COMPOSITION OF FLUCTUATIONS
OF DIFFERENT OBSERVABLES

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We demonstrate that description of fluctuations observed in multiparticle production processes using Tsallis statistics approach (in which fluctuations are described by the nonextensivity parameter q) leads to a specific sum rule for parameters q seen in different observables which can be verified experimentally.

1. Introduction

When analyzing multiparticle production data the standard tool used is statistical modeling [1]. However, this approach does not account for the possible intrinsic nonstatistical fluctuations in the hadronizing system which usually result in a characteristic power-like behavior of the single

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particle spectra and in the broadening of the corresponding multiplicity distributions (and which can signal a possible phase transition(s) [2]). One possibility to include this and still remain in the domain of a statistical approach is to use the so-called Tsallis statistics [3,4,5] (represented by Tsallis distribution, $h_q(E)$) which accounts for such situations by introducing in addition to the temperature T, a new parameter, $q > 1$, directly connected to fluctuations [6,7] (for $q \to 1$ one recovers the usual Boltzmann–Gibbs distribution, $f(E)$)

$$h_q(E) = \frac{2-q}{T} \exp_q\left(-\frac{E}{T}\right) = \frac{2-q}{T}\left[1 - (1-q)\frac{E}{T}\right]^{\frac{1}{1-q}} \tag{1}$$

$$\overset{q \to 1}{\Longrightarrow} \frac{1}{T} \exp\left(-\frac{E}{T}\right) . \tag{2}$$

The most recent applications of this approach come from the PHENIX Collaboration at RHIC [8] and from the CMS Collaboration at LHC [9]. The parameter q is entirely given by intrinsic fluctuations in the system, $cf.$ Eq. (4) below[1].

Before proceeding any further we must emphasize two points. First, the relation between parameter q and fluctuation of temperature was derived in [6], where it was shown that starting from some simple diffusion picture of temperature equalization in a nonhomogeneous heat bath (in which local $\tilde{T}$ fluctuates from point to point around some equilibrium temperature, T) one obtains an evolution of $\tilde{T}$ in the form of Langevin stochastic equation and distribution of $1/\tilde{T}$, $f(1/\tilde{T})$, as a solution of the corresponding Fokker–Planck equation. It turns out that $f(1/\tilde{T})$ has the form of a gamma distribution

$$f(1/\tilde{T}) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \frac{T}{q-1} \left(\frac{1}{q-1}\frac{T}{\tilde{T}}\right)^{\frac{2-q}{q-1}} \exp\left(-\frac{1}{q-1}\frac{T}{\tilde{T}}\right) , \tag{3}$$

where

$$q - 1 = \frac{\mathrm{Var}(1/\tilde{T})}{\langle 1/\tilde{T}\rangle^2} . \tag{4}$$

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[1] See also [10]. One must admit at this point that this approach is subjected to a rather hot debate of whether it is consistent with equilibrium thermodynamics or else it is only a handy way to phenomenologically describe some intrinsic fluctuations in the system [11]. However, as was recently demonstrated on general grounds in [12], fluctuation phenomena can be incorporated into a traditional presentation of thermodynamic and Tsallis distribution, Eq. (1), belongs to the class of general admissible distributions which satisfy thermodynamical consistency conditions and which are therefore a natural extension of the usual Boltzman–Gibbs canonical distribution Eq. (2).

Convoluting $\exp(-E/\tilde{T})$ with such a $f(1/\tilde{T})$ one obtains immediately Tsallis distribution $h_q(E)$ (1) [6]. The parameter q, *i.e.*, according to Eq. (4) also the temperature fluctuation pattern, is therefore fully given by the parameters describing this basic diffusion process (*cf.*, [6] for details)[2].

The second point is that, as was shown in [15], temperature fluctuations in the form given by Eq. (3) result in an automatic broadening of the corresponding multiplicity distributions, $P(N)$, from the Poissonian form for the exponential distributions, Eq. (2)

$$P(N) = \frac{(\bar{N})^N}{N!} \exp\left(-\bar{N}\right), \qquad \text{where} \qquad \bar{N} = \frac{E}{T} \qquad (5)$$

to the negative binomial (NB) form for the Tsallis distributions, Eq. (1) (*cf.*, [15], for details)

$$P(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \frac{\left(\frac{\langle N \rangle}{k}\right)^N}{\left(1 + \frac{\langle N \rangle}{k}\right)^{(N+k)}}, \qquad \text{where} \qquad k = \frac{1}{q-1}. \qquad (6)$$

Notice that in the limiting cases of $q \to 1$ one has $k \to \infty$ and (6) becomes a Poissonian distribution (5), whereas for $q \to 2$ on has $k \to 1$ and (6) becomes a geometrical distribution. It is easy to show that for large values of N and $\langle N \rangle$ one obtains from Eq. (6) its scaling form

$$\langle N \rangle P(N) \cong \psi\left(z = \frac{N}{\langle N \rangle}\right) = \frac{k^k}{\Gamma(k)} z^{k-1} \exp(-kz), \qquad (7)$$

in which one recognizes a particular expression of Koba–Nielsen–Olesen (KNO) scaling [16][3].

2. Results

Proceed to a detailed analysis of different observables, like $P(N)$, dN/dy or dN/dp_{T}, fluctuations in which are expected to differ from each other and therefore to result in different values of the corresponding parameters q. Indeed, from our experience with $p\bar{p}$ collisions [18] we know that one can obtain a very good description of the whole range of p_{T} spectra

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[2] This was recently generalized to account for the possibility of transferring energy from/to a heat bath, which appears to be important for AA applications [4, 13] and for cosmic ray physics [14]; however, we shall not discuss this issue here.

[3] It is worth mentioning at this point that, as shown in [17], fluctuations of $\bar{N}$ in the Poissonian distribution (5) taken in the form of $\psi(\bar{N}/\langle N \rangle)$, Eq. (7), lead to the NB distribution (6).

($\propto \exp_q (-p_{\mathrm{T}}/T)$ with ($T = T_{\mathrm{T}}$ [GeV]; q_{T}) = (0.134; 1.095), (0.135; 1.105) and (0.14; 1.11) for energies (in GeV) 200, 540 and 900, respectively[4]. These values should be compared with the corresponding values of ($T = T_{\mathrm{L}}; q = q_{\mathrm{L}}$) obtained when fitting rapidity distributions ($\propto \exp_q (-\mu_{\mathrm{T}} \cosh y/T)$) at the same energies: (11.74; 1.2), (20.39; 1.26) and (30.79; 1.29). It was noticed there that $q_{\mathrm{L}} - 1$ has the same energy behavior as $1/k$ in the NB distribution fitting the multiplicity distributions at corresponding energies ($q_{\mathrm{L}} - 1 = -0.104 + 0.058 \ln \sqrt{s}$). This means that fluctuations of the total energy are in this case mainly driven by fluctuations in longitudinal phase space. An explanation proposed in [18] was the following. Noticing that $q - 1 = \sigma^2(T)/T^2$ (*i.e.*, it is given by fluctuations of total temperature T) and assuming that $\sigma^2(T) = \sigma^2(T_{\mathrm{L}}) + \sigma^2(T_{\mathrm{T}})$, one can estimate that the resulting values of q should not be too different from

$$q = \frac{q_{\mathrm{L}} T_{\mathrm{L}}^2 + q_{\mathrm{T}} T_{\mathrm{T}}^2}{T^2} - \frac{T_{\mathrm{L}}^2 + T_{\mathrm{T}}^2}{T^2} + 1 \quad \overset{T_{\mathrm{L}} \gg T_{\mathrm{T}}}{\Longrightarrow} \quad \sim q_{\mathrm{L}}. \tag{8}$$

The situation is noticeably different for nuclear collisions, which we shall now address[5]. As shown in Fig. 1, data for dN/dy and $dN/d\mu_{\mathrm{T}}$ can be fitted perfectly by means of Eq. (1). However, the behavior of the q parameters obtained is quite interesting, as displayed in Fig. 2. At first, the parameter q from $P(N)$ turns out to depend on the centrality of the collision defined by the number of participants of projectile, N_{P} (left panel of Fig. 2)

$$q - 1 = \frac{1}{aN_{\mathrm{P}}} \left(1 - \frac{N_{\mathrm{P}}}{A}\right) \tag{9}$$

(A — mass number of colliding nuclei and $a = 0.98$) [13]. Whereas for small centralities it approaches situation encountered in $p\bar{p}$, the more central the event, the smaller is $q - 1$, *i.e.*, the nearer to a Poissonian the corresponding $P(N)$.

This time for each centrality q_{T} are larger than q (both for results based on NA49 data [20] and from fits presented in [22] based on PHENIX data [21]). In the right panel of Fig. 2 we collected all results for the most central events from NA49 [20]. Notice that they clearly display opposite trend to that encountered for the $p\bar{p}$ collisions mentioned above: both, q_{L}

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[4] There is recent compilation of essentially all results for p_{T} spectra, including recent LHC data [19]. It shows that $q(s) = 1.25 - 0.33 s^{-0.054}$, which nicely reproduces results mentioned here.

[5] For this purpose we use mainly NA49 data on Pb+Pb collisions [20] because, at the moment, only this experiment measures both (at least for the most central collisions) multiplicity distributions, $P(N)$, and distributions in rapidity y, transverse momenta, p_{T}, and transverse masses, $\mu_{\mathrm{T}} = \left(m^2 + p_{\mathrm{T}}^2\right)^{1/2}$, and this property is crucial for further considerations. PHENIX results [21] analyzed in [22] are also shown for comparison.
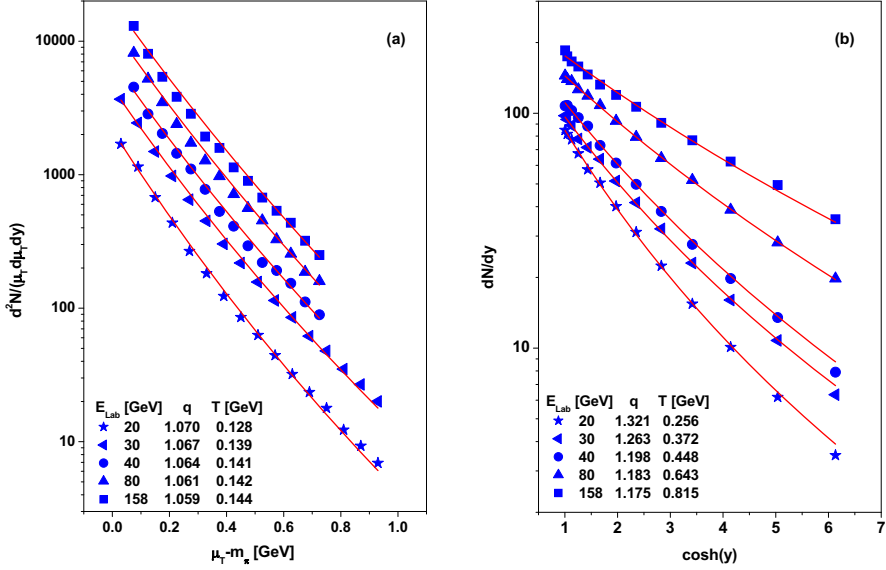
Fig. 1. (Color online) Fits using Eq. (1) to [20] data for $dN/d\mu_T$ (a) and for dN/dy (b) for central collision Pb+Pb at different energies.

and q_T (obtained from p_T distributions are now greater than q and have (approximately) a visible similar dependence on N_P. However, now $q_L < q_T$, again, this is opposite to what was seen in $p\bar{p}$.

A natural question is, what causes such different behavior of the parameter q in this case? The answer we propose: When extracting values of q from the rapidity distributions a tacit assumption was that μ_T in $E = \mu_T \cosh y$ remains constant (*i.e.*, it does not fluctuate). However, this is too crude, because data show that μ_T fluctuates as well. To account for this fact notice that $\exp_q(-E/T) = \exp_q[-(\mu_T/T)\cosh y] = \exp_q(-z\cosh y)$, *i.e.*, that fits to rapidity distributions provide us with fluctuations not so much of partition temperature T but rather of the variable $z = \mu_T/T$. This, in turn, can be written approximately as

$$\mathrm{Var}(z) \simeq \frac{1}{\langle T \rangle^2}\mathrm{Var}(\mu_T) + \frac{\langle \mu_T \rangle^2}{\langle T \rangle^2}\frac{\mathrm{Var}(T)}{\langle T \rangle^2}. \tag{10}$$

Since

$$\langle z \rangle \simeq \frac{\langle \mu_T \rangle}{\langle T \rangle} \qquad \text{and} \qquad \frac{\mathrm{Var}(1/T)}{\langle 1/T \rangle^2} \simeq \frac{\mathrm{Var}(T)}{\langle T \rangle^2} \tag{11}$$

and because

$$\frac{\mathrm{Var}(z)}{\langle z \rangle^2} = \frac{\mathrm{Var}(\mu_T)}{\langle \mu_T \rangle^2} + \frac{\mathrm{Var}(T)}{\langle T \rangle^2} \tag{12}$$

one obtains following sum rule connecting different fluctuations

$$q - 1 \stackrel{\text{def}}{=} \frac{\text{Var}(T)}{\langle T \rangle^2} = \frac{\text{Var}(z)}{\langle z \rangle^2} - \frac{\text{Var}\,(\mu_{\text{T}})}{\langle \mu_{\text{T}} \rangle^2} \,. \tag{13}$$
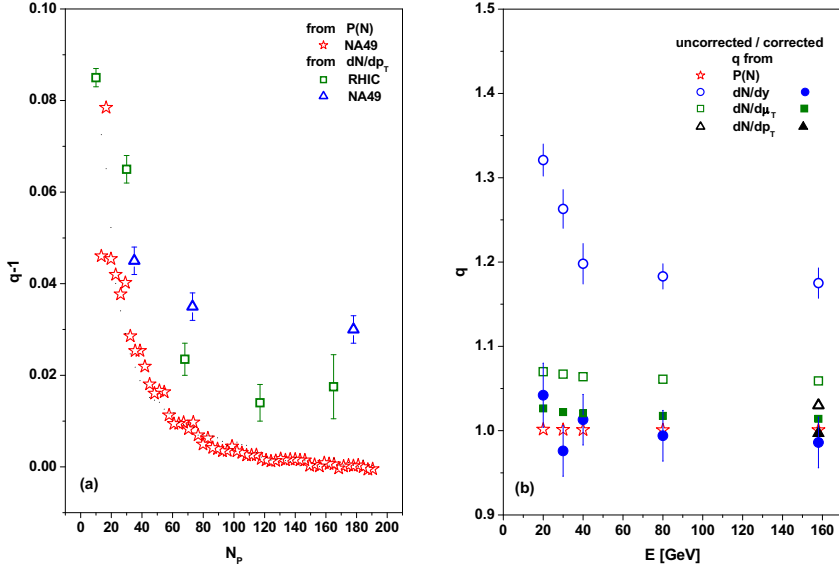

Fig. 2. (Color online) (a) q for different centralities measured by the number of projectile participants, N_{P}. Here q from $P(N)$ are our results from $\text{Var}(N)/\langle N \rangle$ [13] to be compared with $q = q_{\text{T}}$ obtained from dN/dp_{T} data of NA49 (*cf.* first work of [20]). PHENIX results [21] analyzed in [22] are also shown for comparison. (b) Energy dependence of q for the most central events. All results were obtained for the sake of this presentation using distributions provided by [20], *i.e.*, respectively, $dN/d\mu_{\text{T}}$, dN/dy and dN/dp_{T}. The errors are similar to those presented as an example for $q = q_{\text{L}}$ obtained from dN/dy. Open symbols correspond to uncorrected values of q, full symbols to values corrected by means of the procedure proposed in this work and explained in the text.

This sum rule is our main result and its action is presented in the right panel of Fig. 2. It connects total q, which can be obtained from the analysis of the NB form of the measured multiplicity distributions, $P(N)$, with $q_{\text{L}} - 1 = \text{Var}(z)/\langle z \rangle^2$, obtained from fitting rapidity distributions and $\text{Var}\,(\mu_{\text{T}})/\langle \mu_{\text{T}} \rangle^2$ obtained from data on transverse mass distributions. When extracting q from distributions of $dN/d\mu_{\text{T}}$ we proceed analogously with $z = \cosh y/T$.

Some explanatory remarks on the apparent discrepancies between pp and AA data are in order here. When fitting dN/dy data on pp in [17] μ_{T} was kept constant (and given by the $\langle p_{\mathrm{T}} \rangle$ for given energy). This means that all effects related to its fluctuations was attributed to the fluctuations of the "partition temperature" $T = T_{\mathrm{L}}$ (which is therefore only one of the fitted parameters and, for example, it cannot be used to calculate the mean energy). This was possible because, as observed in [22, 8], in the fitting procedure parameters T and q are strongly correlated. To illustrate this fact we present in Fig. 3 an example of the best fit to dN/dy for $E_{\mathrm{CM}} = 540$ GeV together with q-dependency of T/μ_{T} and parameter χ^2 representing the goodness of the fit[6]. Notice that for different values of q and T/μ_{T} we can describe dN/dy with reasonable accuracy. Parameters T and q are strongly correlated, for $E_{\mathrm{CM}} = 540$ GeV shown here $T(q)/\mu_{\mathrm{T}} = 56.5 - 14.5q$. Comparing values of T/μ_{T} and q shown in Fig. 3 to those reported for this energy earlier [17] ($q' = 1.26$ and $T'/\mu_{\mathrm{T}} = 35.1$) one can observe that the difference $\Delta q = q - q' = 0.22$ is roughly the same as the correction caused by fluctuations of μ_{T} discussed above[7].
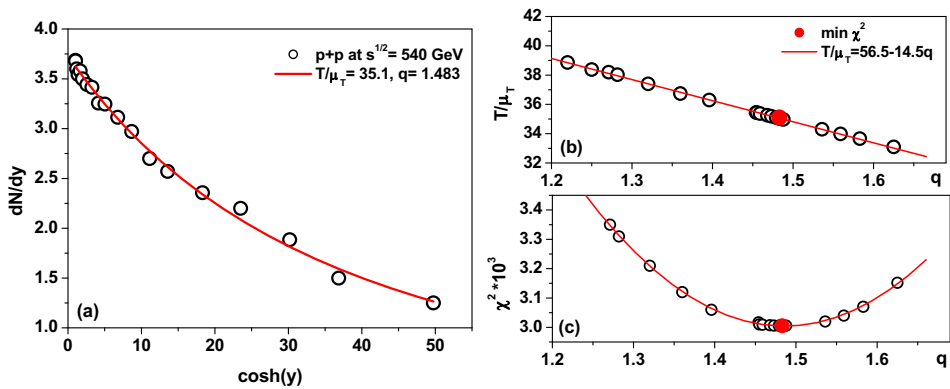

Fig. 3. (Color online) (a) Example of best fit to dN/dy for $E_{\mathrm{CM}} = 540$ GeV (data are the same as in [17]). (b) The q-dependence of the admissible T/μ_{T}. (c) q-dependence of the corresponding χ^2. The full (red) dot shows the minimal value of χ^2, see text for details.

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[6] Note that here χ^2 is not exactly the same as that commonly used in statistics. Namely, for m experimental values x_i compared with values of x_i' given by the fitting formula with m' parameters we use $\chi^2 = \sum_{i=1}^{m} (x_i - x_i')^2 / (m - m')$.

[7] We have chosen for our analysis nuclear collision data from NA49 because all of them (for different energies) are coming from the same experiment, *i.e.*, they were obtained under the same experimental conditions. Because of this fact, comparison of different q (*i.e.*, different fluctuations) was relatively simple. To repeat this procedure for the pp data would be much more complicated and involved and demands a separate investigation.

3. Summary

We have demonstrated that fluctuations of temperature T, together with fluctuations of other variables, result in the sum rule formula, Eq. (13), connecting q obtained from an analysis of different distributions. This allows us to understand why in AA collisions fluctuations observed in multiplicity distributions are much smaller than the corresponding ones seen in the rapidity distribution or in the distribution of transverse momenta (*i.e.*, why the corresponding q parameters evaluated from distributions of different observables are different). This issue should be checked further when complete sets of data become available from the experiments at LHC (especially from ALICE).

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