TRUNCATED MELLIN MOMENTS: USEFUL RELATIONS AND IMPLICATIONS FOR THE SPIN STRUCTURE FUNCTION g_2

Dorota Kotlorz

Division of Physics, Opole University of Technology
Ozimska 75, 45-370 Opole, Poland
d.strozik-kotlorz@po.opole.pl

Andrzej Kotlorz

Division of Mathematics and Applied Informatics
Opole University of Technology
Luboszycka 3, 45-036 Opole, Poland

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We review our previous studies of truncated Mellin moments of parton distributions. We show in detail the derivation of the evolution equation for double truncated moments. The obtained splitting function has the same rescaled form as in a case of the single truncated moments. We apply the truncated moments formalism to QCD analyses of the spin structure functions of the nucleon, g_1 and g_2. We generalize the Wandzura–Wilczek relation in terms of the truncated moments and find new sum rules. We derive the DGLAP-like evolution equation for the twist-2 part of g_2 and solve it numerically. We find also useful relations between the truncated and untruncated moments.

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1. Introduction

Truncated moments (TM) of parton distribution functions (PDFs) were introduced and developed in the QCD analysis by Forte, Latorre, Magnea, Piccione and Ridolfi [1, 2, 3, 4]. The authors obtained the non-diagonal evolution equations, where each n-th truncated moment couples to all higher ones. Then, the idea of TM was successfully applied in the leading $\ln^2 x$

approximation, where we found diagonal solutions [5]. Also Sissakian, Shevchenko and Ivanov used the TM technique in their NLO analyses of SIDIS data, incorporating polynomial expansion [6, 7]. Several years ago, we derived the DGLAP-type diagonal (no mixing between moments of different orders) and exact evolution equation for the TM in a case of a single truncation [8]. Then, we have utilized this approach to the determination of the parton distribution functions from their truncated moments [9]. The idea of the TM was also discussed by Psaker, Melnitchouk, Christy and Keppel in a context of the quark–hadron duality [10]. In this paper, in the continuation of our earlier works, we present in detail a generalization of the TM approach for a double truncation of the integral limits. This problem has been already briefly discussed in [10] and presented in [11]. Here, we also present interesting, novel implications of the TM approach for the analysis of the polarized structure function g_2.

The evolution equations for the truncated Mellin moments of the parton distributions can be a useful additional tool in the perturbative QCD analysis of the structure functions. In the standard Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) formalism [12, 13, 14, 15] a central role play the parton densities, which depend on the kinematic variables Q^2 and x. Then the truncated or untruncated moments, which are *e.g.* contributions to the sum rules, can be obtained by integrating of the parton distribution $q(x, Q^2)$ over the Bjorken-x. Alternatively, one can study directly the Q^2 evolution of the moments. This is sometimes more convenient, particularly in the cases when we know the moments (*e.g.* from direct measurements, calculations on lattice and sum rules constraints), while the PDFs are poorly known. We have shown in [8] that the evolution equation for the n-th truncated at x_0 moment $\int_{x_0}^1 dx\, x^{n-1}\, q(x, Q^2)$ has the same DGLAP form as for the parton density itself, but with a modified splitting function $P'_{ij}(n, x) = x^n P_{ij}(x)$. The TM approach allows one to avoid the problem of the unphysical region $x \to 0$. Furthermore, this approach refers directly to the physical values — moments (rather than to the parton distributions), what enables one to use a wide range of deep-inelastic scattering data in terms of a smaller number of parameters. The evolution equations for the truncated moments are universal — they can be used in each order of the approximation (LO, NLO, NNLO *etc.*) and for unpolarized, as well as polarized parton densities.

The contents of this paper are as follows. In Section 2 we recall the evolution equation for the n-th truncated at x_0 moment and generalize this formula for the double truncation — in both limits of integration. Novel implications of the TM approach for the the polarized structure function g_2 are presented in Section 3. We derive the Wandzura–Wilczek relation in terms of the truncated moments, finding new sum rules. Then we derive the DGLAP-like evolution equation for the twist-2 part of g_2 and present numerical so-

lutions. In Section 4 we derive some useful relations between the truncated and untruncated Mellin moments. Finally, we summarize the main results and discuss future possible applications of the TM approach.

2. The evolution equations for the truncated Mellin moments of the parton densities

The standard perturbative QCD approach is based on the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) evolution equations for the parton densities [12,13,14,15]. In this formalism the main role is played by the parton distribution functions $q(x, Q^2)$, which obey the well-known formula

$$\frac{dq\left(x,Q^2\right)}{d\ln Q^2} = \frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\left(P\otimes q\right)\left(x,Q^2\right)\,, \tag{1}$$

where $\alpha_{\rm s}(Q^2)$ is the running coupling and $\otimes$ denotes the Mellin convolution

$$(A\otimes B)(x) \equiv \int_x^1 \frac{dz}{z}\,A\left(\frac{x}{z}\right)\,B(z)\,. \tag{2}$$

The splitting function $P(z, Q^2)$ can be expanded in a power series of $\alpha_{\rm s}(Q^2)$. For the untruncated n-th Mellin moment, defined for an arbitrary function $f(x)$ as

$$\bar{f}^n = \int_0^1 dx\,x^{n-1}\,f(x)\,, \tag{3}$$

the evolution equation takes the form of an ordinary linear differential one, namely

$$\frac{d\bar{q}^n\left(Q^2\right)}{d\ln Q^2} = \frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\,\gamma^n\left(Q^2\right)\,\bar{q}^n\left(Q^2\right)\,. \tag{4}$$

Here, the anomalous dimension $\gamma^n(Q^2)$ is the untruncated moment of the splitting function $P(z, Q^2)$

$$\gamma^n\left(Q^2\right) = \int_0^1 dz\,z^{n-1}\,P\left(z,Q^2\right)\,. \tag{5}$$

Eq. (4) can be solved analytically

$$\bar{q}^n\left(Q^2\right) = \bar{q}^n\left(Q_0^2\right)\left[\frac{\alpha_{\rm s}\left(Q_0^2\right)}{\alpha_{\rm s}\left(Q^2\right)}\right]^{b\,\gamma^n} \tag{6}$$

and the parton density $q(x,Q^2)$ can be found via the inverse Mellin transform

$$q\left(x,Q^2\right)=\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} dn\, x^{-n}\,\bar{q}^n\left(Q^2\right)\,. \tag{7}$$

Now, let us focus on the approach, in which we can study the Q^2 evolution of the truncated moments of the parton distributions. In [8] we have found that the single truncated moments of the parton distributions $q(x,Q^2)$, defined as

$$\bar{q}^n\left(x_0,Q^2\right)=\int\limits_{x_0}^{1} dx\, x^{n-1}\,q\left(x,Q^2\right)\,, \tag{8}$$

obey the DGLAP-like equation[1]

$$\frac{d\bar{q}^n\left(x_0,Q^2\right)}{d\ln Q^2}=\frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\left(P'\otimes\bar{q}^n\right)\left(x_0,Q^2\right)\,. \tag{9}$$

Here a role of the splitting function plays $P'(n,z)$

$$P'(n,z)=z^n\,P(z)\,. \tag{10}$$

Since the experimental data cover only a limited range of x, except very small $x\to 0$ as well as large $x\to 1$, it is very natural and convenient to deal with the double truncated moments. Truncation at large x is less important in comparison to the small-x limit because of the rapid decrease of the parton densities as $x\to 1$, nevertheless, a comprehensive theoretical analysis requires an equal treatment of the both truncated limits.

It can be shown (see [10,11]), that the double truncated moments

$$\bar{q}^n\left(x_{\rm min},x_{\rm max},Q^2\right)=\int\limits_{x_{\rm min}}^{x_{\rm max}} dx\, x^{n-1}\,q\left(x,Q^2\right) \tag{11}$$

also satisfy the DGLAP-type evolution Eq. (9), namely

$$\frac{d\bar{q}^n\left(x_{\rm min},x_{\rm max},Q^2\right)}{d\ln Q^2}=\frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\int\limits_{x_{\rm min}}^{1}\frac{dz}{z}\,P'(n,z)\,\bar{q}^n\left(\frac{x_{\rm min}}{z},\frac{x_{\rm max}}{z},Q^2\right) \tag{12}$$

[1] For clarity we present only the nonsinglet part.

with P' given again by Eq. (10). Indeed, one can note that the double truncated moment, defined by Eq. (11), is a subtraction of two single truncated ones Eq. (8)

$$\begin{aligned}\bar{q}^n\left(x_{\min}, x_{\max}, Q^2\right) &= \int\limits_{x_{\min}}^{1} dx\, x^{n-1}\, q\left(x, Q^2\right) - \int\limits_{x_{\max}}^{1} dx\, x^{n-1}\, q\left(x, Q^2\right) \\ &= \bar{q}^n\left(x_{\min}, Q^2\right) - \bar{q}^n\left(x_{\max}, Q^2\right) . \end{aligned} \tag{13}$$

Applying the evolution equation for the single truncated moments Eq. (9) to the above formula, one can write

$$\begin{aligned}\frac{d\bar{q}^n\left(x_{\min}, x_{\max}, Q^2\right)}{d\ln Q^2} &= \frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\left[\int\limits_{x_{\min}}^{1} \frac{dz}{z} P'(n,z)\bar{q}^n\left(\frac{x_{\min}}{z}, Q^2\right)\right. \\ &\left. - \int\limits_{x_{\max}}^{1} \frac{dz}{z} P'(n,z)\bar{q}^n\left(\frac{x_{\max}}{z}, Q^2\right)\right] . \end{aligned} \tag{14}$$

Then, the lower limit of integration in the second integral on the r.h.s. can be moved from $x_{\max}$ to $x_{\min}$ since for z below $x_{\max}$ the argument of the truncated moment becomes greater then one and such a moment is equal to zero. This leads immediately to Eq. (12). Our approach Eq. (9)–(12) is valid for the coupled DGLAP equations for quarks and gluons and for any approximation (LO, NLO, NNLO, *etc.*). Only for clarity we present here the nonsinglet and leading order part. Let us emphasize that the evolution equations for the double truncated moments Eq. (12) are in fact a valuable generalization of those for the single truncated and untruncated ones. Setting $x_{\min} = x_0$ or $x_{\min} = 0$ and $x_{\max} = 1$, one obtains Eq. (9) or Eq. (4), respectively. In the next section, we study the application of the presented equations to the polarized structure function g_2.

3. Predictions for the spin structure function g_2 based on the truncated moments

For a complete description of the nucleon spin, one needs two polarized structure functions: g_1 and g_2. Recently, a new generation of experiments with high polarized luminosity, performed at Jefferson Lab, allows more precise study of the polarized structure functions and their moments. This is crucial in our understanding of the QCD spin sum rules, higher-twist effects and quark–hadron duality.

The function g_1 has a simple interpretation in the parton model

$$g_1(x) = \frac{1}{2}\sum_i e_i\,\Delta q_i(x)\,, \tag{15}$$

describing the distribution of quark spin in the nucleon, while function g_2 has no such physical meaning in this classic model. Due to the technical difficulties of obtaining transversely polarized targets, the structure function g_2 has not been a topic of investigations for a long time. Recently, new experimental data at low and intermediate momentum transfers make g_2 also a valuable and hopeful tool to study the spin structure of the nucleon. The function g_2 provides knowledge on higher twist effects, which are a reflection of the quark–gluon correlations in the nucleon. A particular important role in this analysis is played by moments of the spin structure functions

$$\Gamma_1^n = \int\limits_0^1 dx\, x^{n-1} g_1\left(x, Q^2\right)\,, \tag{16}$$

$$\Gamma_2^n = \int\limits_0^1 dx\, x^{n-1} g_2\left(x, Q^2\right)\,. \tag{17}$$

They are a sensitive tool for testing the QCD sum rules and determination of the higher twist contributions (for a review of this problem see, *e.g.* [16]). Here, we would like to focus on the application of the TM and their evolution equations to predictions for the spin structure function g_2.

The experimental value of the function g_2, measured in the small to intermediate Q^2 region, consists of two parts: the twist-2 (leading) and the higher twist term

$$g_2\left(x, Q^2\right) = g_2^{\mathrm{LT}}\left(x, Q^2\right) + g_2^{\mathrm{HT}}\left(x, Q^2\right)\,. \tag{18}$$

The leading-twist term g_2^{LT} can be determined from the other structure function — g_1 via the Wandzura–Wilczek relation [17]

$$g_2^{\mathrm{LT}}\left(x, Q^2\right) = g_2^{\mathrm{WW}}\left(x, Q^2\right) = -g_1\left(x, Q^2\right) + \int\limits_x^1 \frac{dy}{y}\, g_1\left(y, Q^2\right)\,. \tag{19}$$

Then, from the measurements of g_1 and g_2, using the Wandzura–Wilczek approximation Eq. (19), one is able to extract the higher-twist term g_2^{HT}. We find a new equation which is a generalization of the Wandzura–Wilczek relation, for the truncated moments

$$\bar{g}_2^n\left(x_0, Q^2\right) = \frac{1-n}{n}\bar{g}_1^n\left(x_0, Q^2\right) - \frac{x_0^n}{n}\,\bar{g}_1^0\left(x_0, Q^2\right)\,. \tag{20}$$

It is easy to see that for the untruncated moments Eq. (20) takes the well known form

$$\bar{g}_2^n\left(Q^2\right) = \frac{1-n}{n}\bar{g}_1^n\left(Q^2\right) . \tag{21}$$

Here,

$$\bar{g}_{1,2}^n\left(Q^2\right) = \int_0^1 dx\, x^{n-1}\, g_{1,2}\left(x, Q^2\right) , \tag{22}$$

$$\bar{g}_{1,2}^n\left(x_0, Q^2\right) = \int_{x_0}^1 dx\, x^{n-1}\, g_{1,2}\left(x, Q^2\right) , \tag{23}$$

and consequently

$$\bar{g}_1^0\left(x_0, Q^2\right) = \int_{x_0}^1 \frac{dx}{x}\, g_1\left(x, Q^2\right) . \tag{24}$$

For the first moment ($n = 1$) Eq. (20) reads

$$\bar{g}_2^1\left(x_0, Q^2\right) = -x_0\, \bar{g}_1^0\left(x_0, Q^2\right) \tag{25}$$

or equivalently

$$\int_{x_0}^1 dx\, g_2\left(x, Q^2\right) = -x_0 \int_{x_0}^1 \frac{dx}{x}\, g_1\left(x, Q^2\right) . \tag{26}$$

The moments of the spin structure functions g_1 and g_2 are of great importance due to their relations to the fundamental QCD sum rules:

- Bjorken sum rule (BSR) [18]

$$\int_0^1 dx\, \left[g_1^p\left(x, Q^2\right) - g_1^n\left(x, Q^2\right)\right] = \frac{g_A}{6} ; \tag{27}$$

- Efremov–Leader–Teryaev sum rule (ELT) [19]

$$\int_0^1 dx\, x\, \left[g_2^p\left(x, Q^2\right) - g_2^n\left(x, Q^2\right)\right] = -\frac{1}{2}\int_0^1 dx\, x\, \left[g_1^p\left(x, Q^2\right) - g_1^n(x, Q^2)\right] ; \tag{28}$$

- Burkhardt–Cottingham sum rule (BC) [20]

$$\int_0^1 dx\, g_2\left(x,Q^2\right) = 0\,. \tag{29}$$

From Eq. (20), setting $n = 1$ and $x_0 \to 0$, one can obtain automatically the BC sum rule for g_2^{WW}. Furthermore, using the generalization of the Wandzura–Wilczek relation Eq. (20) for $n = 1$ at two different points of the truncation and also applying the BC sum rule Eq. (29), we obtain an interesting relation

$$\int_{x_1}^{x_2} dx\, g_2^{\mathrm{WW}}\left(x,Q^2\right) = (x_2 - x_1)\int_{x_2}^{1}\frac{dx}{x}\, g_1\left(x,Q^2\right) - x_1\int_{x_1}^{x_2}\frac{dx}{x}\, g_1\left(x,Q^2\right)\,. \tag{30}$$

The above formula can be very useful in determination of the partial twist-2 contribution to the BC sum rule. For example, setting $x_1 = 0$ and $x_2 = x_0$, when $x_0 \to 0$, one can get the small-x contribution to the BC sum rule

$$\int_0^{x_0} dx\, g_2^{\mathrm{WW}}\left(x,Q^2\right) = x_0\int_{x_0}^{1}\frac{dx}{x}\, g_1\left(x,Q^2\right)\,. \tag{31}$$

Since it is assumed that for $x < 0.02$ the higher-twist effects are negligible (see, *e.g.* [21]), Eq. (31) allows one to estimate the small-x ($x_0 < 0.02$) contribution as

$$\int_0^{x_0<0.02} dx\, g_2\left(x,Q^2\right) \approx \int_0^{x_0<0.02} dx\, g_2^{\mathrm{WW}}\left(x,Q^2\right)\,. \tag{32}$$

Concluding, the obtained Eqs. (20), (26), (30) and (31) are some sorts of sum rules and can be significant in increasing of the reliability of the data analyses in spin physics.

Now we would like to discuss the problem of the Q^2 evolution of g_2. While a general DGLAP-type equation for g_2 does not exist, for the twist-3 component of g_2 suitable evolution equations have been formulated by Braun, Korchemsky and Manashov in [21, 22, 23]. In the leading twist-2 approximation, the Q^2 evolution of g_2 is governed by the evolution of g_1, according to the Wandzura–Wilczek relation. Since the second term on the r.h.s. of Eq. (19) is the $n = 0$-th truncated moment of the function g_1 Eq. (24), we can rewrite the Wandzura–Wilczek relation in the form

$$g_2^{\mathrm{WW}}\left(x,Q^2\right) = -g_1\left(z,Q^2\right) + \bar{g}_1^0\left(z,Q^2\right) \tag{33}$$

and obtain the evolution equation for g_2^{WW}

$$\frac{dg_2^{\mathrm{WW}}\left(x,Q^2\right)}{d\ln Q^2} = -\frac{dg_1\left(x,Q^2\right)}{d\ln Q^2} + \frac{d\bar{g}_1^0\left(x,Q^2\right)}{d\ln Q^2}\,. \tag{34}$$

It is worth noting that according to Eqs. (9), (10), the $n = 0$-th truncated moment of the parton distribution q evolves in the same way as q itself ($P'(0,z) = P(z)$). Taking this into account in the case of g_1, we obtain from Eqs. (34), (33) the evolution equation

$$\frac{dg_2^{\mathrm{WW}}\left(x,Q^2\right)}{d\ln Q^2} = \frac{\alpha_{\mathrm{s}}\left(Q^2\right)}{2\pi}\int\limits_x^1 \frac{dz}{z}\,P\left(\frac{x}{z}\right)\left[\bar{g}_1^0\left(z,Q^2\right) - g_1\left(z,Q^2\right)\right] \tag{35}$$

or finally[2]

$$\frac{dg_2^{\mathrm{WW}}\left(x,Q^2\right)}{d\ln Q^2} = \frac{\alpha_{\mathrm{s}}\left(Q^2\right)}{2\pi}\int\limits_x^1 \frac{dz}{z}\,P\left(\frac{x}{z}\right)\,g_2^{\mathrm{WW}}\left(z,Q^2\right)\,. \tag{36}$$

The above formula shows that the twist-2 component of the function g_2 obeys the standard DGLAP evolution with the same evolution kernel as g_1. Eq. (36) provides predictions for the leading twist contribution to the g_2 and hence enables determination of the higher twist terms from experimental data as

$$g_2^{\mathrm{HT}}\left(x,Q^2\right) = g_2^{\mathrm{EXP}}\left(x,Q^2\right) - g_2^{\mathrm{WW}}\left(x,Q^2\right)\,. \tag{37}$$

The obtained evolution equation Eq. (36) can be a useful additional tool for the study of the nucleon structure function g_2.

Here, we present numerical solutions of Eq. (36). In Figs. 1 and 2 we show the nonsinglet contributions to the polarized structure function $xg_2^{\mathrm{NS}}(x,Q^2)$, calculated in the leading order. We choose the input parametrization of the structure function $g_1^{\mathrm{NS}}(x,Q_0^2)$ at $Q_0^2 = 1$ GeV^2 in a general form

$$g_1^{\mathrm{NS}}(x,Q_0^2) = N\,x^{\alpha}(1-x)^{\beta}(1+\gamma x)\,, \tag{38}$$

where α, β and γ control the small-, large- and medium-x behavior of g_1 and hence g_2, respectively. The values of the PDF input parameters α, β, γ are usually obtained from a global analysis of data and N must be determined from the sum rules constraints. Here, we use a simple fit with $\beta = 3$, $\gamma = 20$ and different values of α to study its influence on the evolution of g_2. Fig. 1 shows the predictions as a function of x for different scales of Q^2: 1, 10 and

[2] Again, for clarity, we use only the nonsinglet and LO notation.

100 GeV^2. In the input parametrization of g_1 we assume at the initial scale Q_0^2 $\alpha = -0.4$. Note that $xg_2^{\rm NS}$ is positive for low-x, at about $x = 0.1 - 0.2$ changes sign and becomes negative for larger x. Similar results were obtained for light and heavy flavor contributions to g_2 by Bluemlein, Ravindran and van Neerven [24]. This is, as a matter of fact, evidence of agreement with the BC sum rule. From Fig. 1 one can see also that with increasing Q^2, an x-intercept of $g_2^{\rm NS}$ occurs at smaller values of x.

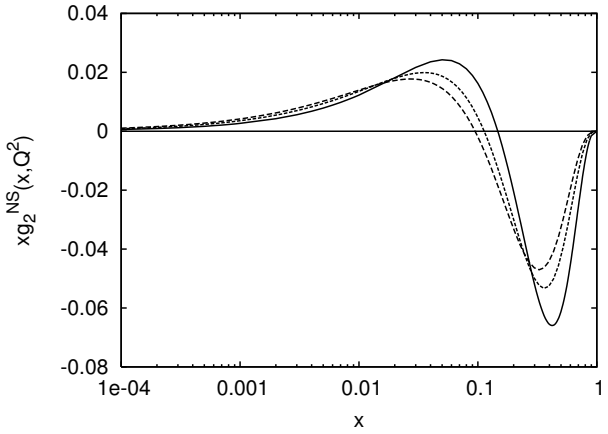


Fig. 1. The nonsinglet LO contributions to the polarized structure function $xg_2^{\rm NS}(x, Q^2)$ as a function of x for different Q^2: 1 GeV^2 (solid), 10 GeV^2 (dashed) and 100 GeV^2 (dotted). Parametrization of g_1 is given by Eq. (38) with $\alpha = -0.4$.

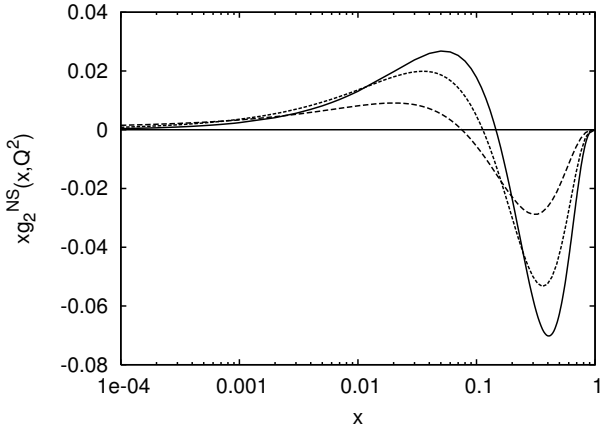


Fig. 2. The nonsinglet LO contributions to the polarized structure function $xg_2^{\rm NS}(x, Q^2)$ at $Q^2 = 10$ GeV^2 as a function of x for different parameterizations of g_1, given by Eq. (38): $\alpha = 0$ (solid), $\alpha = -0.4$ (dashed) and $\alpha = -0.8$ (dotted).

In Fig. 2 we compare the predictions for $g_2^{\rm NS}$ for different small-x behavior of the g_1 parametrization Eq. (38): $\alpha = 0, -0.4, -0.8$. We find that more singular small-x behavior of g_1 implies smaller value of the x-intercept of g_2.

From Figs. 3 and 4, where we plot $g_2^{\rm NS}(x, Q^2)$ *vs.* x, one can see that the evolution of the polarized structure function g_2 is for small-x very sensitive to the value of Q^2 and, of course, to the assumed input parametrization. For $x = 10^{-4}$ the results of $g_2^{\rm NS}$ can differ by a factor of over 2 from $Q^2 = 1$ GeV^2 to $Q^2 = 100$ GeV^2 for typical input with $\alpha = -0.4$. Also the input parametrization itself has a large (dominated) impact on the evolution of g_2. Namely, for $x = 10^{-4}$ and $Q^2 = 10$ GeV^2 the results of $g_2^{\rm NS}$ can differ by a factor of over 4 from the flat input ($\alpha = 0$) to the very steep one ($\alpha = -0.8$).

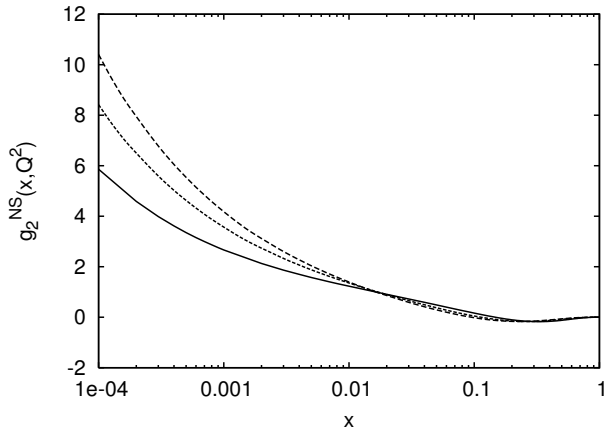


Fig. 3. The nonsinglet LO contributions to the polarized structure function $g_2^{\rm NS}(x, Q^2)$ as a function of x for different Q^2: 1 GeV^2 (solid), 10 GeV^2 (dashed) and 100 GeV^2 (dotted). Parametrization of g_1 is given by Eq. (38) with $\alpha = -0.4$.

Knowledge of the small-x behavior of the structure functions and their moments is crucial in our understanding of the nucleon structure. Presented here uncertainty in the determination of the nonsinglet g_1 and g_2 at low-x is additionally enhanced in the case of the singlet contributions. Namely, the results for the singlet $g_1^{\rm S}$ and $g_2^{\rm S}$ are very sensitive on the polarized gluon densities, which are completely unknown. Therefore, the comprehensive theoretical analysis of the small-x behavior of structure functions is of a great importance to compensate the lack of the experimental data in this region. In the next section, we derive some relations between truncated and untruncated Mellin moments, which can be helpful for such analyses in the future.

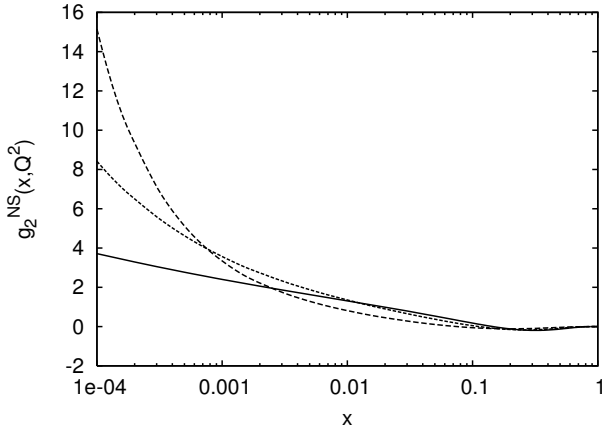


Fig. 4. The nonsinglet LO contributions to the polarized structure function $g_2^{\mathrm{NS}}(x,Q^2)$ at $Q^2 = 10$ GeV^2 as a function of x for different parameterizations of g_1, given by Eq. (38): $\alpha = 0$ (solid), $\alpha = -0.4$ (dashed) and $\alpha = -0.8$ (dotted).

4. Useful relations between truncated and untruncated Mellin moments

The evolution equations for the truncated moments Eq. (9) are very similar to those for the parton densities Eq. (1). In both cases one deals with functions of two variables x and Q^2 (with additionally fixed index n for moments), which obey the differentio-integral Volterra-like equations. The only difference lies in the splitting function, which for moments has the rescaled form Eq. (10). This similarity allows one to solve the equations for truncated moments with use of standard methods of solving the DGLAP equations. Analysis of the evolution, performed in moment space according to Eqs. (3)–(7), when applying to the truncated moments, implies dealing with such an exotic structure like 'Moment of Moment'. Let us discuss this in detail and introduce some useful relations involving untruncated and truncated Mellin moments.

There are in literature several methods for the solution of the integro-differential DGLAP equations. They are based either on the polynomial expansion or on the Mellin transformation — for review see, *e.g.* [25]. In our previous studies on the evolution of the truncated moments we used the Chebyshev polynomial technique [26], earlier widely applied by Kwieciński in many QCD treatments — for details see, *e.g.*, Appendix of [27]. Using this method, one obtains the system of linear differential equations instead of the original integro-differential ones. The Chebyshev expansion provides a robust method of discretising a continuous problem.

An alternative approach is based on the Mellin transformation and the moments factorization. Taking the s-th moment of the evolution equation (9), one obtains

$$\frac{dM^{s,n}\left(Q^2\right)}{d\ln Q^2} = \frac{\alpha_{\rm s}\left(Q^2\right)}{2\pi}\,\gamma^{s+n}\left(Q^2\right)\,M^{s,\,n}\left(Q^2\right)\,, \tag{39}$$

where $M^{s,n}$ denotes s-th (untruncated) moment of n-th truncated moment of the parton density

$$M^{s,n}\left(Q^2\right) = \int\limits_0^1 dx\,x^{s-1}\,\bar{q}^n\left(x,Q^2\right)\,. \tag{40}$$

Analogically to the solution for the parton distribution — Eqs. (6), (7), we can write down immediately solutions for the truncated moments

$$M^{s,n}\left(Q^2\right) = M^{s,n}\left(Q_0^2\right)\left[\frac{\alpha_{\rm s}\left(Q_0^2\right)}{\alpha_{\rm s}\left(Q^2\right)}\right]^{b\,\gamma^{s+n}} \tag{41}$$

and

$$\bar{q}^n\left(x,Q^2\right) = \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} ds\,x^{-s}\,M^{s,\,n}\left(Q^2\right)\,. \tag{42}$$

The quantity $M^{s,n}$, which is rather exotic and has no physical meaning, can be replaced by the usual truncated moment $\bar{q}$. Indeed, the 'Moment of Moment' $M^{s,n}$ can be written in terms of the parton density $q(x,Q^2)$ as

$$M^{s,n} = \int\limits_0^1 dx\,x^{s-1}\int\limits_x^1 dz\,z^{n-1}q(z)\,. \tag{43}$$

For clarity we drop Q^2-dependence of the functions. After simple manipulation with help of the Heaviside function, we can change the order of integration in Eq. (43). Thus, the right-hand side of Eq. (43) takes the following form

$$\int\limits_0^1 dx\,x^{s-1}\int\limits_0^1 dz\,z^{n-1}\Theta(z-x)\,q(z) = \int\limits_0^1 dz\,z^{n-1}q(z)\int\limits_0^1 dx\,x^{s-1}\Theta(z-x)\,. \tag{44}$$

Next, absorbing the $\Theta(z-x)$ into the upper limit of integration gives

$$M^{s,\,n} = \int\limits_0^1 dz\, z^{n-1} q(z) \int\limits_0^z dx\, x^{s-1}\,. \tag{45}$$

Finally, performing integration over x, we obtain

$$M^{s,n} = \frac{1}{s} \int\limits_0^1 dz\, z^{s+n-1} q(z) = \frac{1}{s}\, \bar{q}^{s+n}\,. \tag{46}$$

Now, replacing the $M^{s,n}$ in Eq. (42) by Eq. (46), we obtain

$$\bar{q}^n\left(x, Q^2\right) = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} ds\, \frac{x^{-s}}{s}\, \bar{q}^{s+n}\left(Q^2\right)\,. \tag{47}$$

We also find the invert transformation to Eq. (47). Using Eq. (46) for the 'Moment of Moment' $M^{s-n,n}$

$$M^{s-n,n} = \frac{1}{s-n}\, \bar{q}^s\,, \tag{48}$$

we have

$$\bar{q}^s\left(Q^2\right) = (s-n)\, M^{s-n,\,n}\left(Q^2\right) = (s-n) \int\limits_0^1 dx\, x^{s-n-1}\, \bar{q}^n\left(x, Q^2\right)\,. \tag{49}$$

Eqs. (46), (47) and (49) are useful relations between the truncated and untruncated moments. Particularly Eq. (47) seems to have a large practical meaning and could be applied when the untruncated moments are known *e.g.* from lattice calculations.

5. Summary

This paper is a continuation of our earlier studies on the truncated moments of the parton densities. We have derived the evolution equation for the double truncated moments, which is a generalization of those for the single truncated and untruncated ones. We have obtained the Wandzura–Wilczek relation in terms of the truncated moments and found new sum rules involving the structure functions g_1 and g_2. We have also derived the DGLAP evolution equation for the twist-2 part of g_2. We have presented

numerical predictions for the evolution of $g_2^{\rm NS}$ at different values of Q^2 and for different small-x behavior of the initial parametrization. We have also derived relations between the truncated and untruncated Mellin moments, which can be useful in further studies of spin physics. The method of the truncated moments enables one direct, efficient study of the evolution of the moments (and hence sum rules) within different approximations: LO, NLO, *etc.*, in non-spin as well as in spin sectors. The adaptation of the evolution equations for the restricted experimentally x-region provides a new, additional tool for analysis of the nucleon structure functions.

Finally, let us list a few of the valuable future applications of the presented evolution equation and relations for the truncated moments:

- Studying the fundamental properties of nucleon structure, concerning moments of F_1, F_2 and g_1. These are: the momentum fraction carried by quarks, quark helicities contributions to the spin of nucleon and, what is particularly important, estimation of the polarized gluon contribution ΔG from COMPASS and RHIC data.

- Determination of Higher Twist (HT) effects from the moments of g_2, which will be measured at JLab. This is possible via the generalization of Wandzura–Wilczek relation for the truncated moments and test of Burkhardt–Cottingham and Efremov–Leader–Teryaev sum rules. HT corrections can provide information on the quark–hadron duality.

- Predictions for the generalized parton distributions (GPDs). Moments of the GPDs can be related to the total angular momentum (spin and orbital) carried by various quark flavors. Measurements of DVCS, sensitive to GPDs, will be done at JLab. This would be an important step towards a full accounting of the nucleon spin.

Concluding, in light of the recent progress in experimental program, theoretical developments which improve our knowledge of the nucleon structure functions and their moments are of great importance.

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