ON THE FINSLERIAN EXTENSION OF THE SCHWARZSCHILD METRIC

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We provide a Finslerian extension of the Schwarzschild metric based on heuristic arguments. The proposed metric asymptotically approaches not the Minkowski space-time but the Bogoslovsky locally anisotropic space-time which arises naturally as a deformation of very special relativity.

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Local Lorentz invariance is a fundamental cornerstone of modern physics. Despite extremely tight experimental bounds on Lorentz symmetry violations [1] there still remains a tremendous interest in search for such violations from both experimental and theoretical sides [2].

An ingenious way to break Lorentz invariance was suggested by Cohen and Glashow [3]. According to them, all current experimental limits on violations of Lorentz invariance may not imply that the full Lorentz group is the exact symmetry group of nature. The invariance with respect to its four-parameter subgroup SIM(2), which leaves invariant a preferred null-direction n^μ, will suffice to meet all current experimental limitations [3, 4].

Extended by space-time translations, SIM(2) leads to the eight-parameter subgroup ISIM(2) (first introduced, probably, in [5]) of the Poincaré group which is supposed to be the genuine exact symmetry group of nature.

It is well known that, in isotropic situation, there exists essentially only one way to generalize special relativity, namely by endowing space-time with some constant curvature [6]. The resulting de Sitter and Anti-de Sitter groups, which are the most general isotropic relativity groups, can be considered as deformations of the Poincaré group [7]. Similar deformations of ISIM(2) were considered in [8]. Among deformations of ISIM(2) particularly interesting is a one-parameter family of deformations, called $\mathrm{DISIM}_b(2)$, which does not leave invariant the standard Minkowski metric $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ but the Finslerian metric

$$ds^2 = (n_\sigma dx^\sigma)^{2b}(\eta_{\mu\nu}dx^\mu dx^\nu)^{1-b}\,, \tag{1}$$

which was introduced by Bogoslovsky long ago [9, 10, 11] in his attempts to formulate a generalization of the relativity theory for locally anisotropic space-time.

To investigate cosmological implications of the metric (1), one should ask and answer the natural question: what is a curved-space generalization of this metric? The simplest heuristic guess applied in [12] is to change the Minkowski metric $\eta_{\mu\nu}$ in (1) by the Robertson–Walker (or any other general relativistic) metric.

Although viable, such a procedure is somewhat *ad hoc* and there is no guarantee that it gives the true solution. Of course, more systematic approach implies Finslerian generalization of Einstein field equations. Some such generalizations were suggested (see, for example, [13, 14] and references therein) but at present it is not altogether clear [15] whether any of them is on the correct route in the sense of representing reality.

In this note, applying another kind of heuristic arguments, we consider a Finslerian generalization of Bogoslovsky type of the Schwarzschild metric (units are such that $c = 1$ and $G = 1$)

$$ds^2 = \alpha\, dT^2 - \alpha^{-1}\, dR^2 - R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{2}$$

where

$$\alpha = 1 - \frac{2m}{R}\,.$$

Earlier, a different type of Finslerian generalization of the Schwarzschild metric was considered by Asanov [16, 17].

Firstly, let us concentrate on the radial part of the metric (2). In the Kruskal–Szekeres coordinates t, x [18, 19] this radial part takes the form

$$ds^2 = \frac{32m^3}{R}\, e^{-R/2m}\left(dt^2 - dx^2\right)\,, \tag{3}$$

where (in the exterior region $R > 2m$)

$$\frac{t}{x} = \tanh\left(\frac{T}{4m}\right), \tag{4}$$

and R as a function of $x^2 - t^2$ is implicitly determined through the relation

$$\left(\frac{R}{2m} - 1\right) e^{R/2m} = x^2 - t^2\,. \tag{5}$$

Straightforward Finslerian generalization of the radial metric (3) is

$$ds^2 = \frac{32m^3}{R}\, e^{-R/2m} \left(\frac{dt - dx}{dt + dx}\right)^b \left(dt^2 - dx^2\right) , \tag{6}$$

where the the function $R(x,t)$ is implicitly determined through the relation

$$\left(\frac{R}{2m} - 1\right) e^{R/2m} = \left(\frac{x - t}{x + t}\right)^b \left(x^2 - t^2\right) , \tag{7}$$

while the relation (4) still remains valid.

Note that the r.h.s. of (7) is invariant [10] under the generalized Lorentz transformations (Bogoslovsky transformations)

$$\begin{aligned} x' &= e^{-b\psi} (x \cosh\psi - t \sinh\psi) , \\ t' &= e^{-b\psi} (t \cosh\psi - x \sinh\psi) , \end{aligned} \tag{8}$$

ψ being the rapidity. Under the transformations (8), the original Schwarzschild coordinates R and T transform as follows

$$R' = R , \qquad T' = T - 4m\psi . \tag{9}$$

To rewrite the metric (6) in the Schwarzschild coordinates we proceed as follows. Let us introduce auxiliary variables $u = x + t$ and $v = x - t$. Then (7) takes the form

$$\alpha\, \frac{R}{2m}\, e^{R/2m} = u^{1-b}\, v^{1+b} , \tag{10}$$

which can be used to recast the metric (6) as follows

$$ds^2 = 16m^2\alpha \left(\frac{du}{u}\right)^{1-b} \left(-\frac{dv}{v}\right)^{1+b} . \tag{11}$$

On the other hand, differentiating

$$\left(\frac{R}{2m} - 1\right) e^{R/2m} = u^{1-b}\, v^{1+b}$$

and using (10) in the result, we get

$$\alpha^{-1} \frac{dR}{2m} = (1 - b)\, \frac{du}{u} + (1 + b)\, \frac{dv}{v} . \tag{12}$$

The second equation is obtained if we express (4) in terms of u and v and differentiate the resulting equation. This gives

$$\frac{dT}{2m} = \frac{du}{u} - \frac{dv}{v} \,. \tag{13}$$

Now we can solve (12) and (13) to obtain

$$\begin{aligned} \frac{du}{u} &= \frac{1}{2}\left[(1+b)\frac{dT}{2m} + \alpha^{-1}\frac{dR}{2m}\right] , \\ -\frac{dv}{v} &= \frac{1}{2}\left[(1-b)\frac{dT}{2m} - \alpha^{-1}\frac{dR}{2m}\right] . \end{aligned} \tag{14}$$

Substituting these expressions into (11), we finally get

$$ds^2 = \alpha\,\left[(1+b)dT + \alpha^{-1}\,dR\right]^{1-b}\,\left[(1-b)dT - \alpha^{-1}\,dR\right]^{1+b} ,$$

which can be rewritten as follows

$$ds^2 = \left[\frac{(1-b)\,dT - \alpha^{-1}\,dR}{(1+b)\,dT + \alpha^{-1}\,dR}\right]^b \left[\left(1-b^2\right)\alpha\,dT^2 - \alpha^{-1}\,dR^2 - 2b\,dT\,dR\right] . \tag{15}$$

Let us introduce a generalized tortoise coordinate

$$R_* = R + 2m\,\ln\left(\frac{R}{2m} - 1\right) + bT \,. \tag{16}$$

Then

$$dR_* = \alpha^{-1}\,dR + b\,dT \,, \tag{17}$$

and the metric (15) takes a simpler form

$$ds^2 = \left(\frac{dT - dR_*}{dT + dR_*}\right)^b \alpha\,\left(dT^2 - dR_*^2\right) , \tag{18}$$

that is the same as

$$ds^2 = \left[\sqrt{\alpha}\,(dT - dR_*)\right]^{2b}\,\left[\alpha\,\left(dT^2 - dR_*^2\right)\right]^{1-b} . \tag{19}$$

If the original Schwarzschild radial coordinate R is restored through (17), we get the final form of our radial Finsler metric

$$ds^2 = \left[(1-b)\,\alpha^{1/2}\,dT - \alpha^{-1/2}\,dR\right]^{2b}\,\left[\left(1-b^2\right)\,\alpha\,dT^2 - \alpha^{-1}\,dR^2 - 2b\,dR\,dT\right]^{1-b} . \tag{20}$$

Of course, this can be obtained directly from (15), but the detour through the tortoise coordinate makes it clear that it is $r = R + bT$, the asymptotic form of this (generalized) tortoise coordinate, not R, which retains a well defined Finslerian asymptotic physical meaning. Indeed, $\alpha \to 1$ when $R \to \infty$, and in this limit (18) takes just the Bogoslovsky form [10]

$$ds^2 = \left(\frac{dT - dr}{dT + dr}\right)^b \left(dT^2 - dr^2\right) .$$

But what about the angular part of the metric? Kruskal–Szekeres coordinates are intimately related to the Fronsdal embedding of the Schwarzschild space-time into a six-dimensional pseudo-Euclidean space with signature $(+,-,-,-,-,-)$ [20]. The embedding is given by equations

$$\begin{aligned} z_1 &= 4m\sqrt{\alpha}\,\sinh\left(\frac{T}{4m}\right), \qquad z_2 = 4m\sqrt{\alpha}\,\cosh\left(\frac{T}{4m}\right), \qquad z_3 = g(R)\,, \\ z_4 &= R\sin\theta\cos\phi\,, \qquad z_5 = R\sin\theta\sin\phi\,, \qquad z_6 = R\cos\theta\,, \end{aligned} \tag{21}$$

where the function $g(R)$ satisfies [20, 21]

$$\left(\frac{dg}{dR}\right)^2 = \frac{2m}{R} + \left(\frac{2m}{R}\right)^2 + \left(\frac{2m}{R}\right)^3 = \alpha^{-1}\left[1 - \left(\frac{2m}{R}\right)^4\right] - 1\,. \tag{22}$$

Under this embedding, the Kruskal–Szekeres coordinates are given by [22]

$$t = \frac{1}{4m}\sqrt{\frac{R}{2m}}\exp\left(\frac{R}{4m}\right) z_1, \qquad x = \frac{1}{4m}\sqrt{\frac{R}{2m}}\exp\left(\frac{R}{4m}\right) z_2\,. \tag{23}$$

For the needs of Finslerian generalization, we need the embedding to induce the radial metric $(1-b^2)\,\alpha\,dT^2 - \alpha^{-1}dR^2 - 2b\,dR\,dT$ instead of Schwarzschildian $\alpha\,dT^2 - \alpha^{-1}dR^2$. For this goal, we add one more time-like dimension and modify the Fronsdal embedding in the following way

$$\begin{aligned} z_0 &= bT, \qquad z_3 = f(R)\,, \qquad z_1 = 4m\sqrt{1-b^2}\,\sqrt{\alpha}\,\sinh\left(\frac{T}{4m}\right), \\ z_2 &= 4m\sqrt{1-b^2}\,\sqrt{\alpha}\,\cosh\left(\frac{T}{4m}\right), \qquad z_4 = (R+bT)\sin\theta\cos\phi\,, \\ z_5 &= (R+bT)\sin\theta\sin\phi\,, \qquad z_6 = (R+bT)\cos\theta\,, \end{aligned} \tag{24}$$

where the function $f(R)$ is defined through

$$\left(\frac{df}{dR}\right)^2 = \alpha^{-1}\left[1 - (1-b^2)\left(\frac{2m}{R}\right)^4\right] - 1\,. \tag{25}$$

Evidently, the embedding defined via (24) and (25) reduces to the Fronsdal embedding when $b=0$. The Finslerian variant of the Kruskal–Szekeres coordinates is given by

$$\begin{aligned} t &= \frac{1}{4m}\sqrt{\frac{R}{2m}}\exp\left(\frac{R+bT}{4m}\right)\frac{z_1}{\sqrt{1-b^2}}\,, \\ x &= \frac{1}{4m}\sqrt{\frac{R}{2m}}\exp\left(\frac{R+bT}{4m}\right)\frac{z_2}{\sqrt{1-b^2}}\,, \end{aligned} \tag{26}$$

so that (7) is satisfied.

Let us now consider the Finslerian metric in the ambient space

$$ds^2 = \left(N_A\, dz^A\right)^{2b}\left(\eta_{AB}\, dz^A dz^B\right)^{1-b}\,, \tag{27}$$

where the dummy indices run from 0 to 6 and

$$\eta_{AB} = \mathrm{diag}(+1,+1,-1,-1,-1,-1,-1)\,.$$

We want to choose seven unknown functions $N_A(R,T)$ in such a way that, in the case $\theta=\pi/2$, $\phi=\mathrm{const}$, the metric (27) to induce the radial metric (20) under the embedding (24). First of all, we assume that the induced metric is axially symmetric with symmetry axis coinciding with the z-direction in the usual three-dimensional space, so that the metric does not depend on the angle ϕ. This demands $N_4=N_5=0$. The remaining five N_A functions should ensure the above mentioned radial metric matching condition, along with the requirement that they determine the null-direction, $N_A\,N^A=0$. The resulting functional equations are

$$\begin{aligned} & b(N_0-N_6)+\alpha^{1/2}\sqrt{1-b^2}\left(N_1\cosh\frac{T}{4M}-N_2\sinh\frac{T}{4M}\right)=(1-b)\,\alpha^{1/2}\,, \\ & \left(\frac{2m}{R}\right)^2\sqrt{\frac{1-b^2}{\alpha}}\left(N_1\sinh\frac{T}{4M}-N_2\cosh\frac{T}{4M}\right)-N_3\frac{df}{dR}-N_6=-\alpha^{-1/2}\,, \\ & N_0^2+N_1^2-N_2^2-N_3^2-N_6^2=0\,, \end{aligned} \tag{28}$$

and they have the "natural" solution (as the case $b=1$ indicates)

$$\begin{aligned} N_0 &= N_6\,, \qquad\qquad N_1=\sqrt{\frac{1-b}{1+b}}\cosh\frac{T}{4m}\,, \\ N_2 &= \sqrt{\frac{1-b}{1+b}}\sinh\frac{T}{4m}\,, \qquad N_3=\sqrt{\frac{1-b}{1+b}}\,, \\ N_6 &= \alpha^{-1/2}-N_3\frac{df}{dR}=\alpha^{-1/2}\left[1-\sqrt{\frac{1-b}{1+b}}\sqrt{\frac{2m}{R}}\sqrt{1-(1-b^2)\left(\frac{2m}{R}\right)^3}\,\right]. \end{aligned} \tag{29}$$

We are almost done. Our generalization of the Schwarzschild metric (2) is induced by (27) on the Fronsdal-type sub-manifold parametrically defined through (24). At that, on this manifold,

$$\eta_{AB}\, dz^A dz^B = \left(1-b^2\right)\, \alpha\, dT^2 - \alpha^{-1}\, dR^2 - 2b\, dR\, dT - (R+bT)^2\, \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \quad (30)$$

and

$$N_A\, dz^A = (1-b)\, \alpha^{\,1/2}\, dT - \alpha^{\,-1/2}\, dR + N_6\, d\left[(R+bT)(1-\cos\theta)\right]\,. \quad (31)$$

What remains is to show that this Finslerian generalization of the Schwarzschild metric is asymptotically the flat Bogoslovsky space-time. But in the limit $R\to\infty$, both α and N_6 tend to unity, the generalized tortoise coordinate (16) can be replaced by $r = R + bT$ under differentials, and we have

$$\begin{aligned} \eta_{AB}\, dz^A dz^B &\to dT^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \\ N_A\, dz^A &\to dT - d(r\cos\theta)\,. \end{aligned} \quad (32)$$

Therefore, the asymptotic space-time has the Bogoslovsky metric (1) with $n^\mu = (1,0,0,1)$.

The anisotropy parameter b is expected to be very small and hard to access experimentally [8]. Why all the buzz then? The fact that b arises through the deformation of Cohen and Glashow's very special relativity indicates that the Bogoslovsky space-time parallels de Sitter (or Anti-de Sitter) space-time, not Minkowski space-time. Therefore b parallels, in a sense, the cosmological constant and the question "Why b is so small" is as fundamental as the cosmological constant problem [8]. Maybe both questions are related to each other and have a common origin in quantum gravity.

REFERENCES

[1] D. Mattingly, *Living Rev. Rel.* **8**, 5 (2005) [`arXiv:gr-qc/0502097v2`].

[2] M. Pospelov, M. Romalis, *Phys. Today* **57**, 40 (2004).

[3] A.G. Cohen, S.L. Glashow, *Phys. Rev. Lett.* **97**, 021601 (2006) [`arXiv:hep-ph/0601236v1`].

[4] A. Dunn, T. Mehen, `arXiv:hep-ph/0610202v1`.

[5] J.B. Kogut, D.E. Soper, *Phys. Rev.* **D1**, 2901 (1970).

[6] H. Bacry, J.-M. Lévy-Leblond, *J. Math. Phys.* **9**, 1605 (1968).

[7] M. Lévy-Nahas, *J. Math. Phys.* **8**, 1211 (1967).

[8] G.W. Gibbons, J. Gomis, C.N. Pope, *Phys. Rev.* **D76**, 081701 (2007) [arXiv:0707.2174v2 [hep-th]].

[9] G.Yu. Bogoslovsky, *Dokl. Akad. Nauk SSSR Ser. Fiz.* **213**, 1055 (1973).

[10] G.Yu. Bogoslovsky, *Nuovo Cim.* **B40**, 99 (1977); *Nuovo Cim.* **B43**, 377 (1978).

[11] G.Yu. Bogoslovsky, *Fortsch. Phys.* **42**, 143 (1994).

[12] A.P. Kouretsis, M. Stathakopoulos, P.C. Stavrinos, *Phys. Rev.* **D79**, 104011 (2009) [arXiv:0810.3267 [gr-qc]].

[13] S.F. Rutz, *Gen. Relativ. Gravitation* **25**, 1139 (1993).

[14] S.I. Vacaru, arXiv:0707.1524v3 [gr-qc]; *Int. J. Geom. Meth. Mod. Phys.* **05**, 473 (2008) [arXiv:0801.4958v1 [gr-qc]].

[15] S.I. Vacaru, *Phys. Lett.* **B690**, 224 (2010) [arXiv:1003.0044v2 [gr-qc]].

[16] G.S. Asanov, *Fortsch. Phys.* **40**, 667 (1992).

[17] G.S. Asanov, arXiv:gr-qc/0204070v1.

[18] M.D. Kruskal, *Phys. Rev.* **119**, 1743 (1960).

[19] G. Szekeres, *Publ. Math. Debrecen* **7**, 285 (1960). Reprinted in, *Gen. Relativ. Gravitation* **34**, 2001 (2002).

[20] C. Fronsdal, *Phys. Rev.* **116**, 778 (1959).

[21] D. Wang, R.B. Zhang, X. Zhang, *Class. Quantum Grav.* **26**, 085014 (2009) [arXiv:0809.0614v2 [hep-th]].

[22] E.M. Monte, M.D. Maia, *Int. J. Mod. Phys.* **A17**, 4355 (2002) [arXiv:gr-qc/0212059v1].