DE SITTER INVARIANT VACUUM STATES, VERTEX OPERATORS, AND CONFORMAL FIELD THEORY CORRELATORS

Aharon Casher[a,b†], Paweł O. Mazur[a‡]
Andrzej J. Staruszkiewicz[c§]

[a]Department of Physics and Astronomy, University of South Carolina
Columbia, SC 29208, USA
[b]School of Physics and Astronomy, Tel Aviv University
Tel Aviv 69978, Israel
[c]The Marian Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

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We show that there is only one physically acceptable vacuum state for quantum fields in de Sitter space-time which is left invariant under the action of the de Sitter–Lorentz group $\mathrm{SO}(1,d)$ and supply its physical interpretation in terms of the Poincaré invariant quantum field theory (QFT) on one dimension higher Minkowski space-time. We compute correlation functions of the generalized vertex operator $:e^{i\hat{S}(x)}:$, where $\hat{S}(x)$ is a massless scalar field, on the d-dimensional de Sitter space and demonstrate that their limiting values at time-like infinities on de Sitter space reproduce correlation functions in $(d-1)$-dimensional Euclidean conformal field theory (CFT) on S^{d-1} for scalar operators with arbitrary real conformal dimensions. We also compute correlation functions for a vertex operator $e^{i\hat{S}(u)}$ on the Łobaczewski space and find that they also reproduce correlation functions of the same CFT. The massless field $\hat{S}(u)$ is the nonlocal transform of the massless field $\hat{S}(x)$ on de Sitter space introduced by one of us.

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1. Introduction

The maximally symmetric and homogeneous solution to Einstein's equations with the positive vacuum energy density is the de Sitter space-time. It is therefore important to understand the behavior of quantum fields on this

† ronyc@post.tau.ac.il
‡ mazurmeister@gmail.com, mazur@physics.sc.edu
§ astar@th.if.uj.edu.pl

space. Recently, this subject has been taken up with renewed interest after the new and spectacular evidence for the positive cosmological constant and the accelerating expansion of the Universe has turned up in recent cosmological measurements. Although quantum field theory (QFT) on de Sitter space-time is a rather well studied subject [1,2,3,4,5,6,7,8,9,10,11,12,13,14] there are some aspects of it which attracted considerable interest most recently. One of them is the question of uniqueness of the de Sitter invariant vacuum state [1,2,3,4,5,6,7,8,9,10,11,12] while the second one is the fact first established in [1,2,9,13,14] that the limiting values of correlation functions at time-like infinities on de Sitter space reproduce correlation functions of Euclidean CFT in one dimension lower. This property is now referred to as the de Sitter/CFT correspondence in the recent literature. In some early papers on the subject of QFT on de Sitter space an analysis of solutions to the wave equation for a massive scalar field led to the conclusion that there exists a one-parameter family of de Sitter invariant vacuum states for a massive scalar field $\hat{\Phi}(x)$ [8,9,10]. In the current literature the origin of this one-parameter family of vacuum states is regarded as mysterious [8, 9, 10]. Similarly, the proposal of de Sitter/CFT correspondence based on the analysis of the propagator for massive fields on de Sitter space leads to some difficulties because the scaling dimensions

$$\Delta_{\pm} = \frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 - m^2} \tag{1.1}$$

become complex for $m^2 > (\frac{d-1}{2})^2$ [9, 13].

The purpose of this paper is to address these two aspects of QFT on de Sitter space. In particular, we shall demonstrate the existence of the one-parameter family of de Sitter invariant vacuum states $\mid \lambda >$ by solving equations following from the conditions that they are annihilated by generators of the de Sitter group, $\hat{M}_{ab} \mid \lambda \rangle = 0$ and we will connect their origin to the same property of the Lorentz covariant QFT on Minkowski space. It is well known that the spatial infinity of $d+1$-dimensional Minkowski space-time is the d-dimensional de Sitter space [15, 16]. Quantum field theory of massless and massive fields on de Sitter space inherits its vacuum structure from the Lorentz invariant QFT of massless fields in Minkowski space-time. There exists a one-parameter family of Lorentz invariant vacuum states for QFT on Minkowski space-time. This implies the presence of λ-vacua for QFT on de Sitter space. It turns out that these states are physically unacceptable for $\lambda \neq 0$ as they introduce correlations between the antipodal points on de Sitter space-time. We shall show that there exists only one physically acceptable and unique de Sitter invariant vacuum state for massive and massless fields on de Sitter space, corresponding to $\lambda = 0$, and supply the physical interpretation of this state in terms of the Poincaré invariant QFT of a massless particle in $d+1$-dimensional Minkowski space-time [1, 2, 3, 4, 5, 11, 12]. The positive frequency modes $e^{-ik\cdot x}$ of a massless particle in Minkowski space-time properly projected on de Sitter space lead to the unique positive frequency modes on de Sitter space [1, 2, 3, 4, 5, 6, 7, 11, 12] and to a unique physically acceptable de Sitter invariant vacuum state.

We shall also address the now well known fact [1,2,9,13,14] that behavior of two-point correlation functions at time-like infinities on d-dimensional de Sitter space (and at spatial infinity on Łobaczewski space) is characteristic of the $d-1$-dimensional Euclidean conformal field theory (CFT). In order to make the de Sitter/CFT correspondence as transparent as possible and at the same time satisfy the requirement that conformal dimensions Δ of scalar operators in CFT be real and positive we shall introduce a family of generalized vertex operators $e^{i\hat{S}(x)}$ for a massless scalar field $\hat{S}(x)$ and compute their two-point correlation functions [1, 2, 3, 4, 5, 6, 7]. We demonstrate that every correlation function for a scalar operator of conformal dimension Δ is reproduced by our vertex operators on de Sitter space-time.

2. De Sitter invariant vacuum states

De Sitter space is a single-sheeted hyperboloid in the $d+1$-dimensional Minkowski space

$$x^a\eta_{ab}x^b = x\cdot x = -1\,. \tag{2.1}$$

This representation of de Sitter space makes it transparent that the de Sitter isometry group is $\mathrm{SO}(1,d)$. The generators of this symmetry corresponding to Killing vectors on de Sitter space are

$$M_{ab} = i(x_a\partial_b - x_b\partial_a)\,. \tag{2.2}$$

Using the standard global parametrization of the hyperboloid

$$x^0 = \sinh\tau\,, \qquad x^i = n^i\cosh\tau\,, \qquad \boldsymbol{n}^2 = 1 \tag{2.3}$$

one obtains the induced metric on de Sitter space

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = d\tau^2 - \cosh^2\tau d\Omega^2_{d-1}\,. \tag{2.4}$$

The d'Alambertian on de Sitter space is

$$\begin{aligned}\Box_g &= |\det(g)|^{-\frac{1}{2}}\partial_\mu\left(|\det(g)|^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\right)\\ &= (\cosh\tau)^{1-d}\partial_\tau\left((\cosh\tau)^{d-1}\partial_\tau\right) + \cosh^{-2}\tau\boldsymbol{L}^2\,,\end{aligned} \tag{2.5}$$

where $\boldsymbol{L}^2$ is the Laplace–Beltrami operator on the unit S^{d-1} sphere.

One can show that $\Box_g = -\mathcal{C}$, where $\mathcal{C}$ is the Casimir operator $\mathcal{C} = -\frac{1}{2}M_{ab}M^{ab}$ of the de Sitter group acting on scalars. The Casimir operator $\mathcal{C}$ enters a very useful identity relating the d'Alambertian on Minkowski space $\Box_{d+1} = \eta^{ab}\partial_a\partial_b$ to the one on de Sitter space

$$\mathcal{C} = -\tfrac{1}{2}M_{ab}M^{ab} = x^2\Box_{d+1} - \mathcal{E}(\mathcal{E}+d-1)\,, \tag{2.6}$$

where $\mathcal{E} = x^a\partial_a$ is the Euler operator (a generator of the dilatation subgroup of the conformal group). This identity allows us to describe massive (and massless) scalar fields on de Sitter space in terms of a massless scalar field on Minkowski space. A massless scalar field $\Phi_{\scriptscriptstyle\Delta}(x)$ in Minkowski space with the scaling dimension Δ

$$\Phi_{\scriptscriptstyle\Delta}(\lambda x) = \lambda^{-\Delta}\Phi_{\scriptscriptstyle\Delta}(x)\,, \qquad \mathcal{E}\Phi_{\scriptscriptstyle\Delta}(x) = -\Delta\Phi_{\scriptscriptstyle\Delta}(x)\,, \tag{2.7}$$

corresponds to a massive scalar field on de Sitter space with a mass m such that

$$m^2 = \Delta(d-1-\Delta)\,. \tag{2.8}$$

For a given mass there are two scaling dimensions

$$\Delta_\pm = \frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 - m^2}\,. \tag{2.9}$$

In the following, we choose $\Delta = \Delta_-$ for the scaling dimension of a massless scalar field on Minkowski space corresponding to a massive scalar on de Sitter space.

The dynamics of this massive scalar field $\Phi(x)$ is defined by the following Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\left(g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - m^2\Phi^2\right)\,. \tag{2.10}$$

The Poincaré invariant QFT of a massless scalar field on Minkowski space induces a de Sitter invariant QFT of a massive (or massless) scalar field on de Sitter space with a unique vacuum state. This follows immediately from (2.6) and the observation that the sign of the Klein–Gordon norm for a massless scalar field on Minkowski space is preserved upon projection on the de Sitter hyperboloid [1, 2, 3, 4, 5, 6, 7]. This has two immediately clear implications. First, a positive frequency solution for a massless scalar on Minkowski space

$$\Phi^{(+)}_{\scriptscriptstyle\Delta}(x) = \int \frac{(dk)_d}{k_0} e^{-ik\cdot x} a_{\scriptscriptstyle\Delta}(k)\,, \tag{2.11}$$

where

$$a_{\scriptscriptstyle\Delta}(\lambda k) = \lambda^{-\tilde{\Delta}} a_{\scriptscriptstyle\Delta}(k)\,, \qquad \tilde{\Delta} = d-1-\Delta\,, \tag{2.12}$$

and k is a null momentum vector $k^2 = 0$, when projected on the de Sitter hyperboloid $x{\cdot}x = -1$ becomes a positive frequency solution for the Klein–Gordon wave equation on de Sitter space. Choosing

$$a_{\Delta}(k) = \sum_{nM} N_n {k_0}^{-\tilde{\Delta}} Y_{nM}\left(\hat{k}\right) a_{nM} \,, \tag{2.13}$$

with $\tilde{\Delta} = \Delta_+$, $\hat{k} = \frac{k}{k_0}$, and $Y_{nM}(\hat{k})$ the spherical harmonics on S^{d-1}, one obtains the positive frequency modes of the massive scalar field on de Sitter space

$$f_n(\tau) Y_{nM}(\boldsymbol{n}) = N_n \int \frac{(dk)_d}{k_0} e^{-ik\cdot x} {k_0}^{-\tilde{\Delta}} Y_{nM}\left(\hat{k}\right) . \tag{2.14}$$

The integral (2.14) can be easily evaluated [12] and it is given below in (2.22) with

$$A_n^{(0)} = \frac{1}{2} \sqrt{\frac{\Gamma\left(\frac{n+\Delta_+}{2}\right) \Gamma\left(\frac{n+\Delta_-}{2}\right)}{\Gamma\left(\frac{n+1+\Delta_+}{2}\right) \Gamma\left(\frac{n+1+\Delta_-}{2}\right)}} \,, \qquad B_n^{(0)} = -\frac{i}{2A_n^{(0)}} \,. \tag{2.15}$$

The spherical harmonics $Y_{nM}(\hat{k})$ and $Y_{nM}(\boldsymbol{n})$ are eigenfunctions of the Laplacian on S^{d-1} corresponding to the eigenvalue $n(n+d-2)$ and the degeneracy

$$d_n(d) = \frac{(2n+d-2)}{(d-2)!}(n+1)\dots(n+d-3) \,. \tag{2.16}$$

Second, by promoting the amplitudes a_{nM} to the annihilation operators $\hat{a}_{nM}$ one also obtains the positive frequency part of the massive scalar field operator $\hat{\Phi}^{(+)}(x)$ on de Sitter space. The unique de Sitter invariant vacuum state $\mid 0 >$ is defined by the condition

$$\hat{\Phi}^{(+)}(x) \mid 0 \,\rangle = 0 \,. \tag{2.17}$$

We now construct the generators $\hat{M}_{ab}$ of the de Sitter group acting on the Fock space of a massive scalar field $\hat{\Phi}(x)$. To claim de Sitter invariance of a vacuum state one needs to impose the conditions

$$\hat{M}_{ab} \mid 0 \,\rangle = 0 \,, \tag{2.18}$$

where conserved objects

$$\hat{M}_{ab} = \int (\cosh\tau)^{d-1} d\Omega_{d-1} : \hat{T}^0_\mu(x) : \xi^\mu_{ab} \,, \tag{2.19}$$

are the generators of the de Sitter group evaluated at $\tau = 0$, ξ^{μ}_{ab} are the de Sitter space Killing vectors, and

$$: \hat{T}_{\mu\nu}(x) :=: \partial_{\mu}\hat{\Phi}(x)\partial_{\nu}\hat{\Phi}(x) : -\tfrac{1}{2}g_{\mu\nu} : \mathcal{L}(x) : , \tag{2.20}$$

is the stress-energy-momentum tensor of a massive scalar field on de Sitter space. The field operator of a massive scalar field on the de Sitter space is

$$\hat{\Phi}(x) = \sum_{nM} (f_n(y) Y_{nM}(\boldsymbol{n}) \hat{a}_{nM} + \text{h.c.}) , \tag{2.21}$$

where

$$\begin{aligned} f_n(y) &= (1-y)^{s_-}\left[A_n F\left(s_- + \frac{n}{2}, s_- + \frac{2-d-n}{2}; \frac{1}{2}; y\right)\right. \\ &\quad \left. + B_n y^{\frac{1}{2}} F\left(s_- + \frac{n+1}{2}, s_- + \frac{3-d-n}{2}; \frac{3}{2}; y\right)\right] , \end{aligned} \tag{2.22}$$

$s_- = \frac{1}{2}\Delta_-$, $y = \tanh^2\tau$, and F is the hypergeometric function. The coefficients A_n and B_n satisfy the condition

$$i\left(\overline{A}_n B_n - A_n \overline{B}_n\right) = 1 , \tag{2.23}$$

following from the positivity of the Klein–Gordon norm for the positive frequency modes $f_n(y)Y_{nM}(\boldsymbol{n})$. To find the de Sitter invariant vacuum state it is sufficient to compute the de Sitter group generators $\hat{M}_{0i}$ and demand that they annihilate this vacuum state. The Killing vectors ξ^{μ}_{ab}, $a = 0$, $b = i$, corresponding to these generators are expressed in terms of the $n = 1$ spherical harmonics,

$$\xi_0 = Y_{1M_0}(\boldsymbol{n}) , \qquad \xi_\alpha = -\sinh\tau\cosh\tau\,\partial_\alpha Y_{1M_0}(\boldsymbol{n}) . \tag{2.24}$$

We obtain the following formula for the only non-vanishing terms in the normal ordered expression for the generators $\hat{M}_{0i}$ acting on the vacuum state $\mid 0 >$

$$\tfrac{1}{2}\sum_{nM}\sum_{n'M'} C_{nn'} \int d\Omega_{d-1} Y_{1M_0} \overline{Y}_{nM} \overline{Y}_{n'M'} \hat{a}^{\dagger}_{nM} \hat{a}^{\dagger}_{n'M'} + \dots , \tag{2.25}$$

$$\begin{aligned} C_{nn'} &= \overline{B}_n \overline{B}_{n'} + \Big(m^2 + \tfrac{1}{2}n(n+d-2) \\ &\quad + \tfrac{1}{2}n'(n'+d-2) - \tfrac{1}{2}(d-1)\Big)\overline{A}_n \overline{A}_{n'} . \end{aligned} \tag{2.26}$$

The generators $\hat{M}_{0i}$ annihilate the vacuum state $\mid 0 >$ only when the coefficients in front of two creation operators in (2.25) vanish. The Clebsch–Gordan coupling coefficients for SO(d) do not vanish for $\mid n - n' \mid = 1$. This implies that $C_{nn'}$ for $\mid n - n' \mid = 1$ must vanish. We obtain the following simple equation for $\alpha_n = i\frac{B_n}{A_n}$

$$\alpha_n \alpha_{n+1} = m^2 + n(n + d - 1)\,, \tag{2.27}$$

which together with the K–G norm condition leads to the following result: for n even,

$$A_n = e^{\lambda} A_n^{(0)}\,, \qquad B_n = e^{-\lambda} B_n^{(0)}\,, \tag{2.28}$$

and for n odd,

$$A_n = e^{-\lambda} A_n^{(0)}\,, \qquad B_n = e^{\lambda} B_n^{(0)}\,, \tag{2.29}$$

with λ real. $A_n^{(0)}$ and $B_n^{(0)}$ are the coefficients obtained before (2.15) from the evaluation of the integral (2.14). The closure of the Lorentz–de Sitter Lie algebra implies that the remaining generators will also annihilate the vacuum state. There exists a simple canonical transformation between the field operator $\hat{\Phi}_\lambda(x)$ corresponding to the modes (2.22) and the field operator $\hat{\Phi}(x)$ corresponding to the unique vacuum state (2.14)

$$\hat{\Phi}_\lambda(x) = \cosh \lambda \hat{\Phi}(x) + \sinh \lambda \hat{\Phi}(-x)\,, \tag{2.30}$$

which makes the correlations between antipodal points transparent. The modes (2.22) correspond to the following Lorentz, but not Poincaré, invariant choice for positive frequency modes for a QFT on Minkowski spacetime

$$f_k(x) = \cosh \lambda e^{-ik\cdot x} + \sinh \lambda e^{ik\cdot x}\,. \tag{2.31}$$

This concludes our demonstration of the existence of a one-parameter family of de Sitter invariant vacuum states $\mid \lambda >$ and elucidates their physical meaning.

3. Vertex operators on de Sitter space-time and their correlation functions

The special case of a massless scalar field which is the phase $\hat{S}(x)$ canonically conjugate to the electric charge operator $\hat{Q}$ has been studied extensively in the physically most interesting case $d = 3$ in [1, 2, 3, 4, 5, 6, 7]. It turns out that a special care is needed in the treatment of the zero mode $n = 0$ of the Laplacian on the unit sphere. Indeed, our formula for $A_n^{(0)}$ has a simple pole at $n = 0$ for a massless field and as such is meaningless. We need to solve the d'Alambert equation for the $n = 0$ part of a massless scalar field which from now on we denote $\hat{S}(x)$. The dynamics of $\hat{S}(x)$ is derived from the somewhat more conveniently normalized Lagrangian density

$$\mathcal{L} = \left(2e^2 \Omega_{d-1}\right)^{-1} g^{\mu\nu} \partial_\mu \hat{S} \partial_\nu \hat{S} \,, \tag{3.1}$$

where Ω_{d-1} is the volume of S^{d-1} and e is the coupling constant whose meaning is that of a unit of electric charge in the $d = 3$ case. We find the following expression for the massless field operator $\hat{S}(x)$

$$\hat{S}(x) = \hat{S}_0 + e\hat{Q} f_0(\tau) + \sum_{nM} \left(f_n(\tau) Y_{nM}(\boldsymbol{n}) \hat{a}_{nM} + \text{h.c.}\right) , \tag{3.2}$$

where the sum is over $n \neq 0$ and the zero mode $f_0(\tau)$ satisfies the following equation

$$(\cosh\tau)^{d-1} \partial_\tau f_0(\tau) = 1 \,. \tag{3.3}$$

The operators $\hat{S}_0$ and $\hat{Q}$ are canonically conjugate

$$\left[\hat{S}_0, \hat{Q}\right] = ie \,. \tag{3.4}$$

The annihilation and creation operators satisfy the following commutation relations

$$\left[\hat{a}_{nM}, \hat{a}^\dagger_{n'M'}\right] = e^2 \Omega_{d-1} \delta_{nn'} \delta_{MM'} \,. \tag{3.5}$$

It is easy to show that the de Sitter group generators evaluated for a massless scalar $\hat{S}(x)$ do not depend on the constant phase $\hat{S}_0$ but do depend on the charge operator $\hat{Q}$. The de Sitter invariant vacuum state is then characterized by the following conditions:

$$\hat{Q} \mid 0 \rangle = 0 \,, \qquad \hat{a}_{nM} \mid 0 \rangle = 0 \,. \tag{3.6}$$

In the representation in which the operator $e^{i\hat{S}_0}$ is diagonal and eigenstates of $\hat{Q}$ are normalizable S_0 is a periodic variable with the usual period of 2π. It is a phase conjugate to a charge $\hat{Q}$.

It is convenient, following [1, 2, 3, 4, 5, 6, 7], to introduce the generalized vertex operator on de Sitter space

$$\hat{V}(x) =: e^{i\hat{S}(x)} :, \tag{3.7}$$

and compute its two-point correlation function

$$\langle 0 \,|\!: e^{i\hat{S}(x)} :: e^{-i\hat{S}(y)} :\!|\, 0\rangle = e^{F(x,y)} \,, \tag{3.8}$$

where

$$F(x,y) = e^2 \left(\langle 0 \mid \hat{S}^+(x)\hat{S}^-(y) \mid 0 \rangle - \frac{ie^2}{2}(f_0(x) - f_0(y)) \right) , \qquad (3.9)$$

with $\hat{S}^+(x)$ and $\hat{S}^-(x)$ the positive and negative frequency parts of $\hat{S}(x)$. The master function $F(x,y)$ can be computed exactly but its exact form is of no concern for us here. The remarkable property of this function is its universal asymptotic behavior when $\tau(x) = \tau(y) = \pm\infty$. One finds in this limit

$$F(\hat{x}, \hat{y}) = -C(d)e^2 \left(\frac{1}{(d-3)!} \ln\left(\frac{1-cos\theta}{2}\right) + C'(d) \right) , \qquad (3.10)$$

where

$$\cos\theta = \boldsymbol{n}(x) \cdot \boldsymbol{n}(y) \,, \qquad (3.11)$$

and

$$C(d) \;=\; \frac{2^{d-3}\Gamma^2\left(\frac{d-1}{2}\right)}{(d-2)\pi} \,, \qquad (3.12)$$

$$C'(d) \;=\; \left(1+(-1)^d\right)\ln 2 + (-1)^d \sum_{n=1}^{d-2} (-1)^n \frac{1}{n} \,. \qquad (3.13)$$

This implies that the two-point correlation function of the vertex operator on time-like infinity S^{d-1} is equal to the two-point correlation function of the scalar operator with the conformal dimension $\Delta = C(d)e^2$ in the Euclidean CFT on S^{d-1}. The underlying reason for this relation is the isomorphism of the conformal group of R^{d-1} (or S^{d-1}) and the de Sitter group $\mathrm{SO}(1,d)$ on d-dimensional de Sitter space which is the basic kinematical fact with many different dynamical realizations [1,2,13,14,17,18,19]. Our basic formula (3.8) generalizes the one obtained in the special case of $d = 3$ [1,2]. Unlike the case of the two-point correlation function of a massive scalar field first discussed in [9,13] there are no problems with the vanishing with $\tau \to \infty$ of an overall amplitude of the $d-1$-dimensional CFT two-point function and the complex values of the scaling dimensions $\Delta_\pm$ for $m^2 > (\frac{d-1}{2})^2$. Using our vertex operators we can reproduce all multi-point correlation functions of scalar primary scaling operators of any scaling dimension in the Euclidean $d-1$-dimensional CFT.

4. Correlation functions on the Łobaczewski space

The operator $\hat{S}_0$ can be extracted from $\hat{S}(x)$ by averaging over the sphere S^{d-1}, $\tau = 0$, which is an intersection of $x^0 = 0$ with the de Sitter hyperboloid $x{\cdot}x = -1$. The proper Lorentz invariant generalization of $\hat{S}_0$ is the following nonlocal transform from de Sitter space to the Łobaczewski space of velocities u [1, 2, 3, 4, 5, 6, 7]

$$\hat{S}(u) = \frac{2}{\Omega_{d-1}} \int (dx)_{d+1} \delta(x\cdot x + 1)\delta(u\cdot x)\hat{S}(x)\,, \tag{4.1}$$

where $u\cdot u = 1$. The induced metric on the Łobaczewski space is

$$ds^2 = d\psi^2 + \sinh^2\psi d\Omega^2_{d-1}\,. \tag{4.2}$$

One can show that $\hat{S}(u)$ satisfies the Laplace equation on Łobaczewski space

$$\Delta\hat{S}(u) = 0\,. \tag{4.3}$$

A quick inspection of (4.1) reveals that only the even part of $\hat{S}(x)$,

$$\hat{S}_e(-x) = \hat{S}_e(x)\,, \tag{4.4}$$

enters the formula for $\hat{S}(u)$. This means that for n even only A_n, and for n odd only B_n part of the positive (negative) frequency modes enters the formula for $\hat{S}(u)$. We obtain the following expression for $\hat{S}(u)$

$$\hat{S}(u) = \hat{S}_0 + \sum_{nM} \left(f_n(y) Y_{nM}(\boldsymbol{n})\hat{a}_{nM} + \text{h.c.}\right)\,, \tag{4.5}$$

where the sum is over $n \neq 0$. The $f_n(y)$ modes on Łobaczewski space are the same functions of y as before but here we have $y = \coth^2\psi$. The fact that only even modes in $\hat{S}(x)$ appear in $\hat{S}(u)$ implies that $\hat{Q}$ does not appear in it. The two-point correlation function of the vertex operator

$$\hat{U}(u) = e^{i\hat{S}(u)}\,, \tag{4.6}$$

is a finite function which does not require that the normal ordering prescription be applied in marking contrast to the de Sitter case

$$\langle 0 \mid e^{i\hat{S}(u)} e^{-i\hat{S}(v)} \mid 0\rangle = e^{H(u,v)}\,, \tag{4.7}$$

$$H(u,v) = -\tfrac{1}{2}\langle 0 \mid \left(\hat{S}(u) - \hat{S}(v)\right)^2 \mid 0\rangle + \tfrac{1}{2}\left[\hat{S}(u), \hat{S}(v)\right]\,. \tag{4.8}$$

One can show that the commutator function vanishes on Łobaczewski space. Lorentz invariance implies that $H(u,v)$ is a function of the scalar product $u\cdot v = \cosh\lambda = z$, $H(u,v) = H_d(z)$, which vanishes for $z = 1$ and satisfies an inhomogeneous Laplace equation on Łobaczewski space

$$\Delta_u H(u,v) = \text{const.} \tag{4.9}$$

The complete result of integration of the resulting differential equation on d-dimensional Łobaczewski space is

$$H_d(z) = -e^2 \frac{2^{d-3}(d-1)}{\pi(d-2)!} \Gamma^2 \left(\frac{d-1}{2}\right) \int\limits_1^z dx \left(x^2-1\right)^{-\frac{d}{2}} \int\limits_1^x dy \left(y^2-1\right)^{\frac{d}{2}-1} . \tag{4.10}$$

In particular for $d=3$ we find [1, 2, 3, 4, 5, 20]

$$H_3(\lambda) = -\frac{e^2}{\pi}(\lambda \coth \lambda - 1) , \tag{4.11}$$

and for $d=4$ we have

$$H_4(\lambda) = -\frac{e^2}{2}\left(\ln \cosh \frac{\lambda}{2} + \frac{1}{4}\tanh^2 \frac{\lambda}{2}\right) . \tag{4.12}$$

This completes our demonstration that the proper framework for the discussion of the relation between correlation functions on de Sitter (or Łobaczewski) space and correlation functions of the Euclidean CFT on a time-like infinity S^{d-1} are the generalized vertex operators and this is the context where the so-called dS/CFT correspondence [1, 2, 13, 14, 17, 18] was first discovered as a simple side effect of work on the physical problem [1,2,3,4,5,6,7].

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