BENDER–DUNNE ORTHOGONAL POLYNOMIALS, QUASI-EXACT SOLVABILITY AND ASYMPTOTIC ITERATION METHOD FOR RABI HAMILTONIAN

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We present a method for obtaining the quasi-exact solutions of the Rabi Hamiltonian in the framework of the asymptotic iteration method (AIM). The energy eigenvalues, the eigenfunctions and the associated Bender–Dunne orthogonal polynomials are deduced. We show *(i)* that orthogonal polynomials are generated from the upper limit (*i.e.*, truncation limit) of polynomial solutions deduced from AIM, and *(ii)* prove to have nonpositive norm.

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1. Introduction: formal aspects

A new class of potentials which are intermediate to exactly solvable (ES) ones and non solvable ones are called quasi-exactly solvable (QES) problems [1, 2, 3, 4]. It is well known that a part of their spectrum can be determined algebraically but not the whole spectrum. One of an alternative mathematical language for describing the QES systems is a remarkable set of orthogonal polynomials introduced by Bender and Dunne [5]. These polynomials satisfy the three-term recursion relation and, as a consequence, form an orthogonal set with respect to some weight function $\rho(E)$ depending of the energy [6, 7]. The exact solution of the Schrödinger equation takes the form

$$\chi(x;E)=\sum_{n=0}^{\infty}\chi_n(x)P_n(E)\,, \tag{1.1}$$

in which $P_n(E)$ denotes certain polynomials of E and satisfy [8, 9, 10]

$$P_n(E_k)=0\,;\qquad n\geq\Lambda+1\,,\qquad k=0,1,2,\ldots,\Lambda\,, \tag{1.2}$$

where the parameter Λ is a positive integer value. It follows that $P_n(E)$ are orthogonal

$$\int P_n(E)\rho(E)P_m(E)dE = p_n p_m \delta_{nm}\,, \tag{1.3}$$

where p_n represent the norms of the orthogonal polynomials $P_n(E)$. It is possible to determine the norms p_n (or squared norms $\gamma_n = p_n^2$) by multiplying the recursion relation by $E^{n-1}\rho(E)$ and to integrate with respect to E.

However, the search of the orthogonal polynomials $P_n(E)$ can be generated by means of the asymptotic iteration method (AIM) [11]. This procedure was first introduced as an approximation in order to deduce both energy eigenvalues and eigenfunctions, using some computer-algebra systems. Due to the simplicity of the procedure, various aspects of the method have been employed, successfully, to obtain both ES and QES problems [11, 12, 13, 14, 15, 16]. The systematic procedure of the AIM begins by rewriting a second-order linear differential equation in the following form

$$\chi''(x) = r_0(x)\chi'(x) + s_0(x)\chi(x) \tag{1.4}$$

with $r_0(x)$ and $s_0(x)$ being many times differentiable functions. In order to find the solution of Eq. (1.4), we rely on the symmetric shape of the right-hand side. Thus for $(n+1)^{\text{th}}$ and $(n+2)^{\text{th}}$ derivative of Eq. (1.4), $n = 1, 2, \ldots$, we get

$$\begin{aligned} r_n(x) &= r'_{n-1}(x) + s_{n-1}(x) + r_0(x)r_{n-1}(x)\,, \\ s_n(x) &= s'_{n-1}(x) + s_0(x)r_{n-1}(x)\,, \end{aligned} \tag{1.5}$$

where for sufficiently large iteration number n, the *asymptotic* behavior of the procedure can be applied as

$$\frac{r_n(x,E)}{s_n(x,E)} = \frac{r_{n-1}(x,E)}{s_{n-1}(x,E)} = \alpha(E)\,, \tag{1.6}$$

and which allows us, on the one hand, to calculate the energy eigenvalues by iterations and, on the other hand, to express the so-called *polynomial solutions* of Eq. (1.4).

Many interesting models are obtained by combining two fundamental models of quantum mechanics; namely the two-level system and the harmonic oscillator [17]. The Rabi Hamiltonian is one of the more studied ones [18, 19, 20], and plays an important role in many areas of physics from condensed matter and biophysics to quantum optics. Due to the elegance of the model various aspects of the Hamiltonian have been studied analytically and numerically [21, 22, 23, 24, 25, 26]. The complete Hamiltonian of such system is of the form

$$H = \hbar\omega a^\dagger a + \frac{\hbar\Omega}{2}\sigma_0 + \lambda(\sigma_+ + \sigma_-)\left(a^\dagger + a\right), \tag{1.7}$$

where the parameter ω is the boson field frequency, $\hbar\Omega$ is the atomic level separation and λ is the atom-field coupling constant. Here, $a^\dagger$ (a) is a creation (annihilation) operator for the field mode and the σ are the usual Pauli spin matrices $\sigma_x = \sigma_+ + \sigma_-$, $\sigma_y = -i(\sigma_+ - \sigma_-)$ and $\sigma_z = \sigma_0$. Note that the zero energy level was taken halfway between the two atomic levels, so that the unperturbed atomic energies are $\pm\frac{\hbar\Omega}{2}$. The four terms appearing in the interaction part of the Hamiltonian (1.7) have the following interpretation: $\sigma_+ a$ $(\sigma_- a^\dagger)$ represents an absorbed (emitted) photon and an excited (de-excited) atom from state $|1\rangle$ to $|2\rangle$, while $\sigma_+ a^\dagger$ $(\sigma_- a)$ stands for one photon which is emitted (absorbed) and an excited (de-excited) atom.

In this present paper, we propose an alternative approach to study the Rabi Hamiltonian (1.7) based to the AIM procedure, to obtain both the energy eigenvalues E_n, the eigenfunctions $\chi(x; E)$ and a set of orthogonal polynomials $P_n(E)$ explicitly constructed from the above procedure and known to be orthogonal polynomials of Bender–Dunne of the genus one. We show that polynomial solutions $Y_n(E; \lambda, \beta)$ deduced from AIM procedure are a generating functions of Bender–Dunne orthogonal polynomials $P_n(E)$. In fact, and following our construction, the orthogonal polynomials $P_n(E)$ are deduced from the upper limit (*i.e.*, truncation limit) of $Y_n(E; \lambda, \beta)$ for $n_{\max} \equiv \Lambda = E + \lambda^2$; hence, the useful attribute to the method is that once we get polynomial solutions of any model then its associated Bender–Dunne orthogonal polynomials are easily deduced. We show also that orthogonal polynomials $P_n(E)$ are prone with a nonpositive definite norm, unless the constraint $n \geq E + \lambda^2 + 1$ is checked.

The paper proceeds as the following. In the next section we introduce a unitary transformation with an aim of making the Rabi Hamiltonian simpler. By applying AIM procedure, we deduce the coefficients of the polynomial solutions with the first four iterations. In Sec. 3 we clarify, through an explicit construction, the relation between the calculated eigenfunctions and the orthogonal polynomials leading to the three-term recursion relation which enables us to deduce the orthogonal polynomials $P_n(E)$. We show, in Sec. 4, that the orthogonal polynomials are associated with a nonpositive norm γ_n and the last section is devoted to our final conclusion.

2. Solvability of the Rabi Hamiltonian

In Fock–Bargmann space, the system (1.7) is transformed into the third-order differential equation which is not easy to solve. Therefore, the first step to solve the system (1.7) is to reduce the order of the differential equation using some transformations. The structure of the Rabi Hamiltonian is more easily seen by applying the unitary transformation

$$\mathcal{H} = U^\dagger H U = a^\dagger a + \beta(\sigma_+ + \sigma_-) + \lambda\left(a^\dagger + a\right), \tag{2.1}$$

where $U \equiv U^\dagger = \frac{1}{\sqrt{2}}(\sigma_+ + \sigma_- + \sigma_0)$. Here, $\hbar$ and ω are set to unity, and $\Omega = 2\beta$. By performing a transition to a Fock–Bargmann space $a^\dagger \to x$ and $a \to \frac{d}{dx}$, and writing the stationary Schrödinger equation for the two-component wave-function $\binom{\psi_1(x)}{\psi_2(x)}$, we obtain a system of two first-order linear differential equations for the functions $\psi_1(x)$ and $\psi_2(x)$ [22, 23]

$$\begin{aligned}\frac{d\psi_1(x)}{dx} &= \frac{E-\lambda x}{x+\lambda}\,\psi_1(x) - \frac{\beta}{x+\lambda}\,\psi_2(x)\,,\\ \frac{d\psi_2(x)}{dx} &= -\frac{\beta}{x-\lambda}\,\psi_1(x) + \frac{E+\lambda x}{x-\lambda}\,\psi_2(x)\,,\end{aligned} \tag{2.2}$$

where E are energy eigenvalues of $\mathcal{H}$. The general solution of Eqs. (2.2) can be obtained by transforming them into a second-order form

$$\begin{aligned}&\left(x^2-\lambda^2\right)\psi_1''(x) - \left[(E-\lambda x-1)(x-\lambda) + (x+\lambda)(E+\lambda x)\right]\psi_1'(x)\\ &-\left[\beta^2 - E^2 + \lambda^2 x^2 - \lambda(x-\lambda)\right]\psi_1(x) = 0\,,\end{aligned} \tag{2.3}$$

and substituting

$$x = \lambda(2\xi - 1)\,, \qquad \psi_1(x) = e^{-2\lambda^2\xi}\chi(\xi)\,, \tag{2.4}$$

we obtain a second-order differential equation [22]

$$\begin{aligned}&\xi(1-\xi)\chi''(\xi) + \left[\lambda^2\left(4\xi^2 - 2\xi - 1\right) + E(2\xi-1) - \xi + 1\right]\chi'(\xi)\\ &+\left[\lambda^4(3-4\xi) + 2E\lambda^2(1-2\xi) + \beta^2 - E^2\right]\chi(\xi) = 0\,.\end{aligned} \tag{2.5}$$

Now this equation is similar to Eq. (1.4) and then it is in a suitable form for application of AIM procedure; the initial $r_0(\xi)$ and $s_0(\xi)$ functions are given

$$\begin{aligned}r_0(\xi) &= \frac{\lambda^2\left(4\xi^2-2\xi-1\right) + E(2\xi-1) - \xi + 1}{\xi(\xi-1)}\,,\\ s_0(\xi) &= \frac{\lambda^4(3-4\xi) + 2E\lambda^2(1-2\xi) + \beta^2 - E^2}{\xi(\xi-1)}\,.\end{aligned} \tag{2.6}$$

We may calculate $r_n(\xi)$ and $s_n(\xi)$ using a sequence of Eqs. (1.5) and the calculated energy eigenvalues E_n by the mean of Eq. (1.6) should be independent of the choice of the variable ξ.

We report below the factored form of the first four iterations of Eq. (2.5) which lead to the coefficients of polynomial solutions. By convention n represents the iteration number, while d refers to the degree of polynomial solutions. The coefficients $Y_n(E;\lambda,\beta)$ are polynomials in the energy variable E_n with an even higher-degree $2n+2$.

Note that the energy eigenvalues index is the same as the iteration number.

$$
\begin{aligned}
n=1 \quad & d=2 \quad C_{12} = 16\lambda^4\left(-1+E+\lambda^2\right)\left(E+\lambda^2\right),\\
& d=1 \quad C_{11} = 8\lambda^2\left(-1+E+\lambda^2\right)Y_0(E;\lambda,\beta),\\
& d=0 \quad C_{10} = Y_1(E;\lambda,\beta),\\[1ex]
n=2 \quad & d=3 \quad C_{23} = 64\lambda^6\left(-2+E+\lambda^2\right)\left(-1+E+\lambda^2\right)\left(E+\lambda^2\right),\\
& d=2 \quad C_{22} = 48\lambda^4\left(-2+E+\lambda^2\right)\left(-1+E+\lambda^2\right)Y_0\left(E;\lambda,\beta\right),\\
& d=1 \quad C_{21} = 12\lambda^2\left(-2+E+\lambda^2\right)Y_1\left(E;\lambda,\beta\right),\\
& d=0 \quad C_{20} = Y_2(E;\lambda,\beta),\\[1ex]
n=3 \quad & d=4 \quad C_{34} = 256\lambda^8\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)\left(-1+E+\lambda^2\right)\\
& \qquad\qquad \times\left(E+\lambda^2\right),\\
& d=3 \quad C_{33} = 256\lambda^6\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)\left(-1+E+\lambda^2\right)\\
& \qquad\qquad \times Y_0(E;\lambda,\beta),\\
& d=2 \quad C_{32} = 96\lambda^4\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)Y_1(E;\lambda,\beta),\\
& d=1 \quad C_{31} = 16\lambda^2\left(-3+E+\lambda^2\right)Y_2(E;\lambda,\beta),\\
& d=0 \quad C_{30} = Y_3(E;\lambda,\beta),\\[1ex]
n=4 \quad & d=5 \quad C_{45} = 1024\lambda^{10}\left(-4+E+\lambda^2\right)\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)\\
& \qquad\qquad \times\left(-1+E+\lambda^2\right)\left(E+\lambda^2\right),\\
& d=4 \quad C_{44} = 1280\lambda^8\left(-4+E+\lambda^2\right)\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)\\
& \qquad\qquad \times\left(-1+E+\lambda^2\right)Y_0(E;\lambda,\beta),\\
& d=3 \quad C_{43} = 640\lambda^6\left(-4+E+\lambda^2\right)\left(-3+E+\lambda^2\right)\left(-2+E+\lambda^2\right)\\
& \qquad\qquad \times Y_1(E;\lambda,\beta),\\
& d=2 \quad C_{42} = 160\lambda^4\left(-4+E+\lambda^2\right)\left(-3+E+\lambda^2\right)Y_2(E;\lambda,\beta),\\
& d=1 \quad C_{41} = 20\lambda^2\left(-4+E+\lambda^2\right)Y_3(E;\lambda,\beta),\\
& d=0 \quad C_{40} = Y_4(E;\lambda,\beta).
\end{aligned}
$$

These polynomial solutions have some interesting properties. For example, a polynomial set $Y_n(E;\lambda,\beta)$ can be generated successively for each a new iteration with the constraint $n=d+m$, where $0<m\leq n$. However, if $n=d$ (*i.e.*, $m=0$), then the associated polynomial solution is

$$Y_0(E;\lambda,\beta)=E^2-2\lambda^2E-\beta^2-3\lambda^4\,. \tag{2.7}$$

The factor associated to the coefficient $C_{n,n+1}$ is $(4\lambda^2)^{n+1}$ and $C_{n0}\equiv Y_n(E;\lambda,\beta)$, with $n=1,2,\ldots$ Furthermore, we will see that the polynomials $Y_n(E;\lambda,\beta)$ lead to the Bender–Dunne orthogonal polynomials as reviewed in the next section.

3. Bender–Dunne orthogonal polynomials

The search for a power series solution of Eq. (2.5), $\chi(\xi)=\sum_{n=0}^{\infty}\chi_n(\xi)$, can be explicitly generated by means of the series expansion leading to the three-term recursion relation.

Using the identities

$$\begin{aligned}\chi(\xi) &= \xi^q\sum_{n=0}^{\infty}a_n\xi^n\,,\\ \chi'(\xi) &= \xi^q\sum_{n=0}^{\infty}(n+q)a_n\xi^{n-1}\,,\\ \chi''(\xi) &= \xi^q\sum_{n=0}^{\infty}(n+q)(n+q-1)a_n\xi^{n-2}\,,\end{aligned} \tag{3.1}$$

where the exponent q and the coefficients a_n are still undetermined, and by substituting them into Eq. (2.5), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}4\lambda^2\left(n+q-\lambda^2-E\right)a_n\xi^{n+vq+1}+\sum_{n=0}^{\infty}\Big[3\lambda^4-E^2+\beta^2+2E\lambda^2+(n+q)\\ &\times\left(2E-1\lambda^2-n-q\right)\Big]a_n\xi^{n+q}+\sum_{n=0}^{\infty}(n+q)\left(n+q-\lambda^2\right)a_n\xi^{n+q-1}=0\,,\end{aligned} \tag{3.2}$$

where the coefficients of the left-hand side of Eq. (3.2) must vanish individually. The lowest power of ξ appearing in Eq. (3.2) is ξ^{q-1}, for $n=0$ in the last summation. The requirement that the coefficients vanish yields the indicial equation $q(q-\lambda^2)a_0=0$; hence, we must require either $q=0$ or $q=\lambda^2$, and $a_0\neq0$. Note that if the first summation vanishes, the energy eigenvalues satisfy the constraint

$$E_{n,q}(\lambda)=n+q-\lambda^2\,. \tag{3.3}$$

Therefore, if we replace $n \to j$ in the second summation and $(n-1) \to j$ in the last (they are independent summations), this results in the two-term recursion relation

$$a_{j+1} = -\frac{\beta^2}{(j+q+1)\,(j+q+1-\lambda^2)}\,a_j\,. \tag{3.4}$$

Substituting j by n and choosing the indicial equation root $q = 0$, the mathematical induction leads us to write

$$a_n = \frac{(-1)^n\beta^{2n}}{n!}\,\frac{\Gamma\left(1-\lambda^2\right)}{\Gamma\left(n+1-\lambda^2\right)}\,a_0\,, \tag{3.5}$$

with $a_0 = 1$. The solution of Eq. (2.5), following Eq. (1.1)

$$\chi(\xi;E) = \sum_{n=0}^{\infty}\chi_n(\xi)P_n(E) = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^{2n}}{n!}\,\frac{\Gamma\left(1-\lambda^2\right)}{\Gamma\left(n+1-\lambda^2\right)}P_n(E)\xi^n\,, \tag{3.6}$$

can be considered as the generating function for the polynomials $P_n(E)$. However, for positive integer values Λ, the series expansion in Eq. (3.6) are truncated when E is a zero of $P_\Lambda(E)$. We know that $\Gamma(n+1-\lambda^2)$ has simple poles at $n = \lambda^2-1, \lambda^2-2, \ldots,$ [27] and taking into account the constraint (3.3) with $q = 0$, leading to the upper limit for n, *i.e.*, $n_{\max} \equiv \Lambda = E + \lambda^2 > 0$. Indeed, this can be observed in the coefficients $C_{n,d}$ of the polynomial solutions as it was reported above. Therefore, the corresponding exact eigenfunctions are given

$$\chi_\Lambda(\xi;E) = \sum_{n=0}^{\Lambda}\frac{(-1)^n\beta^{2n}}{n!}\,\frac{\Gamma\left(1-\lambda^2\right)}{\Gamma\left(n+1-\lambda^2\right)}P_n(E)\xi^n\,. \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (2.5) leads, after some straightforward but lengthy calculation, to the following three-term recursion relation for $P_n(E)$

$$\begin{aligned}&4\lambda^2\left(n-E-\lambda^2\right)P_{n+1}(E) + \left[\left(n-E-\lambda^2\right)\left(E-3\lambda^2-n\right)-\beta^2\right]\\&\times P_n(E) + 4\lambda^2\left(n-E-\lambda^2\right)P_{n-1}(E) = 0\end{aligned} \tag{3.8}$$

with the normalization condition $P_0(E) = 1$.

The polynomial $P_n(E)$ vanishes for $n \geq E + \lambda^2 + 1$ as occurred in the third term of Eq. (3.8). The recursion relation (3.8) generates a set of polynomials, where the first five of them are reported below in Table I. Actually, the deduced orthogonal polynomials are written as a product of $P_n(\lambda)$ by $(-1)^{n-1}\beta^2$. One can see that orthogonal polynomials $P_n(E)$ of the Table I are generated from polynomial solutions $Y_n(E;\lambda,\beta)$ deduced from

AIM procedure. In fact, the orthogonal polynomials $P_n(E)$ are deduced from the upper limit (*i.e.*, truncation limit) of $Y_n(E;\lambda,\beta)$ for $n_{\max}\equiv \Lambda = E+\lambda^2$, *i.e.*,

$$\lim_{E\to n-\lambda^2} Y_n(E;\lambda,\beta) = (-1)^{n-1}\beta^2 P_n(E)\,. \tag{3.9}$$

TABLE I

Bender–Dunne orthogonal polynomials for Rabi Hamiltonian and the associated energy eigenvalues.

n	$E(\lambda)$	$P_n(\lambda)$
1	$1-\lambda^2$	$4\lambda^2+(\beta^2-1)$
2	$2-\lambda^2$	$32\lambda^4+4(3\beta^2-8)\lambda^2+(\beta^2-1)(\beta^2-4)$
3	$3-\lambda^2$	$384\lambda^6+16(11\beta^2-54)\lambda^4+8(3\beta^4-29\beta^2+54)\lambda^2$ $+(\beta^2-1)(\beta^2-4)(\beta^2-9)$
4	$4-\lambda^2$	$6144\lambda^8+128(25\beta^2-192)\lambda^6+16(35\beta^4-542\beta^2+1728)\lambda^4$ $+8(5\beta^6-115\beta^4+722\beta^2-1152)\lambda^2$ $+(\beta^2-1)(\beta^2-4)(\beta^2-9)(\beta^2-16)$
5	$5-\lambda^2$	$122880\lambda^{10}+512(137\beta^2-1500)\lambda^8+64(225\beta^4-5036\beta^2$ $+24000)\lambda^6+16(85\beta^6-2867\beta^4+27518\beta^2-72000)\lambda^4$ $+4(15\beta^8-670\beta^6+9551\beta^4-49216\beta2+72000)\lambda^2$ $+(\beta^2-1)(\beta^2-4)(\beta^2-9)(\beta^2-16)(\beta^2-25)$

These orthogonal polynomials are exactly similar to those obtained by the Juddian isolated method and agree with the first roots of the Kuś series [21, 22]. In other words, the system has energy $E_n=n-\lambda^2$, $n=1,2,\ldots$, only if the atomic-level separation 2β and the boson–atom field coupling λ obey condition $P_n(\lambda)=0$.

4. Norms of orthogonal polynomials

As the polynomials $P_n(E)$ are orthogonal of a discrete variable E_n, it is possible to determine their squared norms. The procedure is to apply Eq. (1.3); *i.e.*, multiplying the recursion relation (3.8) by $E^{n-1}\rho(E)$, where $\rho(E)$ is the weight function, and to integrate with respect to E using the fact that both $P_n(E)$ and E^k, $k<n$, are orthogonal. We obtain

$$\gamma_n = 2\lambda^2\frac{n-\lambda^2}{n+\lambda^2}\gamma_{n-1}\,, \tag{4.1}$$

with $\gamma_n = p_n^2$ and $\gamma_0 = 1$. It is obvious that by mathematical induction, we get

$$\gamma_n = \prod_{k=1}^{n} 2\lambda^2 \frac{k-\lambda^2}{k+\lambda^2}\gamma_0 = 2^n \lambda^{2n} \frac{\left(1-\lambda^2\right)_n}{\left(1+\lambda^2\right)_n}\,, \tag{4.2}$$

where $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$ are the often-used Pochhammer symbol [27], with $(a)_0 = 1$.

Using the well-known identity $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$, we have

$$\gamma_n = 2^n \lambda^{2n} f(\lambda) \frac{\Gamma\left(n+1-\lambda^2\right)}{\Gamma\left(n+1+\lambda^2\right)}\,. \tag{4.3}$$

with $f(\lambda) = \frac{\sin \pi\lambda^2}{\pi}\Gamma(\lambda^2)\Gamma(1+\lambda^2)$.

This identity reveals an important result that the orthogonal polynomials are associated with a nonpositive definite norm represented by $\Gamma(n+1+\lambda^2)$. However, it shows in the Γ-function of the numerator that γ_n has in the complex plan poles which are given at the negative integers $N = 0, -1, -2, \ldots,$ (or equivalently: $n = \lambda^2 - 1, \lambda^2 - 2, \ldots$), leading to the upper limit $\Lambda = E + \lambda^2 > 0$ as conjectured above. Consequently, γ_n vanishes for each $n \geq \Lambda + 1$.

5. Summary and conclusions

In this paper, we have shown that there exists another approach of introducing the Bender–Dunne orthogonal polynomials which characterizes the quasi-exact solvable problems. The model given here is related to the AIM procedure and we have taken a new look at the solution of the Rabi Hamiltonian in order to generate its energy eigenvalues as well as the eigenfunctions and their Bender–Dunne orthogonal polynomials.

The main aim of this article was to consider the possibility of generating the Bender–Dunne orthogonal polynomials $P_n(\lambda)$ via another method different from the standard approach which consists in applying the eigenfunctions to the three-term recursion relation. Our method enables us to generate them directly starting from polynomial solutions $Y_n(E;\lambda,\beta)$ deduced by AIM procedure. In fact, we have shown that orthogonal polynomials $P_n(\lambda)$ are deduced from the upper limit (*i.e.*, truncation limit) of polynomial solutions $Y_n(E;\lambda,\beta)$ deduced from AIM procedure for $n_{\max} \equiv \Lambda = E + \lambda^2$. Therefore $Y_n(E;\lambda,\beta)$ can be considered as a generating functions of $P_n(\lambda)$. In other words, this means that $P_n(\lambda)$ and $Y_n(E;\lambda,\beta)$ are similar if the system has the associated energy eigenvalues $E_n = n - \lambda^2$, with $n = 1, 2, \ldots$. We have also shown that the orthogonal polynomials arising from a quasi-exactly solvable Rabi model are associated with a nonpositive definite norm whose the iteration number n exceeds a critical value mentioned above, *i.e.*, $n \geq \Lambda + 1$. Consequently, the quasi-exact energy eigenvalues of the Rabi Hamiltonian are the zeros of the polynomials $P_n(\lambda)$.

For other applications, we believe that similar results can be obtained via the approach worked out here to the various atomic systems such as Jaynes–Cummings and $E \otimes \epsilon$ Jahn–Teller Hamiltonian.

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