BRIGHT AND DARK SOLITONS OF AN INTEGRABLE EQUATION GOVERNING SHORT WAVES IN A LONG-WAVE MODEL WITH PERTURBATION TERMS

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We consider an integrable equation governing short waves in a long-wave model, derived recently by Faquir *et al.* [M.J. Faquir, M.A. Manna, A. Neveu, *Proc. R. Soc.* **A463**, 1939 (2007)]. The study is conducted in presence of perturbation terms. The perturbation terms that are considered are non-linear dispersion terms and fourth order dispersion. The solitary wave Ansatz is used to carry out the integration of the considered perturbed evolution equation. Both bright and dark solitons solutions are obtained. The physical parameters in the soliton solutions are obtained as function of the dependent model coefficients. The conditions of the existence of the derived solitons are derived.

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1. Introduction

In the past decades, studies have been made on solutions and integrability of different non-linear partial differential equations (NLPDEs), with constant and variable coefficients. One main reason for giving such a great interest is that this class of non-linear wave equations describes several phenomena in non-linear systems in a variety of scientific applications. Understanding these phenomena is better with the help of exact solutions when these solutions exist. Besides, the application of non-linear systems comprises many physics areas such as non-linear optics, plasmas, fluid mechanics, condensed matter, electro magnetic and many more. Therefore, finding explicit solutions offers a rich knowledge on the mechanism of the complicated physical phenomena and dynamical processes modeled by these non-linear evolution equations.

A large variety of powerful methods were developed for calculating exact solutions of NLPDEs of all kinds *e.g.* the non-linear Schrödinger equation, the Korteweg–de Vries equation, the Boussinesq equation and many others. Among these methods, we can cite the coupled amplitude-phase formalism [1, 2], the hyperbolic tangent method [3], Hirota bilinear method [4, 5], the sub-ODE method [6, 7], the solitary wave Ansatz method [8, 9, 10, 11, 12, 13, 14] and other analytical methods as well. Moreover, several numerical methods, such as the Petrov–Galerkin method [15], the collocation method [16], were employed for numerical treatments of the non-linear problems. However, some of these analytical and numerical solutions methods are not easy to use and sometimes suffer from tedious works and calculations [17, 18].

It is then essential to use appropriate techniques that do not require complicated calculations for the determination of explicit solutions of NLPDEs of physical relevance. What is important here is whether the method is efficient to construct closed form solutions of a given non-linear evolution equation. The solitary wave Ansatz method [8,9,10,11,12,13,14], and other methods of integrability have shown great success and progress in this area of research. The solitary wave Ansatz method is rather heuristic and possesses significant features that make it practical for the determination of soliton-type solutions for a wide class of NLPDEs in a direct method. This technique has recently been applied successfully to a wide range of NLPDEs [8, 9, 10, 11, 12, 13, 14].

Very recently, Faquir *et al.* [19] derived an integrable equation governing short waves in a long-wave model in the form

$$u_{xt} = u - uu_{xx} - \frac{1}{2}u_x^2 + \frac{\gamma}{2}u_x^2 u_{xx}\,, \tag{1}$$

where $u(x,t)$ is the fluid velocity at the surface, is a non-dimensional measure of the surface tension, and subscripts denote partial derivatives. The integrability of Eq. (1) for $\gamma = 0$ has already been discussed in Refs. [20, 21, 22]. More recently, Kraenkel and Senthilvelan [23] studied Lie symmetries and similarly reductions of Eq. (1) and reported some particular solutions, including static solution, traveling wave solution, and separable solution for different values of the system parameter.

In this work, we deal with the existence of exact soliton solutions for a perturbed form of Eq. (1) as follows

$$u_{xt} - u + uu_{xx} + \frac{1}{2}u_x^2 - \frac{\gamma}{2}u_x^2 u_{xx} = \alpha u^2 u_x^2 + \beta u^3 u_{xx} - \delta u_{xxxx}\,, \qquad \delta > 0\,, \quad (2)$$

where α, β and δ are constants. The first two perturbation terms on the right-hand side of Eq. (2) may be regarded as a combination of non-linear terms, while the last term represents the fourth-order dispersion term. To our knowledge, the exact analytic soliton solutions of Eq. (2) have not been previously obtained. It is always useful and desirable to construct exact analytical solutions (in particular soliton solutions) by using appropriate techniques. By applying the solitary wave Ansatz method, we find the exact bright and dark soliton solutions for the considered model. All the physical parameters in the soliton solutions are obtained as functions of the model coefficients. It is worth noting that the existence or the non-existence of solitary wave solutions depends on the dependent model coefficients, and therefore on the specific non-linear and dispersive features of the medium.

2. Bright soliton solution

We start the analysis by assuming a solitary wave Ansatz of the form

$$u(x,t) = \frac{A}{\cosh^p\left[B\left(x - vt\right)\right]}\,, \quad (3)$$

where A, B and v are, respectively, the amplitude, the inverse width and the velocity of the soliton. The exponents p is unknown at this point and its value will fall out in the process of deriving the soliton solution of the considered equation.

From Ansatz (3), one obtains

$$u_{xt} = -\frac{Ap^2B^2v}{\cosh^p\tau} + \frac{AB^2vp\left(p+1\right)}{\cosh^{p+2}\tau}\,, \quad (4)$$

$$uu_{xx} = \frac{A^2p^2B^2}{\cosh^{2p}\tau} - \frac{A^2B^2p\left(p+1\right)}{\cosh^{2p+2}\tau}\,, \quad (5)$$

$$u_x^2 = \frac{A^2p^2B^2}{\cosh^{2p}\tau} - \frac{A^2B^2p^2}{\cosh^{2p+2}\tau}\,, \quad (6)$$

$$u_x^2u_{xx} = \frac{A^3p^4B^4}{\cosh^{3p}\tau} - \frac{A^3B^4p^3\left(2p+1\right)}{\cosh^{3p+2}\tau} + \frac{A^3B^4p^3\left(p+1\right)}{\cosh^{3p+4}\tau}\,, \quad (7)$$

$$u^2u_x^2 = \frac{A^4p^2B^2}{\cosh^{4p}\tau} - \frac{A^4B^2p^2}{\cosh^{4p+2}\tau}\,, \quad (8)$$

$$u^3 u_{xx} = \frac{A^4 p^2 B^2}{\cosh^{4p}\tau} - \frac{A^4 B^2 p\,(p+1)}{\cosh^{4p+2}\tau}\,, \tag{9}$$

$$u_{xxxx} = \frac{A p^4 B^4}{\cosh^{p}\tau} - \frac{A B^4 p\,(p+1)\left\{p^2+(p+2)^2\right\}}{\cosh^{p+2}\tau} + \frac{A B^4 p\,(p+1)\,(p+2)\,(p+3)}{\cosh^{p+4}\tau}\,, \tag{10}$$

where

$$\tau = B\,(x - vt)\ . \tag{11}$$

Substituting Eqs. (3)–(11) into Eq. (2), we get

$$\begin{aligned}
&-\frac{A p^2 B^2 v}{\cosh^{p}\tau} + \frac{A B^2 v p\,(p+1)}{\cosh^{p+2}\tau} - \frac{A}{\cosh^{p}\tau} + \frac{A^2 p^2 B^2}{\cosh^{2p}\tau}\\
&-\frac{A^2 B^2 p\,(p+1)}{\cosh^{2p+2}\tau} + \frac{A^2 p^2 B^2}{2\cosh^{2p}\tau} - \frac{A^2 B^2 p^2}{2\cosh^{2p+2}\tau}\\
&-\frac{\gamma}{2}\left\{\frac{A^3 p^4 B^4}{\cosh^{3p}\tau} - \frac{A^3 B^4 p^3\,(2p+1)}{\cosh^{3p+2}\tau} + \frac{A^3 B^4 p^3\,(p+1)}{\cosh^{3p+4}\tau}\right\}\\
&= \alpha\left\{\frac{A^4 p^2 B^2}{\cosh^{4p}\tau} - \frac{A^4 B^2 p^2}{\cosh^{4p+2}\tau}\right\} + \beta\left\{\frac{A^4 p^2 B^2}{\cosh^{4p}\tau} - \frac{A^4 B^2 p\,(p+1)}{\cosh^{4p+2}\tau}\right\}\\
&-\delta\left\{\frac{A p^4 B^4}{\cosh^{p}\tau} - \frac{A B^4 p\,(p+1)\left\{p^2+(p+2)^2\right\}}{\cosh^{p+2}\tau}\right.\\
&\left.+\frac{A B^4 p\,(p+1)\,(p+2)\,(p+3)}{\cosh^{p+4}\tau}\right\}\ .
\end{aligned} \tag{12}$$

From (12), equating the exponents $3p+4$ and $4p+2$ gives

$$3p + 4 = 4p + 2 \tag{13}$$

so that

$$p = 2 \tag{14}$$

which is also obtained by equating the exponents $3p+2$ and $4p$.

If we put $p = 2$ in Eq. (12), we can determine the soliton parameters by setting the corresponding coefficients of $\frac{1}{\cosh^2\tau}$, $\frac{1}{\cosh^4\tau}$, $\frac{1}{\cosh^6\tau}$, $\frac{1}{\cosh^8\tau}$ and $\frac{1}{\cosh^{10}\tau}$, respectively, to zero such that

$$Ap^2B^2v + A = \delta Ap^4B^4 \,, \tag{15}$$

$$AB^2vp\,(p+1) + A^2p^2B^2 + \frac{A^2p^2B^2}{2} = \delta AB^4p(p+1)\left\{p^2 + (p+2)^2\right\} \,, \tag{16}$$

$$A^2B^2p\,(p+1) + \frac{A^2B^2p^2}{2} + \frac{\gamma}{2}A^3p^4B^4 = \delta AB^4p(p+1)\,(p+2)\,(p+3) \,, \tag{17}$$

$$\frac{\gamma}{2}A^3B^4p^3\,(2p+1) = \alpha A^4p^2B^2 + \beta A^4p^2B^2 \,, \tag{18}$$

$$\frac{\gamma}{2}A^3B^4p^3\,(p+1) = \alpha A^4B^2p^2 + \beta A^4B^2p\,(p+1) \,. \tag{19}$$

By multiplying (15) by $p+1$ and (16) by p, then extracting the resulting two equations, we get the expression

$$\tfrac{3}{2}A^2p^3B^2 - (p+1)A = \delta AB^4p^2(p+1)\,(p+2)^2 \,. \tag{20}$$

This latter equation gives the inverse width B of the soliton pulse as

$$B^2 = \frac{3Ap^2 \pm \sqrt{9A^2p^4 - 16\delta(p+1)^2\,(p+2)^2}}{4\delta p(p+1)\,(p+2)^2} \,. \tag{21}$$

Substituting $p = 2$ in equation (21) we find

$$B^2 = \frac{A \pm \sqrt{A^2 - 16\delta}}{32\delta}$$

that gives

$$B = \frac{\sqrt{A \pm \sqrt{A^2 - 16\delta}}}{4\sqrt{2\delta}} \,, \qquad \delta > 0 \,.$$

This equation shows that the relation between A and δ is given by

$$A^2 \geq 16\delta \,, \qquad \delta > 0 \,.$$

To examine the relation between A and B, we select for simplicity $\delta = 1$, then by graphing

$$B = \frac{\sqrt{A + \sqrt{A^2 - 16}}}{4\sqrt{2}}$$

we can easily observe that B is an increasing function with respect to A, where the lower bound for $B = 0.35$, and $B \to \infty$, when $A \to \infty$. This is illustrated by Fig. 1.

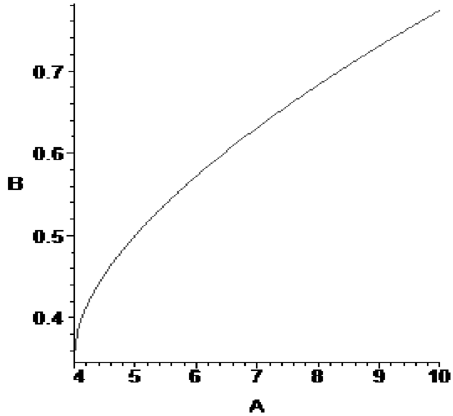

Fig. 1. The relation between the amplitude A and the inverse width B for the first value of B.

However, by graphing the second expression for B

$$B = \frac{\sqrt{A - \sqrt{A^2 - 16}}}{4\sqrt{2}}$$

we notice that the B is a decreasing function with respect to A, where the upper bound for $B = 0.35$ and B approaches 0 as A goes to infinity. This can be seen from Fig. 2.

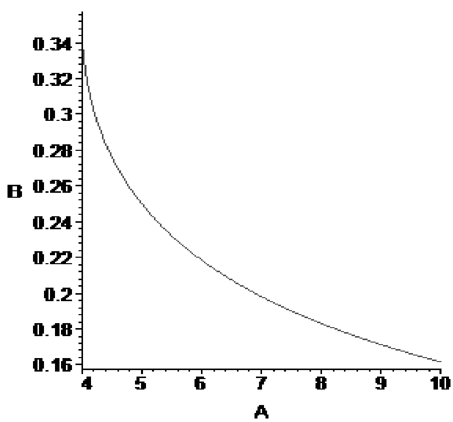

Fig. 2. The relation between the amplitude A and the inverse width B for the second value of B.

Substituting (21) into (16), one gets the velocity v of the soliton as

$$v = \frac{\left\{3Ap^2 \pm \sqrt{9A^2p^4 - 16\delta(p+1)^2 (p+2)^2}\right\}\left\{p^2 + (p+2)^2\right\}}{4(p+1)(p+2)^2} - \frac{3Ap}{2(p+1)} \,. \tag{22}$$

From (18), we have

$$\frac{\gamma}{2}B^2 = \frac{A(\alpha+\beta)}{p(2p+1)} \,. \tag{23}$$

From (19), we find

$$\frac{\gamma}{2}B^2 = \frac{A[\alpha p + \beta(p+1)]}{p^2(p+1)} \,. \tag{24}$$

Equating the two quantities gives the constraint condition

$$\frac{\alpha}{\beta} = -\frac{9}{4} \,. \tag{25}$$

Moreover, substituting $p = 2$ in (22) yields

$$v = \frac{5\left(A \pm \sqrt{A^2 - 16\delta}\right)}{8} - A \,.$$

Graphing the first value of the velocity v, given by

$$v = \frac{5\left(A + \sqrt{A^2 - 16}\right)}{8} - A \,, \qquad \delta = 1 \,,$$

we notice that v is an increasing function with respect to A with lower limit $v = -1.5$ as shown by Fig. 3.

However, by graphing the second value of v given by

$$v = \frac{5\left(A - \sqrt{A^2 - 16}\right)}{8} - A \,, \qquad \delta = 1 \,,$$

we notice that v is a decreasing function with respect to A with upper limit $v = -1.5$ as shown below by Fig. 4.

For this case, there is a possibility that the velocity v may become 0. In this case we find that

$$\delta = \frac{1}{25}A^2 \,.$$

Thus, finally, the bright soliton solution to the model (2) is given by

$$u(x,t) = \frac{A}{\cosh^2[B(x - vt)]} \,, \tag{26}$$

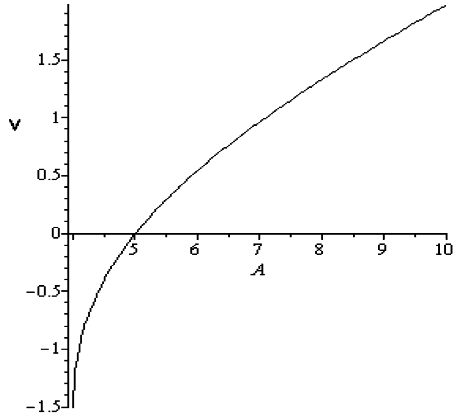

Fig. 3. The relation between the amplitude A and the velocity v for the first value of v.

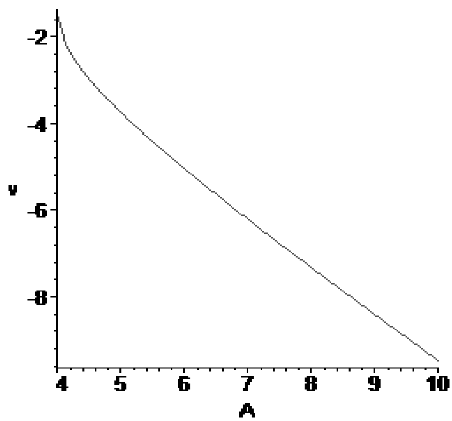

Fig. 4. The relation between the amplitude A and the velocity v for the second value of v.

where the relation between the free parameters A and B is given by (21) and the velocity v of the soliton is given by (22). Finally, the constraint relation between the full non-linear parameters α and β is displayed in (25).

The following remarks can be concluded from the discussion presented above:

Remark 1. It is clear from the obtained results that the coefficient δ of the dispersive term plays a major role in the results. However, the coefficients α and β of the perturbation terms were found to have a constant ratio as in (25). This means that this is not the unique generalization of (1). Other dispersive terms can be used as well. From the results we obtained, equation (2) gives bright and dark soliton solutions. Other solitary waves solutions such as compactons, peakons and cuspons do not exist for equation (2).

Remark 2. To show that (2) is integrable or not, we should find the Lax pair for this equation. No Lax pairs can be found for this generalization in (2). Based on this, the perturbation and the dispersive terms killed the integrability of the original integrable equation (1).

3. Dark soliton solution

In this section, we are interested in finding the dark soliton solution (expressed as hyperbolic tangent function) for the considered equation (2). One main reason for pursuing such a goal could be the possible applications of dark solitons in such a system taking advantage of their interesting properties. Note that to date (to our knowledge) no dark soliton solution for the equation (2) has been obtained.

For dark solitons, we assume an Ansatz solution of the form

$$u(x,t) = A \tanh^p \left[B\left(x - vt\right)\right] , \tag{27}$$

where A, B and v are unknown dependent parameters representing the amplitude, the inverse width and the velocity of the soliton, respectively, that will be determined. Also, the unknown exponent p will be determined during the course of the derivation of the dark soliton solution to (2). Therefore, from (27), we get after transforming *sech* terms to *tanh* terms the following expressions:

$$\begin{aligned}
u_{xt} &= -pvAB^2\left\{(p-1)\tanh^{p-2}\tau - 2p\tanh^{p}\tau + (p+1)\tanh^{p+2}\tau\right\} , &(28)\\
uu_{xx} &= pA^2B^2\left\{(p-1)\tanh^{2p-2}\tau - 2p\tanh^{2p}\tau + (p+1)\tanh^{2p+2}\tau\right\} , &(29)\\
u_x^2 &= p^2A^2B^2\left\{\tanh^{2p-2}\tau + \tanh^{2p+2} - 2\tanh^{2p}\tau\right\} , &(30)\\
u_x^2u_{xx} &= p^3A^3B^4\left\{(p-1)\tanh^{3p-4}\tau + (p+1)\tanh^{3p+4}\tau + 6p\tanh^{3p}\right.\\
&\quad \left.-2(2p+1)\tanh^{3p+2} - 2(2p-1)\tanh^{3p-2}\right\} , &(31)\\
u^2u_x^2 &= p^2A^4B^2\left\{\tanh^{4p-2}\tau + \tanh^{4p+2} - 2\tanh^{4p}\tau\right\} , &(32)\\
u^3u_{xx} &= pA^4B^2\left\{(p-1)\tanh^{4p-2}\tau - 2p\tanh^{4p}\tau + (p+1)\tanh^{4p+2}\tau\right\} , &(33)\\
u_{xxxx} &= pAB^4\left\{(p-1)(p-2)\left(p-3\right)\tanh^{p-4} + (p+1)(p+2)\left(p+3\right)\tanh^{p+4}\right.\\
&\quad -2\left\{p^2+(p-2)^2\right\}\left\{p-1\right\}\tanh^{p-2}\tau - 2\left\{p^2+(p+2)^2\right\}\\
&\quad \left.\times\left\{p+1\right\}\tanh^{p+2}\tau + \left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}\tanh^{P}\right\} . &(34)
\end{aligned}$$

Substituting Eqs. (28)–(34) into Eq. (2), we have

$$-pvAB^2\left\{(p-1)\tanh^{p-2}\tau-2p\tanh^{p}\tau+(p+1)\tanh^{p+2}\tau\right\}-A\tanh^{p}\tau$$
$$+pA^2B^2\left\{(p-1)\tanh^{2p-2}\tau-2p\tanh^{2p}\tau+(p+1)\tanh^{2p+2}\tau\right\}$$
$$+\tfrac{1}{2}p^2A^2B^2\left\{\tanh^{2p-2}\tau+\tanh^{2p+2}-2\tanh^{2p}\tau\right\}$$
$$-\frac{\gamma}{2}p^3A^3B^4\left\{(p-1)\tanh^{3p-4}\tau+(p+1)\tanh^{3p+4}\tau+6p\tanh^{3p}\right.$$
$$\left.-2(2p+1)\tanh^{3p+2}-2(2p-1)\tanh^{3p-2}\right\}$$
$$=\alpha p^2A^4B^2\left\{\tanh^{4p-2}\tau+\tanh^{4p+2}-2\tanh^{4p}\tau\right\}$$
$$+\beta pA^4B^2\left\{(p-1)\tanh^{4p-2}\tau-2p\tanh^{4p}\tau+(p+1)\tanh^{4p+2}\tau\right\}$$
$$-\delta pAB^4\left\{(p-1)(p-2)\,(p-3)\tanh^{p-4}+(p+1)(p+2)\,(p+3)\tanh^{p+4}\right.$$
$$-2\left\{p^2+(p-2)^2\right\}\{p-1\}\tanh^{p-2}\tau-2\left\{p^2+(p+2)^2\right\}\{p+1\}\tanh^{p+2}\tau$$
$$\left.+\left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}\tanh^{P}\right\}. \tag{35}$$

By equating the highest exponents of $\tanh^{3p+4}\tau$ and $\tanh^{4p+2}\tau$ terms in (35), one gets

$$3p+4=4p+2 \tag{36}$$

which yields the following value of p

$$p=2\,. \tag{37}$$

It should be remarked that the same value (37) arises also from equating the exponents of $\tanh^{3p+2}\tau$ and $\tanh^{4p}\tau$ functions (35).

Setting the coefficients of $\tanh^{0}\tau$, $\tanh^{2}\tau$, $\tanh^{4}\tau$, $\tanh^{6}\tau$, $\tanh^{8}\tau$, $\tanh^{10}\tau$ and $\tanh^{-2}\tau$ functions, respectively, equal to zero, one obtains the following equations

$$-pvAB^2(p-1)=2\delta pAB^4\left\{p^2+(p-2)^2\right\}\{p-1\}\,, \tag{38}$$
$$2p^2vAB^2-A+pA^2B^2(p-1)+\tfrac{1}{2}p^2A^2B^2-\frac{\gamma}{2}p^3A^3B^4(p-1)$$
$$=-\delta pAB^4\left\{4p^3+(p-1)^2(p-2)+(p+1)^2(p+2)\right\}\,, \tag{39}$$
$$-pvAB^2(p+1)-2p^2A^2B^2-p^2A^2B^2+\gamma p^3A^3B^4(2p-1)$$
$$=2\delta pAB^4\left\{p^2+(p+2)^2\right\}\{p+1\}\,, \tag{40}$$
$$(p+1)pA^2B^2+\tfrac{1}{2}p^2A^2B^2-3\gamma p^4A^3B^4p=\alpha p^2A^4B^2+\beta pA^4B^2(p-1)$$
$$-\delta pAB^4(p+1)(p+2)\,(p+3)\,, \tag{41}$$
$$\gamma p^3A^3B^4(2p+1)=-2\alpha p^2A^4B^2-2p\beta pA^4B^2\,, \tag{42}$$
$$-\frac{\gamma}{2}p^3A^3B^4(p+1)=\alpha p^2A^4B^2+(p+1)\beta pA^4B^2\,, \tag{43}$$
$$-\delta pAB^4(p-1)(p-2)\,(p-3)=0\,. \tag{44}$$

From (38) and (40), we get

$$B = \sqrt{\frac{3pA}{2\delta(p+1)\left\{(p-2)^2 - (p+2)^2\right\} + \gamma p^2 A^2 (2p-1)}}\,. \tag{45}$$

Inserting (45) into (38), one obtains

$$v = -\frac{6pA\delta\left\{p^2 + (p-2)^2\right\}}{2\delta(p+1)\left\{(p-2)^2 - (p+2)^2\right\} + \gamma p^2 A^2 (2p-1)}\,. \tag{46}$$

Eq. (44) is satisfied for $p = 2$ which coincides with the value of p in (37). Inserting this value in (42) and (43), we get the condition

$$\frac{\alpha}{\beta} = -\frac{9}{4} \tag{47}$$

for the dark soliton to exist.

Lastly, we can determine the dark soliton solution for Eq. (2) as

$$u(x,t) = A\tanh^2\left[B\left(x - vt\right)\right]\,, \tag{48}$$

where the amplitude B is given by (45) and the velocity v of the soliton pulse by (46). Note that this solution exists provided that the constraint condition (47) is satisfied.

4. Conclusion

We have derived the exact bright and dark soliton solutions of an equation governing short waves in a long-wave model in the presence of additional non-linear dispersive terms and fourth order dispersion term. The bright and the dark soliton solutions have been derived by using the solitary wave Ansatz method. The physical parameters in the obtained soliton solutions are calculated in course of the derivation of the exact solutions as function of the dependent model coefficients. Conditions for the existence of soliton solutions have also been reported. We believe that the soliton solutions obtained by the used Ansatze are entirely new and have not been previously presented. Note that it is always useful and desirable to construct exact analytical solutions for understanding most non-linear physical phenomena. It should be noted that the research of new integrable models and their soliton solutions is of great interest, because the soliton approach is universal in different fields of modern physics. A more general model having higher-order effects and time-dependent coefficients will be studied is the future works.

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