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ON SPECTRAL ACTION OVER BIEBERBACH MANIFOLDS

Piotr Olczykowski, Andrzej Sitarz†

The Marian Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

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We compute the leading terms of the spectral action for orientable three dimensional Bieberbach manifolds using two different methods: the Poisson summation formula and the perturbative expansion. Assuming that the cut-off function is not necessarily symmetric we find that the scale invariant part of the perturbative expansion might only differ from the spectral action of the flat three-torus by the value of the eta invariant.

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1. Introduction

Bieberbach manifolds are compact manifolds, which are quotients of the Euclidean space by a free, properly discontinuous and isometric action of a discrete group. The classification of all Bieberbach manifolds is a complex problem in itself (see [1] and the references therein). The non-trivial lowest dimensional examples appear in three dimensions and have been first described in [2, 3].

In this paper we study whether the non-perturbative form of the spectral action as introduced by Connes [4] and calculated for several examples in [5] and [6] can distinguish between the torus itself and the chosen Bieberbach three-manifolds.

2. Three-dimensional Bieberbach manifolds

In this section, we briefly recall the description of three-dimensional Bieberbach manifolds as quotients of three-dimensional tori by the action of a finite discrete group. We use the algebraic language, starting with the algebra of the functions on the three-torus $\mathbb{T}^3$, which we view as generated by three mutually commuting unitaries U, V, W.

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There are, in total, 10 different Bieberbach three-dimensional manifolds, six orientable (and that includes the three-torus) and four non-orientable ones. They all could be defined as quotients of the three-torus by the following free actions of a discrete group on the three-torus. We restrict our attention here to the orientable manifolds different from the three-torus. The respective action of discrete groups on the unitary generators of the algebra of functions on the three-torus is summarized in Table I.

TABLE I

Name	Group	Generators	Action on the generators of $\mathbb{T}^3$
G_2	$\mathbb{Z}_2$	e	$e \triangleright U = -U$, $e \triangleright V = V^*$, $e \triangleright W = W^*$
G_3	$\mathbb{Z}_3$	e	$e \triangleright U = e^{\frac{2}{3}\pi i}U$, $e \triangleright V = W$, $e \triangleright W = W^*V^*$
G_4	$\mathbb{Z}_4$	e	$e \triangleright U = iU$, $e \triangleright V = W$, $e \triangleright W = V^*$
G_5	$\mathbb{Z}_6$	e	$e \triangleright U = e^{\frac{1}{3}\pi i}U$, $e \triangleright V = W$, $e \triangleright W = WV^*$
G_6	$\mathbb{Z}_2 \times \mathbb{Z}_2$	e_1, e_2	$e_1 \triangleright U = -U$, $e_1 \triangleright V = V^*$, $e_1 \triangleright W = W^*$ $e_2 \triangleright U = U^*$, $e_2 \triangleright V = -V$, $e_2 \triangleright W = -W^*$

3. The spectra of the Dirac operator

The spectrum of the Dirac operator on Bieberbach manifolds has been first calculated by Pfäffle [7]. We shall use his results, though rewritten in a form more suitable for our purposes. As the covering three-torus we shall choose the equilateral one (with lengths of three fundamental circles equal).

The spectrum can be also easily derived using the formalism of real spectral triples (which we shall present elsewhere [8]).

Let us fix here the notation. By D^3_τ we denote the Dirac operator on the three-torus with equal lengths of three circles, with the eigenvalues

$$\lambda_{k,l,m} = \pm\sqrt{(k+\epsilon_1)^2 + |l+\epsilon_2+\tau(m+\epsilon_3)|^2}\,, \qquad k,l,m \in \mathbb{Z}\,,$$

where ϵ_1, ϵ_2, ϵ_3 are 0 or $\frac{1}{2}$ and depend on the choice of the spin structure and $|\tau| = 1$, τ not real. The choice of $\tau = i$ corresponds to the usually assumed Dirac operator. We shall denote the spectrum of D^3_τ, counted with the multiplicities, by $\mathcal{S}p^3_\tau$. Further, we shall need the generalized Dirac operator on the circle, taking the standard one, with eigenvalues

$$\lambda_k = \alpha k + \beta\,, \qquad k \in \mathbb{Z}, \alpha, \beta \in \mathbb{R}\,,$$

we shall denote its spectrum by $\mathcal{S}p^1_{\alpha,\beta}$.

Whenever we write a coefficient in front of the spectrum set we mean the same set but with the multiplicities reduced by that factor (of course, if the coefficient is $\frac{1}{n}$ this requires that the multiplicities must be divisible by n).

As above the spin structures of the three-torus are parametrized by $\epsilon_1, \epsilon_2, \epsilon_3$. Furthermore, the representation of the discrete group on the Hilbert space of spinors can add an additional spin structure on the quotient, labelled here by $\delta = \pm 1$.

We have the following spectra of the Dirac operator:

- G_2

  Here we must have $\epsilon_1 = \frac{1}{2}$ and have eight possible spin structures, parametrized by choice of ϵ_2, ϵ_3 and δ. As the Dirac operator on three torus we take D^3_i, with eigenvalues $\pm\sqrt{(k+\frac{1}{2})^2 + (l+\epsilon_2)^2 + (m+\epsilon_3)^2}$, $k, l, m \in \mathbb{Z}$

$$\mathcal{S}p_{G_2} = \begin{cases} \frac{1}{2}\mathcal{S}p^3_i & \text{if } \epsilon_2 = \frac{1}{2} \text{ or } \epsilon_3 = \frac{1}{2}\,, \\ \frac{1}{2}\left(\mathcal{S}p^3_i \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{2,1-\frac{1}{2}\delta} & \text{if } \epsilon_2 = \epsilon_3 = 0\,,\ \delta = \pm 1\,. \end{cases}$$

  Observe that only in the $\epsilon_2 = \epsilon_3 = 0$ case the spectrum is not the same as for the torus.

- G_3

  In this case only the spin structures with $\epsilon_2 = \epsilon_3 = 0$ could be projected to the quotient space. The parameter δ is fixed by the choice of the spin structure ϵ_1. As the projectable Dirac we take $D^3_{e^{\frac{2\pi i}{3}}}$

$$\mathcal{S}p_{G_3} = \begin{cases} \frac{1}{3}\left(\mathcal{S}p^3_{e^{\frac{2\pi i}{3}}} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{3,\frac{1}{2}} & \text{if } \epsilon_1 = \frac{1}{2}\,,\ \delta = 1\,, \\ \frac{1}{3}\left(\mathcal{S}p^3_{e^{\frac{2\pi i}{3}}} \setminus 2\mathcal{S}p^1_{1,0}\right) \cup 2\mathcal{S}p^1_{3,2} & \text{if } \epsilon_1 = 0\,,\ \delta = -1\,. \end{cases}$$

- G_4

  For the action of Z_4 only $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \epsilon_3$ spin structures could be projected onto the quotient, the Dirac operator which commutes with the action of the discrete group is $D^3_{e^{\frac{\pi i}{4}}}$. There are four possible spin structures and the corresponding spectra are:

$$\mathcal{S}p_{G_4} = \begin{cases} \frac{1}{4}\left(\mathcal{S}p^3_{e^{\frac{\pi i}{4}}}\right) & \text{if } \epsilon_2 = \epsilon_3 = \frac{1}{2}\,, \\ \frac{1}{4}\left(\mathcal{S}p^3_{e^{\frac{\pi i}{4}}} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{4,\frac{3}{2}-\delta} & \text{if } \epsilon_2 = \epsilon_3 = 0\,,\ \delta = \pm 1\,. \end{cases}$$

- G_5

  Here, the situation is similar as in the G_3 case and only the spin structures with $\epsilon_1 = \frac{1}{2}$ and $\epsilon_2 = \epsilon_3 = 0$ could be projected to the quotient space. The parameter δ is free and gives us two possible spin structures. The projectable Dirac we take $D^3_{e^{\frac{2\pi i}{3}}}$

$$\mathcal{S}p_{G_5} = \begin{cases} \frac{1}{6}\left(\mathcal{S}p^3_{e^{\frac{2\pi i}{3}}} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{6,\frac{1}{2}} & \text{if } \epsilon_1 = \frac{1}{2}\,,\ \delta = 1\,, \\ \frac{1}{6}\left(\mathcal{S}p^3_{e^{\frac{2\pi i}{3}}} \setminus 2\mathcal{S}p^1_{1,\frac{1}{2}}\right) \cup 2\mathcal{S}p^1_{6,\frac{7}{2}} & \text{if } \epsilon_1 = \frac{1}{2}\,,\ \delta = -1\,. \end{cases}$$

- G_6

  In this case the only projectable spin structure are those with $\epsilon_1 = \epsilon_2 = \epsilon_3 = \frac{1}{2}$, the projectable Dirac operator is D^3_i and the spectrum of D remains the same (apart from rescaled multiplicities) for each of four spin structures over G_6

$$\mathcal{S}p_{G_6} = \mathcal{S}p^3_i\,.$$

4. The spectral action

Since we have split the spectra of the Dirac operators into the sets, which corresponds to the known cases, we shall begin by calculating the spectral action of the corresponding three-dimensional and one-dimensional tori.

We assume here that the spectral action depends on D and not on D^2 (that is we do not restrict ourselves to the even functions over the spectrum), therefore there is a slight change of notation when compared to [6].

We begin with the action for the torus with the Dirac D^3_τ

$$\begin{aligned} \mathcal{S}\left(D^3_\tau, \Lambda\right) &= 2\sum_{k,l,m} f\left(\pm\frac{\sqrt{(k+\epsilon_1)^2 + |l+\epsilon_2+\tau(m+\epsilon_3)|^2}}{\Lambda}\right) \\ &= 2\widehat{f}(0,0,0) + o\left(\Lambda^{-1}\right) = 2\int dx \int dy \int dz f\left(\frac{\sqrt{x^2+|y+\tau z|^2}}{\Lambda}\right) \\ &= \frac{8\pi}{\sin\phi}\Lambda^3 \int\limits_0^\infty d\rho\, f(\rho)\rho^2 + o\left(\Lambda^{-1}\right)\,, \end{aligned}$$

where $\tau = e^{i\phi}$, $\widehat{f}$ denotes the Fourier transform of f considered as a function of three variables

$$\widehat{f}(k_x, k_y, k_z) = \int\limits_{\mathbb{R}^3} f(x,y,z) e^{2\pi i(k_x x + k_y y + k_z z)} dx dy dz\,,$$

and we have used the Poisson summation formula.

For the Dirac operator $D^1_{\alpha,\beta}$ over the circle we have

$$\begin{aligned}\mathcal{S}\left(D^1_{\alpha,\beta},\Lambda\right) &= \sum_k f\left(\frac{\alpha k+\beta}{\Lambda}\right)\\ &= \hat{f}(0)+o\left(\Lambda^{-1}\right) = \Lambda\int_{\mathbb{R}} f\left(\frac{\alpha k+\beta}{\Lambda}\right)dx + o\left(\Lambda^{-1}\right)\\ &= \frac{1}{\alpha}\Lambda\int_{\mathbb{R}} f(x)dx + o\left(\Lambda^{-1}\right)\,.\end{aligned}$$

In the formula above $\hat{f}$ is the usual Fourier transform of f. Let us observe that the following identity occurs

$$\mathcal{S}\left(D^1_{1,\gamma},\Lambda\right) = \alpha\mathcal{S}\left(D^1_{\alpha,\beta},\Lambda\right)$$

independently of the values of α,β and γ.

We can now formulate

Theorem 4.1. *The non-perturbative spectral action over the orientable Bieberbach manifolds with the Dirac operator projected from the equilateral Dirac operator over the three-torus is (up to a scaling and up to order $o(\Lambda^{-1})$) indistinguishable from the spectral action over the three-torus.*

Proof. Of course, only the cases when the spectrum differs significantly for the spectrum of the Dirac over three torus may give rise to some differences. However, observe that the difference in the spectra is always of the form

$$-\frac{1}{n}2\left(\mathcal{S}p^1_{1,\gamma}\right)+2\left(\mathcal{S}p^1_{n,\beta}\right)\,.$$

The coefficient denotes the multiplicity of the eigenvalues from the part of the spectrum, whereas its sign denotes whether part of the spectrum is present in the spectrum of chosen Bieberbach manifold (then it is $+$) or absent from it (then it is $-$). The constants β and γ vary from case to case, n depends on the order of the discrete group G so that the manifold if T^3/G.

From the observation above, however, we know that the resulting spectral action components will not depend on β and γ and we will obtain

$$-\frac{2}{n}\left(\int_{\mathbb{R}} f(x)dx\right)+\frac{2}{n}\left(\int_{\mathbb{R}} f(x)dx\right)=0\,,$$

and hence will not contribute to the leading terms of the spectral action. □

5. The perturbative spectral action

We can explicitly calculate the difference between the spectral action on the Bieberbach manifolds and the spectral action on the three-torus, expanding then the result in Λ.

Consider first an even function f. Taking f be a Laplace transform of h

$$f(s) = \int_0^\infty e^{-sx} h(x) dx \,,$$

we can write

$$\mathrm{Tr}\; f\left(\frac{|D|}{\Lambda}\right) = \int_0^\infty \mathrm{Tr}\; e^{-x\frac{|D|}{\Lambda}} h(x) dx \,.$$

Knowing the spectra of the Dirac operator over Bieberbach manifolds when compared to the three torus we can calculate the difference.

First we need the technical lemma. Let $D^1_{\alpha,\beta}$ be (as denoted before) the spectrum of the rescaled Dirac over the circle. We calculate exactly the exponential $\mathrm{Tr}\; e^{-p|D^1_{\alpha,\beta}|}$, assuming that $|\beta| < \alpha$.

$$\begin{aligned} \mathrm{Tr}\; e^{-p|D^1_{\alpha,\beta}|} &= e^{-p|\beta|} + \sum_{k=1}^\infty e^{-p(\alpha k+\beta)} + \sum_{k=1}^\infty e^{-p(\alpha k-\beta)} \\ &= e^{-p|\beta|} + \left(\frac{e^{-p\alpha}}{1-e^{-p\alpha}}\right) 2\cosh(p\beta) \,. \end{aligned}$$

Taking into account that $p = \frac{x}{\Lambda}$ we can take the Laurent expansion for large values of Λ

$$e^{-p|\beta|} + \left(\frac{e^{-p\alpha}}{1-e^{-p\alpha}}\right) 2\cosh(p\beta) \sim \frac{2}{\alpha}\frac{\Lambda}{x} + o\left(\frac{x}{\Lambda}\right) \,. \tag{1}$$

We can now state

Theorem 5.1. *The even component of the function f in the spectral action is the same up to order $o(\Lambda^{-1})$ on all three-dimensional Bieberbach manifolds (including three-torus).*

Proof. As we have seen only for some of the spin structures the spectra of the Dirac operator differ from the spectrum of the three torus (apart from the trivial factor of multiplicities). In general, we have, for the Bieberbach G_x, which is a quotient of the torus T^3 the following relation of the spectra

$$\mathcal{S}p(G_x) = \frac{1}{n_x}\Big(\mathcal{S}p\left(T^3\right) \setminus 2\mathcal{S}p_{1,\epsilon}\Big) + 2\mathcal{S}p_{n_x,\epsilon'} \,,$$

where n_x are integer constants, which are $n_2 = 2, n_3 = 3, n_4 = 4, n_5 = 6$ and ϵ, ϵ' depend on the choice of spin structure.

So the difference between the spectral actions on the three-torus $\mathbb{T}^3$ and on the Bieberbach G_x could be calculated from this difference of the spectra

$$\mathcal{S}(G_x, \Lambda) - \frac{1}{n_x}\mathcal{S}\left(T^3, \Lambda\right) = 2 \sum_{\lambda \in \mathcal{S}p_{n_x,\epsilon'}} f\left(\frac{\lambda}{\Lambda}\right) - \frac{2}{n_x} \sum_{\lambda \in \mathcal{S}p_{1,\epsilon}} f\left(\frac{\lambda}{\Lambda}\right). \quad (2)$$

As the spectra $\mathcal{S}p_\pm$ are the spectra of rescaled Dirac on the circle, using the result (1) we see that only the Λ component in the perturbative expansion could appear. Calculating it explicitly

$$\mathcal{S}(G_x, \Lambda) - \frac{1}{n_x}\mathcal{S}\left(T^3, \Lambda\right) = \Lambda\left(2\frac{2}{n_x} - \frac{2}{n_x}2\right)\int \frac{1}{x} f(x)dx + o\left(\Lambda^{-1}\right) = o\left(\Lambda^{-1}\right).$$

Therefore, irrespective of the chosen spin structure and Bieberbach manifold, even component of the function determining the spectral action gives the same result. □

The situation is different for the odd component of f. We can always write, for an odd function f

$$f\left(\frac{D}{\Lambda}\right) = \frac{D}{|D|}\phi\left(\frac{|D|}{\Lambda}\right),$$

where ϕ is an even function. Assuming that ϕ is a Laplace transform of h the odd part of the spectral action becomes

$$\mathrm{Tr}\ f\left(\frac{D}{\Lambda}\right) = \mathrm{Tr}\ \frac{D}{|D|}\phi\left(\frac{|D|}{\Lambda}\right) = \int_0^\infty \mathrm{Tr}\ \frac{D}{|D|}e^{-x\frac{|D|}{\Lambda}}h(x)dx\,.$$

For the spectra of Dirac operators, which we know, we can calculate the function under the integral

$$\mathrm{Tr}\ \frac{D}{|D|}e^{-x\frac{|D|}{\Lambda}}$$

and obtain (again we denote $p = \frac{x}{\Lambda}$)

$$\begin{aligned}
\mathrm{Tr}\ \mathrm{sign}(D^1_{\alpha,\beta})e^{-p|D^1_{\alpha,\beta}|} &= \mathrm{sign}(\beta)e^{-p|\beta|} + \sum_{k=1}^\infty e^{-p(\alpha k+\beta)} - \sum_{k=1}^\infty e^{-p(\alpha k-\beta)} \\
&= \mathrm{sign}(\beta)e^{-p|\beta|} - \left(\frac{e^{-p\alpha}}{1-e^{-p\alpha}}\right)2\sinh(p\beta)\,.
\end{aligned}$$

We can expand the function for small p around 0

$$\mathrm{sign}(\beta)e^{-p|\beta|} - \left(\frac{e^{-p\alpha}}{1-e^{-p\alpha}}\right) 2\sinh(p\beta) \sim \mathrm{sign}(\beta)\frac{\alpha - 2|\beta|}{\alpha} + o(p)\,.$$

Therefore, only (up to terms of order $o(\Lambda^{-1})$) only scale invariant term can appear. We have

Theorem 5.2. *The odd component of the function f gives rise to a difference in the spectral action on the Bieberbach manifolds in the scale invariant part of the action. The difference equals the eta invariant of the Dirac operator on the Bieberbach manifold.*

Proof. First of all, observe that for the rescaled Dirac operator on the circle $D_{\alpha,\beta}$ the term

$$\mathrm{sign}(\beta)\frac{\alpha - 2|\beta|}{\alpha}\,,$$

is the eta invariant $\eta(D^1_{\alpha,\beta})$, which measures the antisymmetry between the positive and negative parts of the spectrum of $D^1_{\alpha,\beta}$. Therefore, for any of the spin structures of the circle, the term vanishes for the standard Dirac operator (that is, $D^1_{1,\frac{1}{2}}$ or $D^1_{1,0}$, using the notation of the paper). As a consequence, the difference between the (rescaled) spectral action on the three-torus $\mathbb{T}^3$ and on the Bieberbach G_x is (up to order $o(\Lambda^{-1})$

$$\mathcal{S}(G_x,\Lambda) - \frac{1}{n_x}\mathcal{S}\left(T^3,\Lambda\right) = 2\eta\left(D^1_{n_x,\epsilon'_x}\right)\phi(0)\,,$$

where n_x is as before and ϵ'_x depends on the chosen spin structure, and we have used that ϕ is a Laplace transform of h, so that

$$\int_0^\infty h(x)dx = \phi(0)\,.$$

As this is, however, the only component of the spectrum of the Bieberbach manifolds, we have

$$2\eta\left(D^1_{n_x,\epsilon'_x}\right) = \eta\left(D^3_{G_x,\epsilon}\right)\,,$$

and, finally

$$\mathcal{S}(G_x,\Lambda) = \eta\left(D^3_{G_x,\epsilon}\right)\Phi(0)\,.$$

The value of η invariant can be calculated explicitly for the manifolds G_2, G_3, G_4, G_5 and the chosen spin structures for which it does not vanish, Table II shows the results (A and B denote the spin structures, giving rise to asymmetric spectrum, in the order presented in Section 3) □

TABLE II

Name	A	B
G_2	1	-1
G_3	$\frac{4}{3}$	$-\frac{2}{3}$
G_4	$\frac{3}{2}$	$-\frac{1}{2}$
G_5	$\frac{5}{3}$	$-\frac{1}{3}$

In fact, the result is not entirely surprising. From the general results of Bismut and Freed [9] one knows the small-t asymptotic of the following function of the Dirac operator on the odd dimensional manifolds

$$\mathrm{Tr}\, \frac{D}{|D|} e^{-t|D|} = \eta(D) + \sum_{l=0}^{\infty} (A_l + B_l \log t) t^{2l+2} .$$

We can calculate then the leading term of the spectral action arising from an odd function to be

$$\mathcal{S}(D, \Lambda, f_o) = \eta(D)\phi(0) + o\left(\Lambda^{-1}\right) .$$

We shall finish this section by observing why this effect was not picked by the methods used earlier, which involved sum over the entire spectrum with the help of the Poisson summation formula.

Observe that the η invariant would appear if $\phi(0) \neq 0$. Since our function $f(x) = \mathrm{sign}(x)\phi(|x|)$ that means that f is odd, but discontinuous at $x = 0$. Therefore, the previous considerations were valid but since were (implicitly) assuming continuity of f we could not have obtained any deviation from the spectral action over the torus.

6. Conclusions

We have shown that apart from the possible difference arising from the eta invariant the perturbative spectral action is exactly the same for all three dimensional Bieberbach manifolds as for the three torus. This is not at all surprising as all terms in the perturbative expansion (for the symmetric cut-off) depend on the Riemann curvature and Bieberbach manifolds are flat. The new result is the appearance of slight modifications when the cut-off function has an asymmetric part. Although here we obtain an invariant, it would be interesting to see if such a term might appear in some more complicated models, with some extra degrees of freedom coming from discrete spectral triples, for instance.

Note added: After the paper was submitted, similar results appeared in a paper [10] by Marcolli, Pierpaoli and Teh.

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