} + a\Phi + O(a^2). \quad (24)$$

Up to a ϕ independent term we then have

$$V(\phi) = \bar{V}(\Phi), \quad \bar{V}(\Phi) \equiv \frac{\Lambda_{\text{cdt}}\Phi - \frac{1}{6}\Phi^3}{2G_s}, \quad (25)$$

where $V(\phi)$ is the potential given by (6). Thus we can write

$$Z(g, g_s) = a^{N^2} Z(\Lambda_{\text{cdt}}, G_s), \quad Z(\Lambda_{\text{cdt}}, G_s) := \int d\Phi e^{-N \text{tr} \bar{V}(\Phi)}. \quad (26)$$

The change of variable (24) explains immediately and in a trivial way the scaling (22) of $w(z)$

$$\frac{1}{z - \phi} = \frac{1}{a} \frac{1}{Z_{\text{cdt}} - \Phi} \quad \Rightarrow \quad w(z) = \frac{1}{a} W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s). \quad (27)$$

This relation is obviously correct to all orders in N . What is truly surprising is that the new scaling limit is itself a matrix model defined by $\bar{V}(\Phi)$ [11]. The continuum limit $a \rightarrow 0$ is thus described by a matrix model. This bears some resemblance with the Kontsevich matrix model, even the matrix potential is somewhat similar. But contrary to that model, where the continuum objects are the modular spaces of Riemann surfaces it turns out that the present matrix model actually describes a set of “real” continuum surfaces as we will describe below.

We have

$$\bar{V}(\Phi) \propto 2\Lambda_{\text{cdt}}\Phi - \frac{1}{3}\Phi^3, \quad (28)$$

and the cubic potential is shown in Fig. 3. It has a local minimum ℓ_0 determined by

$$\bar{V}'(\ell_0) = 0 \quad \Rightarrow \quad \ell_0 = -\sqrt{2\Lambda_{\text{cdt}}}. \quad (29)$$

Thus the “classical” limit of the matrix integral with potential $\bar{V}(\Phi)$, when only the minimum plays a role, leads to the following expectation value

$$\frac{1}{N} \left\langle \text{tr} \frac{1}{Z_{\text{cdt}} - \Phi} \right\rangle = \frac{1}{Z_{\text{cdt}} + \sqrt{2\Lambda_{\text{cdt}}}} = \lim_{G_s \rightarrow 0} W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s), \quad (30)$$

so in this way one can use this matrix model in a very explicit way to obtain the classical limit. As we will show in the next section even this classical limit has a non-trivial representation as a sum over certain random surfaces.

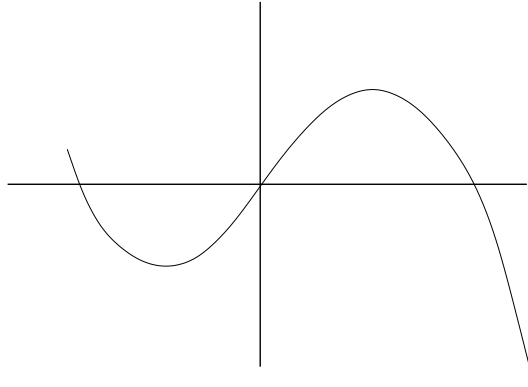


Fig. 3. The potential $\bar{V}(\ell)$, (25), with the local minimum $\ell_0 = -\sqrt{2\Lambda_{\text{cdt}}}$.

6. Geometric interpretation

Let us define some geometric objects, related by Laplace transformations, $W_{\lambda, g_s}(\ell)$ and $W_{\lambda, g_s}(x)$. We have now a little abuse of notation. Above we used capital letters for continuum, dimensionful variables and small letters for dimensionless (lattice) variables. Now we will use λ and g_s , x for continuum, dimensionful variables. In the end they will be identified with the continuum variables Λ_{cdt} , G_s and Z_{cdt} *etc.*, in the same way as the object $W_{\lambda, g_s}(x)$ in the end will be identified with $W_{\text{cdt}}(Z_{\text{cdt}}, \Lambda_{\text{cdt}}, G_s)$. However, we do it in order to stress that we are now starting from scratch with a geometric theory which in principle has nothing to do with the matrix model above. And in fact that was how these “quantum geometric” objects to first found and analyzed [12, 13].

$$W_{\lambda, g_s}(x) = \int_0^\infty d\ell e^{-x\ell} W_{\lambda, g_s}(\ell). \quad (31)$$

The objects are intended to represent the disk amplitude in the theory of two-dimensional quantum gravity based on causal dynamical triangulated random surfaces (CDT). The idea is to start by summing over all random surfaces with a boundary of length ℓ which admit a proper time foliation. The action used is just the area action (*i.e.* the cosmological term, since the Einstein term is trivial in 2d as long as we do not allow topology change. And we will not allow that presently. Thus the topology of the surface is just the trivial topology of the disk.) We denote this sum the disk amplitude $W_\lambda^{(0)}(\ell)$. It is called the CDT disk amplitude. If we add a boundary cosmological constant x at the boundary, we should add a boundary action $x\ell$, and thus the Laplace transformation (31) can be viewed as changing the path integral

over surfaces with a fixed boundary length to a path integral where we also integrate over all boundary lengths and instead keep fixed a boundary cosmological constant x .

We now allow a larger class of surfaces by allowing branching, *i.e.* we allow the spatial surface at a proper time t to split in two, the splitting assigned a weight g_s . g_s is clearly like a string coupling constant. The process is shown in Fig. 4. We are allowing the splitting of a spatial universe in two, but presently we do not allow for topology change of the two-dimensional surface, so we do not allow the two universes to join again, since that would create a handle and then change the two-dimensional topology. Had we allowed it, the total coupling constant associated with this process would have been g_s^2 , one factor for splitting, one factor for joining, like in string theory. The unshaded disk amplitude is $W_\lambda^{(0)}(\ell)$, while the full disk amplitude is denoted $W_{\lambda, g_s}(\ell)$ and is shown as the shaded graph on the l.h.s. of the equality sign.

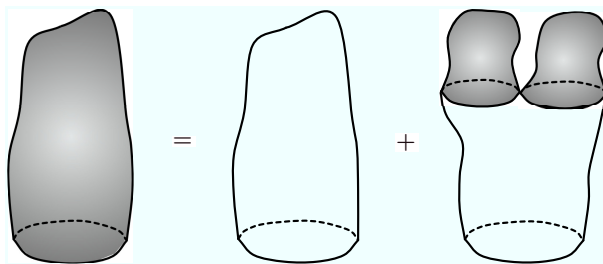


Fig. 4. The graphic representation of the integral equation satisfied by the full disk amplitude, Eq. (32).

Fig. 4 is a graphic representation of the following integral equation

$$W_{\lambda, g_s}(x) = W_\lambda^{(0)}(x) + g_s \int_0^\infty dt \int_0^\infty d\ell_1 d\ell_2 (\ell_1 + \ell_2) G_\lambda^{(0)}(x, \ell_1 + \ell_2; t) W_{\lambda, g_s}(\ell_1) W_{\lambda, g_s}(\ell_2). \quad (32)$$

In this equation the object $G_\lambda^{(0)}(\ell_1, \ell_2; t)$ denotes the “propagator” in CDT. It is defined in analogy with $W_\lambda^{(0)}(\ell)$. We sum over all two-dimensional geometries where we have a spatial entrance loop of length ℓ_1 and a spatial exit loop of length ℓ_2 , with the further constraint that all point on the exit loop is separated a geodesic distance t from the entrance loop. Again we assume that all points separated a geodesic distance $t' \leq t$ from the entrance loop form a connected one-dimensional space. $G_\lambda^{(0)}(x, \ell'; t)$ denotes the Laplace transform of $G_\lambda^{(0)}(\ell, \ell'; t)$ with respect to ℓ .

In the same way as we generalized the geometries which entered into path integral defining $W_\lambda^{(0)}(\ell)$, by allowing space to split, and in this was obtained $W_{\lambda,g_s}(\ell)$, we can allow for the spatial hyper-surface at any time $t' \leq t$ to separate in two. One of these will then be connected to the exit loop while the other will eventually vanish in the vacuum. This is shown in Fig. 5. We denote this generalized propagator $G_{\lambda,g_s}(\ell_1, \ell_2; t)$. For both $G_{\lambda,g_s}(\ell_1, \ell_2; t)$ and $G_\lambda^{(0)}(\ell_1, \ell_2; t)$ we introduce, again in analogy with the definitions for $W_{\lambda,g_s}(\ell)$ and $W_\lambda^{(0)}(\ell)$, the Laplace transforms

$$G_{\lambda,g_s}(x, y; t) = \int_0^\infty d\ell_1 \int_0^\infty d\ell_2 e^{-x\ell_1} e^{\ell_2 y} G_{\lambda,g_s}(\ell_1, \ell_2; t), \quad (33)$$

as well as the hybrid forms $G_{\lambda,g_s}(x, \ell_2; t)$ and $G_{\lambda,g_s}(\ell_1, y; t)$. Here x and y denotes boundary cosmological constants at the entry and exit boundaries.

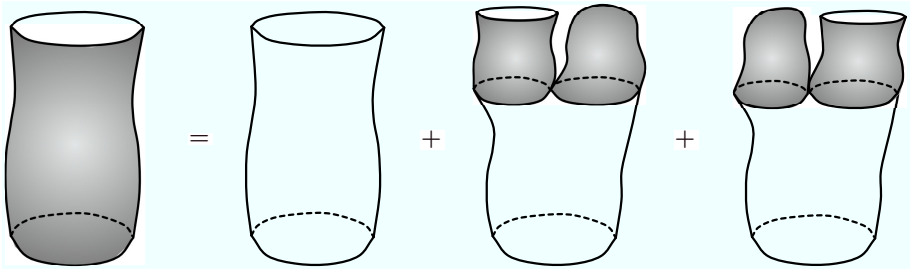


Fig. 5. The graphic illustration of the geometries which contribute to the propagator $G_{\lambda,g_s}(x, y; t)$ in Eq. (34).

The shaded parts of graphs in Fig. 5 represent the full, g_s -dependent propagator and the full g_s -dependent disc amplitude, and the non-shaded parts the CDT propagator, where $g_s = 0$. In all four graphs, the geodesic distance from the final to the initial loop is given by t , while the “baby-universes” which end in the vacuum can terminate their life at any positive time after they have been created, also at a time *larger* than t .

From Fig. 5 one can write down an integral equation much like Eq. (32) for $W_{\lambda,g_s}(x)$. However it is convenient to differentiate this equation with respect to t and we then obtain

$$a^\varepsilon \frac{\partial}{\partial t} G_{\lambda,g_s}(x, y; t) = - \frac{\partial}{\partial x} \left[\left(a(x^2 - \lambda) + 2g_s a^\delta a^{\eta-1} W_{\lambda,g_s}(x) \right) G_{\lambda,g_s}(x, y; t) \right]. \quad (34)$$

In this equation we have explicitly assumed that we have some kind of regularized theory, on a lattice, say. The equation is then first written in terms of

the dimensionless lattice variables and the translated into continuum notation by inserting the relation between the dimensionless variables and their continuum counterparts, in this way introducing the lattice cut-off a . This cut-off will then appear with a power determined by the dimension of the continuum variables, except for a subtlety related to W , to be explained now. The lattice cut-off a is assigned length dimension 1, the time t assigned the unspecified length dimension ε , and the disk amplitude W the length dimension $-\eta$. We assume x , as the coupling constant conjugate to the length ℓ has length dimension -1 . We leave the length dimension of g_s unspecified as $-\delta$.

Denote the dimensionless lattice variables and observables by $t_{\text{reg}}, W_{\text{reg}}$ etc. We thus have $t_{\text{reg}} = t/a^\varepsilon$. Similarly with W and W_{reg} : $W_{\text{reg}} = W a^\eta$, except that we allow for the possibility that W_{reg} is not scaling when $a \rightarrow 0$ (i.e. that $\eta > 0$). Although at first sight a little strange this was precisely what happened in the ordinary scaling limit of the matrix models, as we noted in the discussion above (see Eq. 11) which shows that in the ordinary scaling limit we have $\eta = 3/2$). Thus we will allow for this possibility. We can summarize as follows

$$W_{\text{reg}} \xrightarrow{a \rightarrow 0} a^\eta W_\lambda(x), \quad \eta < 0, \quad (35)$$

$$t_{\text{reg}} \xrightarrow{a \rightarrow 0} t/a^\varepsilon, \quad \varepsilon = 1. \quad (36)$$

$$W_{\text{reg}} \xrightarrow{a \rightarrow 0} \text{const.} + a^\eta W_\lambda(x), \quad \eta = 3/2, \quad (37)$$

$$t_{\text{reg}} \xrightarrow{a \rightarrow 0} t/a^\varepsilon, \quad \varepsilon = 1/2. \quad (38)$$

The values of the exponents ε and η are written to the right in equations (35)–(38). They tell us that when $\eta < 0$, i.e. when $W_{\lambda, g_s}(x)$ scales, then $\varepsilon = 1$, while if $\eta > 0$ then $\varepsilon = 1/2$ and $\eta = 3/2$. The reason these exponent are uniquely determined is that we have the geometric picture shown in Fig. 6, which couples W and G and leads to consistency relations for the scaling of W

$$-\frac{\partial W_{\lambda, g_s}(x)}{\partial \lambda} = \int_0^\infty dt \int_0^\infty d\ell G_{\lambda, g_s}(x, \ell; t) \ell W_{\lambda, g_s}(\ell). \quad (39)$$

We know the scaling dimension of $G_{\lambda, g_s}(x, \ell; t)$. It is zero. It all goes back to the fact that G as a propagator has to satisfy the composition rule

$$G(\ell_1, \ell_2; t_1 + t_2) = \int_0^\infty d\ell G(\ell_1, \ell; t_1) G(\ell, \ell_2; t_2), \quad (40)$$

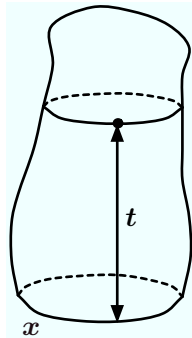


Fig. 6. The differentiation of the disk amplitude with respect to the cosmological constant leads to the disk amplitude with one marked point. Such a geometry has the unique decomposition shown in the figure, where the loop has a geodesic distance t from the boundary loop. The equation graphically represented in the figure is (39).

valid for both $G_{\lambda, g_s}(\ell_1, \ell_2; t)$ and $G_{\lambda}^{(0)}(\ell_1, \ell_2; t)$. This means that $G(\ell_1, \ell_2; t)$ has to scale like a^{-1} and the Laplace transform $G(x, \ell; t)$ thus as a^0 . Combined with that fact that the cosmological constant λ has length dimension -2 , we are led to right-hand side values of ε and η in Eqs. (35)–(38). It is a beautiful example of the constraints imposed by *quantum geometry*, and it is remarkable that one is able to derive the non-trivial, ordinary matrix model value $\eta = 3/2$ in Eq. (11) from such simple geometric considerations.

If we choose the solution (35)–(36) then we can solve the geometric equations (32) and (34). A glance at Eq. (34) shows that *if* the term involving g_s is going to play a role we have to take $\delta = 3$, the same result as in the new scaling limit of the matrix models. If we make this choice the solution for $W_{\lambda, g_s}(x)$ is precisely the one given by the new scaling limit of the matrix models. Thus the new scaling limit has indeed a geometric representation in terms of random surfaces, and even a regularized lattice representation where a lattice spacing a is taken to zero. All this is discussed in detail in [12, 13]. The main characteristic is here that the geodesic distance (the proper time t) has canonical scaling dimension, identical to that of space and there is a smooth limit $g_s \rightarrow 0$ where one obtains the original CDT solution [9].

If we choose the solution (37)–(38) and solve the geometric equations (32) and (34) we find a completely different solution. It is characterized by a different scaling of the geodesic distance or proper time t . We see that Eq. (34) is only non-trivial *if* we choose the dimension of g_s to be zero, and the first term on the r.h.s. of Eq. (34) is then irrelevant. The equation is thus reflecting an excessive branching off of baby universes. There is nothing else!

Thus, looking at Figs. 4 and 5, we have no unshaded parts of the graphs. They simply play no role in the scaling limit where the lattice spacing $a \rightarrow 0$. The typical geometry which appears in the path integral will be very fractal and will have Hausdorff dimension $d_h = 4$, not $d_h = 2$ as one would expect from a “nice” two-dimensional geometry. This is described in detail in [15]. The wild branching of baby universes is illustrated in Fig. 7. This limit with $d_h = 4$ corresponds to the “ordinary” scaling limit of the matrix model.

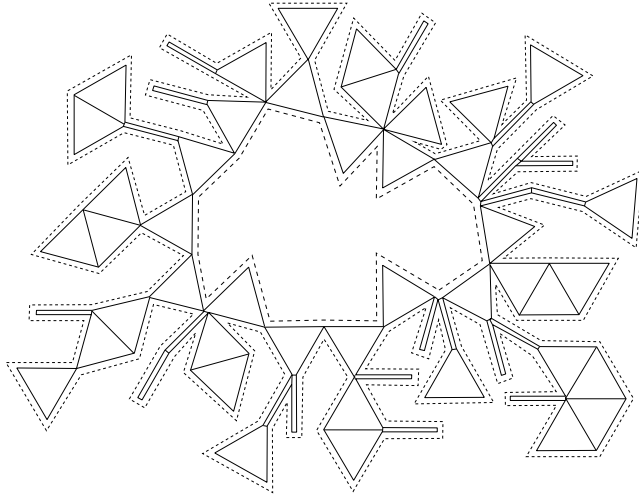


Fig. 7. Part of a “typical” triangulation of the disk which one will encounter in the “ordinary scaling” limit of the matrix models, where equations (37) and (38) are satisfied. The dashed and dotted lines represent two “spatial” curves separated by one (proper)-time step, plus all the baby universes which are cut off at the this time step.

This is in sharp contrast to the solution provided by conditions (35)–(36). There the typical geometry has $d_h = 2$ and the total number of baby universes created is finite. This implies that a typical geometry will look as shown in Fig. 8. In Fig. 8 we have drawn an amplitude which is more complicated than the disk amplitude (we have two entrance loops, and the surface also have a handle, *i.e.* it is a higher genus surface). This gives us the opportunity to emphasize that while we have mainly discussed geometries of the simplest topology, the whole discussion of the new scaling generalizes to any genus surface and with any number of boundary loops, and it matches precisely the $1/N^2$ expansion in the matrix models of expectation values of multiple trace operators [12, 14]

$$\omega(z_1, \dots, z_n) = \left\langle \frac{1}{N} \text{tr} \frac{1}{z_1 - \phi} \cdots \frac{1}{N} \text{tr} \frac{1}{z_n - \phi} \right\rangle_{\text{connected}}. \quad (41)$$

These multiple trace operators are obvious generalizations of the single trace operator defined in Eq. (7).

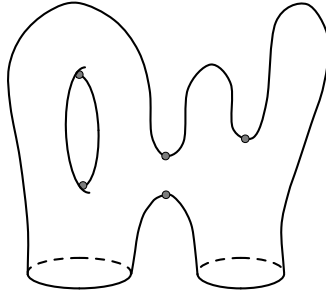


Fig. 8. A “typical” geometry (in the continuum limit) of the disk which one will encounter in the “new” scaling limit of the matrix models, where equations (35) and (36) are satisfied. We have shown a surface where the topology of the surface is also changed from that of a simple disk to a disk with a handle, see discussion in the main text.

7. Unfinished stuff

For the ordinary matrix models we have a description of conformal matter coupled to 2d quantum gravity by multicritical one-matrix models and by two-matrix models.

Similarly, it is easy to couple matter to the “plain” CDT model. Ising models and multiple Pott models coupled to CDT have been studied numerically [16].

Now that we have a matrix model description of the generalized CDT models it seems natural to apply the same technique as was applied for the ordinary matrix models and in this way use matrix models to study the matter systems defined on the CDT-like set of random surfaces. From the computer simulations referred to, it seems that one obtain the flat space-time exponents. It would be very interesting if one could obtain a simple proof of the any conformal field theory exponent from a matrix integral. It would provide us with an explicit realization of these critical systems, even at a regularized level. Work in this direction is in progress.

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