ON THE GRAVITO-ELECTROMAGNETIC ANALOGY

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Earlier research by Zel'manov and by Hönl and Dehnen has shown how the geodesic equation for a charged test particle can be written as a Lorentz force law in which the four-velocity u^i of an observer in the physical three-space $\gamma_{\alpha\beta} = -g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00}$ is regarded as a gravitational vector potential. Analysing this analogy further, we write the four $\binom{i}{0}$ components of the Einstein equations in a form resembling a non-linear Maxwell system, which, for a stationary field, is most clearly understood from the Kaluza–Klein perspective, the projection being from four dimensions to three, rather than from five dimensions to four. For the vacuum theory defined by vanishing energy-momentum tensor, $T_{ij} = 0$, these equations exhibit the structure of a non-linear sigma model, found by Ernst, and investigated by Gibbons and Hawking and by Sanchez, the scalar potentials of which we here relate to the gravito-electromagnetic fields. The non-stationary gravitational field is also considered in the normal coordinate system introduced by de Donder and Lanczos, in which case a gravitational displacement current occurs in the three $\binom{\alpha}{0}$ field equations, converting the $((_{00})/g_{00})$ and $\binom{\alpha}{0}$ components into a dynamical system. Finally, we discuss the vacuum degeneracy of the superstring theory, arguing from the quantum-gravitational path-integral method that Minkowski space is favoured probabilistically over the stringy vacuum state, in agreement with observation.

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1. Introduction

Energy conservation laws are very important in physics. It was emphasized by Einstein [1] that with the advent of the general theory of relativity, the definition of energy becomes problematic, the concept of an integral, global energy conservation law being superseded by local differential laws, expressing rather the continuity of energy and momentum. Nevertheless, under certain geometric conditions, the notion of global energy remains meaningful, and as a first step to the mathematical formulation of this problem, Einstein [2] introduced the energy-momentum pseudo-tensor $t_i^{\ j}$, whereby the four-dimensional covariant energy conservation equation

$$T_{i\ ;j}^{\ j} = 0\,, \tag{1}$$

defined using the covariant derivatives $_{;j}$, is converted into the equation

$$\left(\mathscr{T}_i^{\ j} + \mathfrak{t}_i^{\ j}\right)_{,j} = 0 \tag{2}$$

expressed in terms of partial derivatives $_{,j}$ alone and the densitized quantities $\mathscr{T}_i^{\ j} \equiv \sqrt{-g}\,T_i^{\ j}$ and $\mathfrak{t}_i^{\ j} \equiv \sqrt{-g}t_i^{\ j}$, where g_{ij} is the metric and $g = \det g_{ij}$.

Although the pseudo-tensor defining gravitational energy does not transform covariantly under changes of the coordinate system, and is therefore not a true tensor, Eq. (2) has the advantage over Eq. (1) that it can be integrated, and thus converted into a statement about global energy.

This idea was put on a precise footing by a Nordström[3] (see also Tolman[4]), who noted, in particular for a stationary gravitational field satisfying $g_{ij,0} = 0$, that

$$\left(\mathscr{T}_i^{\ 0} + \mathfrak{t}_i^{\ 0}\right)_{,0} = 0\,, \tag{3}$$

leading to the identification of the energy density as

$$\rho_{\rm N} = 2\left(T_0^0 + t_0^0\right)\,. \tag{4}$$

Assuming that T_i^j and t_i^j are non-vanishing only within a finite bounded region Ω, which is equivalent to assuming asymptotic flatness, we can then define the total energy of the distribution of matter plus gravitational field by the integral

$$M_{\rm N} = 2\int_{\Omega} d^3x\sqrt{-g}\left(T_0^0 + t_0^0\right)\,. \tag{5}$$

Now the pseudo-tensor is defined in terms of the Ricci scalar R by Eq. (20) of Ref. [2],

$$t_i^j = \frac{1}{2\kappa^2}\left(R\delta_i^j - \frac{\partial R}{\partial g^{kl}_{\ ,j}}g^{kl}_{\ ,i}\right)\,, \tag{6}$$

where $\kappa^2 \equiv 8\pi G_{\rm N}$ is the gravitational coupling and $G_{\rm N}$ is the Newton constant, and hence for a stationary gravitational field we have

$$t_0^0 = \frac{1}{2\kappa^2} R = -\frac{1}{2} T \,, \tag{7}$$

substitution of which into Eq. (4) yields

$$\rho_{\rm N} = 2T_0^0 - T = \frac{2R_0^0}{\kappa^2} \,, \tag{8}$$

where R_{ij} is the Ricci tensor. Thus, the total energy $M_{\rm N}$ is given by an integral over the matter distribution alone[3],

$$M_{\rm N} = \int_{\Omega} d^3x \sqrt{-g} \left(T_0^0 - T_\alpha^\alpha \right) \,, \tag{9}$$

where $\alpha = 1, 2, 3$.

The fact that we obtain a definite value for the energy can also be understood quantum mechanically, from the Heisenberg[5] indeterminacy principle relating time and energy,

$$\Delta t \Delta E \sim \hbar \,. \tag{10}$$

A stationary gravitational field cannot be localized in time at all, so that $\Delta t = \infty$ and hence $\Delta E = 0$, allowing a precise evaluation of $M_{\rm N}$.

Gravity is a spin-2 field obeying the principle of equivalence, according to which the effect of an arbitrary gravitational field can be counteracted locally by transformation to an accelerating frame of reference, whilst remaining invariant under transformation to a different frame moving at uniform relative velocity. From the position-momentum indeterminacy relation

$$\Delta x \Delta p \sim \hbar \,, \tag{11}$$

it therefore follows that the gravitational field cannot be localized at all in space either. For by choosing $\Delta p = 0$, as implied by stationarity, we have $\Delta x = \infty$, which explains heuristically why we are free to identify the contribution to $M_{\rm N}$ deriving from the gravitational field either with the matter distribution T_{ij} or with empty space.

Classically, of course, this spatial non-localizability of the gravitational field energy reflects the non-tensorial character of the t_{ij}. In this regard, gravitational energy differs from the energy of the spin-1 electromagnetic field, which is defined uniquely by the distribution of the electric and magnetic fields, according to well-known formulae. It is of some interest to study this phenomenon from the viewpoint of gravito-electromagnetism, which is the subject of this paper.

2. Gravito-electromagnetic fields

In order to understand why vector fields assume importance in gravity, which is an intrinsically tensorial theory, let us first consider the force on a test particle of rest mass m and electric charge e, in gravitational and electromagnetic fields defined by four-tensor and four-vector potentials g_{ij} and A_i, respectively, and the corresponding line element

$$ds^2 = g_{ij} dx^i dx^j \, . \tag{12}$$

The force on the particle is obtained by applying the principle of least action to the function

$$S = -\int m ds + \int e A_i dx^i = \int \left(-m \frac{ds}{d\lambda} + e A_i \frac{dx^i}{d\lambda} \right) d\lambda = \int dx^0 L \left(x^0, x^\alpha, \frac{dx^\alpha}{dx^0} \right) , \tag{13}$$

where λ is an affine parameter.

We first write the line element in the form

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j = h \left(dx^0 - \gamma_\alpha dx^\alpha \right)^2 + \left(g_{\alpha\beta} - h \gamma_\alpha \gamma_\beta \right) dx^\alpha dx^\beta \\ &= dx_0^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta = d\tau^2 - dl^2 , \end{aligned} \tag{14}$$

where $h = g_{00}$, $\gamma_\alpha = -g_{0\alpha}/g_{00}$ and the physical three-metric defined on the time-slice $dx_0 = 0$ is

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + h \gamma_\alpha \gamma_\beta \, . \tag{15}$$

Tensor indices of three-dimensionally covariant quantities referred to $\gamma_{\alpha\beta}$ are raised by the inverse three-metric $\gamma^{\alpha\beta} \equiv -g^{\alpha\beta}$, so that $\gamma^\alpha = -g^{0\alpha}$ and $g^{00} = h^{-1} - \gamma_\alpha \gamma^\alpha$.

Thus, the Lagrangian function defined from the action (13) is (see also Møller[6], p. 384)

$$L = -m \left[h \left(1 - \boldsymbol{\gamma} \cdot \boldsymbol{v} \right)^2 - v^2 \right]^{1/2} + e \left(A_0 + \boldsymbol{A} \cdot \boldsymbol{v} \right) , \tag{16}$$

where $v^\alpha \equiv dx^\alpha / dx^0$ is the coordinate velocity.

The path of the particle is given by $\delta S = 0$, which, since the integrand is a homogeneous function of first degree of the four variables $dx^i/d\lambda$, is equivalent to a variational problem with only three dependent variables x^α, and with global time x^0 as independent variable (see Courant and Hilbert [7]). Therefore, the motion of the particle is determined by a Hamilton principle, and we may now write

$$\delta \int dx^0 L \left(x^0, x^\alpha, v^\alpha \right) = 0 \, . \tag{17}$$

Differentiation of L with respect to the velocities v^α, keeping x^α fixed, yields

$$\frac{\partial L}{\partial v^\alpha} = m\Gamma(v)\left[v_\alpha + h\left(1 - \boldsymbol{\gamma}\cdot\boldsymbol{v}\right)\gamma_\alpha\right] + eA_\alpha\,, \tag{18}$$

where the curved-space gamma factors are defined, setting $\beta = dl/d\tau$, as

$$\Gamma(v) = \frac{dx^0}{ds} = \left[h\left(1 - \boldsymbol{\gamma}\cdot\boldsymbol{v}\right)^2 - v^2\right]^{-1/2}\,, \qquad \gamma(\beta) = \frac{d\tau}{ds} = \left(1 - \beta^2\right)^{-1/2}\,. \tag{19}$$

From expressions (16), (18) and (19), we obtain the Hamiltonian

$$\begin{aligned} H &= v^\alpha\frac{\partial L}{\partial v^\alpha} - L = mh\Gamma(v)\left(1 - \boldsymbol{\gamma}\cdot\boldsymbol{v}\right) - eA_0 = m\sqrt{h}\gamma(\beta) - eA_0 \\ &= \varepsilon\phi^{(\mathrm{g})} + e\phi\,, \end{aligned} \tag{20}$$

where $\varepsilon \equiv m\gamma(\beta)$ is the locally measured (rest-mass plus kinetic) energy, and we have introduced the gravito-electric and electric scalar potentials

$$\phi^{(\mathrm{g})} = \sqrt{h}\,, \qquad \phi = -A_0\,. \tag{21}$$

For the application of expression (20) to the theory of a fermionic spin-1/2 particle, see Ref. [8].

The energy ε is chronometrically invariant, in the terminology of Zel'manov (see Appendix A), that is it remains invariant under the action of the group of coordinate transformations

$$x^0 \to x'^0\left(x^i\right)\,, \qquad x^\alpha \to x'^\alpha\left(x^\beta\right)\,, \tag{22}$$

the time coordinate x^0 transforming arbitrarily, while the spatial coordinates x^α transform only amongst themselves. The three-metric $\gamma_{\alpha\beta}$ is the fundamental chronometrically covariant tensor, but the metric component $g_{00} \equiv h$ is not a chronometrically invariant scalar, and therefore the Hamiltonian (20) is not chronometrically invariant either. In a stationary gravitational field, however, the Noether conservation theorem implies that H is constant, as noted by Landau and Lifschitz[10] for the purely gravitational case $\phi = 0$, confirming the physical significance of expression (20) and the interpretation of $\sqrt{h}$ as the gravito-electric potential $\phi^{(\mathrm{g})}$ on the same footing as the electric scalar potential ϕ, just as the chronometrically invariant energy ε is analogous to the charge e.

This symmetry of expression (20) between gravity and electromagnetism, together with the metrical origin of $\phi^{(\mathrm{g})}$, suggests an explanation via the theory of Kaluza[11] and Klein[12], in which the electric scalar potential ϕ is also given a metrical interpretation. We shall return to this observation below.

Here, we note that the five-dimensional line element written in Kaluza–Klein (4+1)-dimensional form,

$$\begin{aligned} d\hat{s}^2 &= \hat{g}_{AB}dx^A dx^B = \hat{g}_{44}\left(dx^4 - \alpha_i dx^i\right)^2 + \left(\hat{g}_{ij} - \alpha_i\alpha_j\hat{g}_{44}\right)dx^i dx^j \\ &= dx_4^2 + g_{ij}dx^i dx^j\,, \end{aligned} \tag{23}$$

is the analogue of the (3+1)-dimensional form (14) of the four-dimensional line element. Thus, the four-vector potential $A_i \equiv \sqrt{\hat{g}_{44}/2\kappa^2}\alpha_i$ plays the same rôle in the (4+1)-dimensional metric (23) as $\sqrt{h}\gamma_\alpha$ plays in the (3+1)-dimensional metric (14). While the electric scalar potential is defined from expression (23) as $\phi = -A_0 = \hat{g}_{04}/\sqrt{2\kappa^2\hat{g}_{44}}$, however, its gravito-electric counterpart defined from expression (14) is $\phi^{(\mathrm{g})} = \sqrt{g_{00}}$, the different functional dependences on the metric components reflecting the fact that the four-Hamiltonian (20) is canonically conjugate to x^0, not x^4.

In this paper, we shall clarify the precise nature of this analogy. The starting point for this discussion is the expression for the force f_α on the test particle, obtained from the equation of motion

$$\frac{d}{dx^0}\left(\frac{\partial L}{\partial v^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0\,. \tag{24}$$

Differentiating Eqs. (16) and (18) appropriately, we have

$$\begin{aligned} \frac{\partial L}{\partial x^\alpha} &= -m\mathit{\Gamma}(v)\left\{\sqrt{h}\,(1-\boldsymbol{\gamma}\cdot\boldsymbol{v})^2\frac{\partial\sqrt{h}}{\partial x^\alpha} - h\,(1-\boldsymbol{\gamma}\cdot\boldsymbol{v})\,v^\beta\frac{\partial\gamma_\beta}{\partial x^\alpha} - \tfrac{1}{2}v^\beta v^\gamma\frac{\partial\gamma_{\beta\gamma}}{\partial x^\alpha}\right\} \\ &\quad + e\left(\frac{\partial A_0}{\partial x^\alpha} + v^\beta\frac{\partial A_\beta}{\partial x^\alpha}\right) \end{aligned} \tag{25}$$

and

$$\begin{aligned} \frac{d}{dx^0}\left(\frac{\partial L}{\partial v^\alpha}\right) &= \frac{d}{dx^0}\left[m\mathit{\Gamma}(v)v_\alpha\right] + \sqrt{h}\gamma_\alpha\frac{d}{dx^0}\left[m\mathit{\Gamma}(v)\sqrt{h}\,(1-\boldsymbol{\gamma}\cdot\boldsymbol{v})\right] \\ &\quad + m\mathit{\Gamma}(v)\sqrt{h}\,(1-\boldsymbol{\gamma}\cdot\boldsymbol{v})\,\frac{d}{dx^0}\left(\sqrt{h}\gamma_\alpha\right) + e\frac{dA_\alpha}{dx^0}\,. \end{aligned} \tag{26}$$

Upon substitution of expressions (25) and (26) into Eq. (24), recalling that

$$\frac{d}{dx^0} = \frac{\partial}{\partial x^0} + v^\beta\frac{\partial}{\partial x^\beta} \tag{27}$$

and that the three-dimensional covariant derivative with respect to x^0 of a vector X_α is therefore

$$\frac{DX_\alpha}{dx^0} = \frac{dX_\alpha}{dx^0} - \frac{1}{2}\left[\frac{\partial\gamma_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial\gamma_{\alpha\beta}}{\partial x^\gamma} - \frac{\partial\gamma_{\alpha\gamma}}{\partial x^\beta}\right]X^\beta v^\gamma\,, \tag{28}$$

after some rearrangement we obtain for the coordinate force the expression

$$\begin{aligned} f_\alpha &\equiv \frac{D\left[m\Gamma(v)v_\alpha\right]}{dx^0} + \sqrt{h}\gamma_\alpha \frac{d\varepsilon}{dx^0} \\ &= \varepsilon\left[-\frac{\partial\sqrt{h}}{\partial x^\alpha} - \frac{\partial(\sqrt{h}\gamma_\alpha)}{\partial x^0} + v^\beta\left(\frac{\partial(\sqrt{h}\gamma_\beta)}{\partial x^\alpha} - \frac{\partial(\sqrt{h}\gamma_\alpha)}{\partial x^\beta}\right)\right] \\ &\quad + e\left[-\frac{\partial\phi}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^0} + v^\beta\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right)\right]. \end{aligned} \tag{29}$$

The term $\sqrt{h}\gamma_\alpha d\varepsilon/dx^0$ is placed on the left-hand side of Eq. (29), as part of the force, rather than on the right-hand side, in order to secure the correct special relativistic limit.

Expression (29) suggests that the quantities $\sqrt{h}$ and $\sqrt{h}\gamma_\alpha$ are on a par with the potentials of electricity. Generalizing Eqs. (21), we make the identifications

$$\sqrt{h} = \phi^{(\mathrm{g})} = -A_0^{(\mathrm{g})}\,, \qquad \sqrt{h}\gamma_\alpha = A_\alpha^{(\mathrm{g})}\,, \tag{30}$$

from which we see that the four-vector $A_i^{(\mathrm{g})}$ implicit in the definitions (30) is simply (minus) the unit vector u_i normal to the time lines, that describes an arbitrary observer in the physical three-space (assuming the space-time to admit a time foliation at every point). For in time-(three-space) notation, we have

$$u^i = -A^{(\mathrm{g})i} = \left(1/\sqrt{h}, O_\alpha\right)\,, \qquad u_i = -A_i^{(\mathrm{g})} = \left(\sqrt{h}, -\sqrt{h}\gamma_\alpha\right)\,, \tag{31}$$

such that $u_i u^i = 1$, and hence also

$$A_i^{(\mathrm{g})} A^{(\mathrm{g})i} = 1\,. \tag{32}$$

In this way we are led to introduce the three-dimensional vectors $\boldsymbol{E}^{(\mathrm{g})}$ and $\boldsymbol{B}^{(\mathrm{g})}$ as gravitational analogues of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$, these four quantities being

$$\boldsymbol{E}^{(\mathrm{g})} = -\nabla\phi^{(\mathrm{g})} - \partial\boldsymbol{A}^{(\mathrm{g})}/\partial x^0\,, \qquad \boldsymbol{B}^{(\mathrm{g})} = \mathrm{curl}\boldsymbol{A}^{(\mathrm{g})} \tag{33}$$

and

$$\boldsymbol{E} = -\nabla\phi - \partial\boldsymbol{A}/\partial x^0\,, \qquad \boldsymbol{B} = \mathrm{curl}\boldsymbol{A}\,, \tag{34}$$

respectively, whereupon Eq. (29) takes the three-vector form

$$\boldsymbol{f} = \varepsilon\left(\boldsymbol{E}^{(\mathrm{g})} + \boldsymbol{v}\times\boldsymbol{B}^{(\mathrm{g})}\right) + e\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right)\,. \tag{35}$$

Eq. (35), which is equivalent to Eq. (3.5) of Hönl and Dehnen[13], is valid for an arbitrary gravitational field. It not only illustrates how the Lorentz law for the electromagnetic force on a charged test particle remains unchanged in the presence of a gravitational field, but also shows the way in which the influence of the gravitational field itself on the particle is described by a law of force of the familiar Lorentz form. In this connexion the energy ε of the particle takes the place of the electric charge e, as it does also in the expression (20) for the Hamiltonian, while $\boldsymbol{E}$ and $\boldsymbol{B}$ are replaced by $\boldsymbol{E}^{(\mathrm{g})}$ and $\boldsymbol{B}^{(\mathrm{g})}$, respectively.

Although e is constant, note that the energy ε is not constant in general — only H is constant, in a stationary gravitational field, as remarked above.

Note also that the abbreviations ∇, div and curl will always refer to the three-dimensional space $\gamma_{\alpha\beta}$, in which they are defined for arbitrary scalar ξ and three-vector $\boldsymbol{X}$ by

$$\nabla_\alpha \xi = \frac{\partial \xi}{\partial x^\alpha}\,, \qquad \nabla^\alpha \xi = \gamma^{\alpha\beta}\frac{\partial \xi}{\partial x^\beta}\,, \tag{36}$$

$$\mathrm{div}\boldsymbol{X} = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}X^\alpha\right) \tag{37}$$

and

$$(\mathrm{curl}\boldsymbol{X})_\alpha = \epsilon_{\alpha\beta\gamma}\gamma^{\beta\delta}\gamma^{\gamma\epsilon}\frac{\partial X_\epsilon}{\partial x^\delta}\,, \qquad (\mathrm{curl}\boldsymbol{X})^\alpha = \epsilon^{\alpha\beta\gamma}\frac{\partial X_\gamma}{\partial x^\beta}\,, \tag{38}$$

where $\epsilon_{\alpha\beta\gamma}$ is defined in terms of the Levi–Civita three-index symbol $\delta_{\alpha\beta\gamma}$, whose values are ±1 or 0, by $\epsilon_{\alpha\beta\gamma} = \sqrt{\gamma}\delta_{\alpha\beta\gamma}$, $\epsilon^{\alpha\beta\gamma} = \delta^{\alpha\beta\gamma}/\sqrt{\gamma}$.

3. The connexion between three-dimensional and four-dimensional field equations

The vector and tensor equations of the first Maxwell system of electromagnetism are

$$\mathrm{div}\boldsymbol{B} = 0\,, \qquad \mathrm{curl}\boldsymbol{E} + \frac{1}{\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\boldsymbol{B})}{\partial x^0} = 0 \tag{39}$$

and

$$F_{[ij;k]} = 0\,, \tag{40}$$

respectively, where $F_{ij} \equiv \partial A_j/\partial x^i - \partial A_i/\partial x^j$ is the electromagnetic field tensor and

$$E_\alpha = -F_{0\alpha}\,, \qquad B^\alpha = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma}\,. \tag{41}$$

Prior to writing down the second Maxwell system, we have to introduce the electric current four-vector J^i. Consider an element of electrical charge dq confined to a spatial volume which is $d\mathring{V}$ as reckoned in a local rest frame and dV in a global frame. The corresponding measures of electrical charge density are $\mathring{\rho} = dq/d\mathring{V}$ and $\rho = dq/dV$, and therefore the four-current may be expressed as

$$J^i = \mathring{\rho} dx^i/ds = \rho dx^i/\sqrt{h} dx^0 \,. \tag{42}$$

The volume elements $d\mathring{V}$ and dV are related as

$$\sqrt{\gamma} d\mathring{x}^1 d\mathring{x}^2 d\mathring{x}^3 ds = \sqrt{\gamma}\, d\mathring{V} ds = \sqrt{-g} dV dx^0 \,, \tag{43}$$

which, since $\sqrt{-g} = \sqrt{h}\sqrt{\gamma}$ and $dx^0/ds = \Gamma(v)$, becomes

$$d\mathring{V} = \sqrt{h}\Gamma(v) dV \,. \tag{44}$$

The tensorial second Maxwell system relates F^{ij} to the source J^i as

$$\frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g} F^{ik}\right)}{\partial x^k} = 4\pi J^i \,. \tag{45}$$

Comparison of Eqs. (42) and (45) shows that in order to retrieve the familiar three-dimensional equations, we have to define the vectors of displacement current and magnetic intensity as

$$D^\alpha = \sqrt{h} F^{0\alpha} \,, \qquad H_\alpha = \tfrac{1}{2}\epsilon_{\alpha\beta\gamma}\sqrt{h} F^{\beta\gamma} \,, \tag{46}$$

respectively, for then Eqs. (45) yield

$$\mathrm{div}\boldsymbol{D} = 4\pi\rho \,, \qquad \mathrm{curl}\boldsymbol{H} = \frac{1}{\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\,\boldsymbol{D})}{\partial x^0} + 4\pi\rho\boldsymbol{v} \,, \tag{47}$$

while the current conservation equation

$$\frac{1}{\sqrt{-g}} \frac{\partial\left(\sqrt{-g} J^k\right)}{\partial x^k} = 0 \,, \tag{48}$$

which is a consequence of the anti-symmetry of F^{ij}, reduces to

$$\mathrm{div}(\rho\boldsymbol{v}) + \frac{1}{\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\rho)}{\partial x^0} = 0 \,. \tag{49}$$

Re-expressing the electric field as

$$\begin{aligned} E_\alpha &= -F_{0\alpha} = -g_{0i} g_{\alpha j} F^{ij} = -g_{00} g_{\alpha\beta} F^{0\beta} - g_{\alpha 0} g_{\beta 0} F^{\beta 0} + g_{\alpha\beta} g_{0\gamma} F^{\beta\gamma} \\ &= g_{00}\gamma_{\alpha\beta} F^{0\beta} - \gamma_{\alpha\beta} g_{0\gamma} F^{\beta\gamma} \end{aligned} \tag{50}$$

and using the definitions (46), we have

$$E_\alpha = \sqrt{h}\left(D_\alpha + \epsilon_{\alpha\beta\gamma}\gamma^\beta H^\gamma\right) . \tag{51}$$

Similarly, the magnetic field can be expanded as

$$\begin{aligned} B^\alpha &= \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}F_{\beta\gamma} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}g_{\beta i}g_{\gamma j}F^{ij} \\ &= \tfrac{1}{2}\epsilon^{\alpha\beta\gamma}\left(g_{\beta\delta}g_{\gamma\epsilon}F^{\delta\epsilon} + g_{\beta 0}g_{\gamma\epsilon}F^{0\epsilon} + g_{\beta\delta}g_{\gamma 0}F^{\delta 0}\right) \\ &= \epsilon^{\alpha\beta\gamma}\left[\tfrac{1}{2}\gamma_{\beta\delta}\gamma_{\gamma\epsilon}F^{\delta\epsilon} + g_{\beta 0}\left(-\gamma_{\gamma\epsilon}F^{0\epsilon} + \gamma_\delta\gamma_{\gamma\epsilon}F^{\delta\epsilon}\right)\right] , \end{aligned} \tag{52}$$

or, again using the definitions of $\boldsymbol{D}$ and $\boldsymbol{H}$,

$$B^\alpha = \frac{1}{\sqrt{h}}H^\alpha + \sqrt{h}\,\epsilon^{\alpha\beta\gamma}\gamma_\beta\left(D_\gamma + \epsilon_{\gamma\delta\epsilon}\gamma^\delta H^\epsilon\right) . \tag{53}$$

Eqs. (51) and (53) may be written as the vector equations (see §90 of Ref. 10)

$$\boldsymbol{D} = \frac{1}{\sqrt{h}}\boldsymbol{E} - \boldsymbol{\gamma}\times\boldsymbol{H} , \qquad \boldsymbol{B} = \frac{1}{\sqrt{h}}\boldsymbol{H} + \boldsymbol{\gamma}\times\boldsymbol{E} . \tag{54}$$

4. The Einstein field equations

The density and motion of electricity at each point in space-time are described by the four-current J^i, which is the source for the second system of Maxwell equations (45). Matter-energy, on the other hand, is represented by the four-tensor T_{ij}, which acts as the source for the ten Einstein equations

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = \kappa^2 T_{ij} , \tag{55}$$

which we do not expect to be completely equivalent to a four-component Maxwellian system. Due to the similarity with electromagnetism manifest in the Hamiltonian (20) and in the law of force (35) on a massive charged test particle, however, it becomes of interest to study the form taken by these equations when written, as far as possible, in terms of the gravito-electromagnetic fields $\boldsymbol{E}^{(\mathrm{g})}$, $\boldsymbol{B}^{(\mathrm{g})}$ and the corresponding intensities $\boldsymbol{D}^{(\mathrm{g})}$, $\boldsymbol{H}^{(\mathrm{g})}$, defined analogously to Eqs. (54) for $\boldsymbol{D}$ and $\boldsymbol{H}$, namely

$$\boldsymbol{D}^{(\mathrm{g})} = \frac{1}{\sqrt{h}}\boldsymbol{E}^{(\mathrm{g})} - \gamma\times\boldsymbol{H}^{(\mathrm{g})} , \qquad \boldsymbol{B}^{(\mathrm{g})} = \frac{1}{\sqrt{h}}\boldsymbol{H}^{(\mathrm{g})} + \gamma\times\boldsymbol{E}^{(\mathrm{g})} . \tag{56}$$

The first thing to note is that the Einstein equations (55) do not contain a gravitational term corresponding to the displacement current $\partial\boldsymbol{D}/\partial x^0$ of electromagnetism. For $\boldsymbol{D}^{(\mathrm{g})}$ is constructed from the metric components g_{00}

and $g_{0\alpha}$, time derivatives of which do not occur in the $\binom{0}{0}$ or $\binom{0}{\alpha}$ components of Eq. (55) — see §95 of Ref. [10]. Such a term can only be introduced into the equations of gravitation upon imposition of a coordinate (that is, a gauge) condition, as discussed in Sec. 8. Therefore, in the following exposition we shall initially specialize to the case of time-independence.

It is straightforward to obtain these equations in coordinate form. From the equations and definitions of chronometrically invariant quantities given in Ref. [9], for example, we find, in the stationary space-time satisfying $\partial\left[g_{ij}(x^k)\right]/\partial x^0 = 0$, that the $h^{-1}({}_{00})$ and $\binom{\alpha}{0}$ components of Eqs. (55) are

$$\operatorname{div}\left(h^{-\frac{1}{2}}\boldsymbol{E}^{(\mathrm{g})}\right) = h^{-1}\left(E^{(\mathrm{g})2} + \tfrac{1}{2}H^{(\mathrm{g})2}\right) - 4\pi G_{\mathrm{N}}\left(\rho^{(\mathrm{m})} + \sigma\right) \tag{57}$$

and

$$\operatorname{curl}\boldsymbol{H}^{(\mathrm{g})} = h^{-\frac{1}{2}}\boldsymbol{E}^{(\mathrm{g})} \times \boldsymbol{H}^{(\mathrm{g})} - 16\pi G_{\mathrm{N}}\,\boldsymbol{j}^{(\mathrm{m})}\,, \tag{58}$$

respectively, where the chronometrically invariant density, stress tensor and stress scalar of matter energy, and the matter current density, are defined as

$$\rho^{(\mathrm{m})} = T_{00}/h\,, \qquad \sigma^{\alpha\beta} = T^{\alpha\beta}\,, \qquad \sigma = \gamma_{\alpha\beta}\sigma^{\alpha\beta}\,, \qquad j^{(\mathrm{m})\alpha} = T^{\alpha}_{0}\,. \tag{59}$$

Keeping in mind the definitions (56) of $\boldsymbol{D}^{(\mathrm{g})}$ and $\boldsymbol{H}^{(\mathrm{g})}$, we see that addition to both sides of Eq. (57) of the quantity

$$\operatorname{div}\left(-\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\right) = -h^{-\frac{1}{2}}\boldsymbol{B}^{(\mathrm{g})}\cdot\boldsymbol{H}^{(\mathrm{g})} + 2h^{-\frac{1}{2}}\boldsymbol{\gamma}\cdot\boldsymbol{E}^{(\mathrm{g})}\times\boldsymbol{H}^{(\mathrm{g})} - 16\pi G_{\mathrm{N}}\,\gamma\cdot\boldsymbol{j}^{(\mathrm{m})} \tag{60}$$

yields the equation

$$\operatorname{div}\boldsymbol{D}^{(\mathrm{g})} = h^{-\frac{1}{2}}\boldsymbol{D}^{(\mathrm{g})}\cdot\boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}h^{-1}\boldsymbol{H}^{(\mathrm{g})2} - 4\pi G_{\mathrm{N}}\left(\rho^{(\mathrm{m})} + \sigma + 4\gamma\cdot\boldsymbol{j}^{(\mathrm{m})}\right). \tag{61}$$

Note that Eq. (58) for $\operatorname{curl}\boldsymbol{H}^{(\mathrm{g})}$ contains the Poynting-vector source-current $h^{-\frac{1}{2}}\boldsymbol{E}^{(\mathrm{g})}\times\boldsymbol{H}^{(\mathrm{g})}$, non-linear in the fields, that we might have guessed at for gravity on the basis of analogy, whereas Eq. (61) for $\operatorname{div}\boldsymbol{D}^{(\mathrm{g})}$ does not contain precisely the non-linear field density $h^{-\frac{1}{2}}\left(\boldsymbol{D}^{(\mathrm{g})}\cdot\boldsymbol{E}^{(\mathrm{g})} + \boldsymbol{B}^{(\mathrm{g})}\cdot\boldsymbol{H}^{(\mathrm{g})}\right)$ that we might have predicted. Here we re-encounter the problem of non-localizability of the energy of the gravitational field.

The situation becomes completely clear if we introduce the modified field $\bar{\boldsymbol{D}}^{(\mathrm{g})}$, defined as

$$\bar{\boldsymbol{D}}^{(\mathrm{g})} \equiv \boldsymbol{D}^{(\mathrm{g})} - h^{-\frac{1}{2}}\boldsymbol{E}^{(\mathrm{g})} = -\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\,, \tag{62}$$

and if we also rescale the fields $\bar{\boldsymbol{D}}^{(\mathrm{g})}$ and $\boldsymbol{H}^{(\mathrm{g})}$ as

$$\bar{\boldsymbol{D}}^{(\mathrm{g})} \to \widetilde{\boldsymbol{D}}^{(\mathrm{g})} = \bar{\boldsymbol{D}}^{(\mathrm{g})}/4G_{\mathrm{N}}\,, \qquad \boldsymbol{H}^{(\mathrm{g})} \to \widetilde{\boldsymbol{H}}^{(\mathrm{g})} = \boldsymbol{H}^{(\mathrm{g})}/4G_{\mathrm{N}}\,. \tag{63}$$

Including electromagnetism in the source T_{ij}, we can then write Eqs. (61) and (58), after some substitution, as

$$\mathrm{div}\widetilde{\boldsymbol{D}}^{(\mathrm{g})} = \tfrac{1}{2}h^{-\frac{1}{2}}\Big[\Big(\widetilde{\boldsymbol{D}}^{(\mathrm{g})}\cdot\boldsymbol{E}^{(\mathrm{g})}+\boldsymbol{B}^{(\mathrm{g})}\cdot\widetilde{\boldsymbol{H}}^{(\mathrm{g})}\Big)-(\boldsymbol{D}\cdot\boldsymbol{E}+\boldsymbol{B}\cdot\boldsymbol{H})\Big] + \tfrac{1}{4}G_{\mathrm{N}}^{-1}P - 4\pi\rho_{(\mathrm{mat})} \tag{64}$$

and

$$\mathrm{curl}\widetilde{\boldsymbol{H}}^{(\mathrm{g})} = h^{-\frac{1}{2}}\Big[\boldsymbol{E}^{(\mathrm{g})}\times\widetilde{\boldsymbol{H}}^{(\mathrm{g})} - \boldsymbol{E}\times\boldsymbol{H}\Big] - 4\pi\boldsymbol{j}_{(\mathrm{mat})}\,, \tag{65}$$

where

$$\rho_{(\mathrm{mat})} = T^{0}_{(\mathrm{mat})0}\,, \qquad j^{\alpha}_{(\mathrm{mat})} = T^{\alpha}_{(\mathrm{mat})0}\,, \tag{66}$$

the suffix (mat) denoting all non-electromagnetic matter.

Eqs. (64) and (65) show precisely how far the analogy between gravitation and electromagnetism is valid, for a stationary gravitational field. Note that in addition to the material source term T_{ij}, Eq. (64) contains the geometrical quantity $P \equiv P_{\alpha\beta}\gamma^{\alpha\beta} \equiv -\frac{3}{2}h^{-1}H^{(g)2} + 16\pi G_{\mathrm{N}}\rho^{(\mathrm{m})}$, where the three-dimensional Ricci curvature tensor $P_{\alpha\beta}$ is defined from $\gamma_{\alpha\beta}$ in exactly the same way as R_{ij} is defined from g_{ij}, and which cannot be expressed in terms of $\boldsymbol{E}^{(\mathrm{g})}$ or $\widetilde{\boldsymbol{D}}^{(\mathrm{g})}$. Note also that $\widetilde{\boldsymbol{D}}^{(\mathrm{g})}$ vanishes when either $\boldsymbol{\gamma}$ or $\boldsymbol{H}^{(\mathrm{g})}$ vanishes.

Assuming the space-time to admit a foliation by time lines at each point, we recall the definition (31) of the unit time-like vector u_i, from which it is possible to define the tensor

$$h_{ij} = g_{ij} - u_i u_j = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma_{\alpha\beta} \end{pmatrix}, \qquad h^{ij} = -\begin{pmatrix} \gamma^2 & \gamma^\alpha \\ \gamma^\alpha & \gamma^{\alpha\beta} \end{pmatrix}. \tag{67}$$

From the four-dimensional viewpoint, h_{ij} is a tensor projecting into the three-space $\gamma_{\alpha\beta}$, with the properties

$$\mathrm{det}h_{ij} = 0\,, \qquad h_{ij}u^j = 0\,, \qquad h_{ij}h^{ij} = \gamma_{\alpha\beta}\gamma^{\alpha\beta} = 3\,. \tag{68}$$

Now we can see that the fields $\widetilde{\boldsymbol{D}}^{(\mathrm{g})}$ and $\widetilde{\boldsymbol{H}}^{(\mathrm{g})}$ are in fact combined into the projected field tensor defined as

$$f^{(\mathrm{g})}_{ij} = \tfrac{1}{4}G_{\mathrm{N}}^{-1}h_{ik}h_{jl}F^{(\mathrm{g})kl}\,, \tag{69}$$

in the same way that the fields $\boldsymbol{D}$ and $\boldsymbol{H}$ of electromagnetism are combined into F_{ij}. Therefore, Eqs. (64) and (65) can be written in the alternative form

$$\begin{aligned} f^{(\mathrm{g})ij}{}_{;j} = \Big[&-\Big(f^{(\mathrm{g})ik}F^{(\mathrm{g})}{}_{jk} - \tfrac{1}{4}\delta^i_j f^{(\mathrm{g})kl}F^{(\mathrm{g})}{}_{kl}\Big) + \Big(F^{ik}F_{jk} - \tfrac{1}{4}\delta^i_j F^{kl}F_{kl}\Big) \\ &+ \tfrac{1}{4}G_{\mathrm{N}}^{-1}P\delta^i_j - 4\pi T^i_{(\mathrm{mat})j}\Big]u^j\,. \end{aligned} \tag{70}$$

This result is best understood from the Kaluza–Klein perspective.

The conciseness and non-linearity of Eqs. (70) also recall the fact that the stationary vacuum Einstein equations admit a non-linear sigma-model structure, analysed by Gibbons and Hawking[14], and which has been used by Sanchez[15] to determine new solutions to these equations (generalized in Ref. [16] to non-stationary space-times). We shall describe this formalism in the next section, relating the potentials of the non-linear sigma model to the gravito-electromagnetic fields.

5. The stationary vacuum gravitational field as a non-linear sigma model

A space-time possessing one time-like Killing vector field $\xi_{(0)} \equiv \partial/\partial x^0$ can be represented by a metric in the canonical form[14]

$$ds^2 = h\left(dx^0 - \gamma_\alpha dx^\alpha\right)^2 - h^{-1}\widetilde{\gamma}_{\alpha\beta} dx^\alpha dx^\beta \equiv dx_0^2 - h^{-1}\widetilde{\gamma}_{\alpha\beta} dx^\alpha dx^\beta \,. \qquad (71)$$

For mathematical reasons, it is convenient to work in the three-space $\widetilde{\gamma}_{\alpha\beta}$, which is related to the physical three-space $\gamma_{\alpha\beta}$ by the conformal transformation

$$\widetilde{\gamma}_{\alpha\beta} = h\gamma_{\alpha\beta}\,, \qquad \widetilde{\gamma}^{\alpha\beta} = h^{-1}\gamma^{\alpha\beta}\,. \qquad (72)$$

If we introduce the scalar potentials Φ and Ψ, so defined that

$$\Phi = h\,, \qquad \widetilde{\nabla}\Psi = \Phi^2 \widetilde{\mathrm{curl}}\boldsymbol{\gamma}\,, \qquad (73)$$

where the tilda-ed operators are defined in the three-metric $\widetilde{\gamma}_{\alpha\beta}$, then it can be shown that the vacuum Einstein equations reduce to[15, 16]

$$\widetilde{\nabla}^2\left(\Phi + i\Psi\right) - \Phi^{-1}\left[\widetilde{\nabla}\left(\Phi + i\Psi\right)\right]^2 = 0 \qquad (74)$$

and

$$\widetilde{P}_{\alpha\beta} - \Phi^{-2}\left[\left(\widetilde{\nabla}_\alpha\Phi\right)\left(\widetilde{\nabla}_\beta\Phi\right) - \left(\widetilde{\nabla}_\alpha\Psi\right)\left(\widetilde{\nabla}_\beta\Psi\right)\right] = 0\,. \qquad (75)$$

Eq. (74) contains the equations $R_{00} = R_{0\alpha} = 0$. It is the Ernst[17] equation, and describes an O(2,1) non-linear sigma model for the complex potential $\mathscr{V} \equiv \Phi + i\Psi$. Note that no significant generalization of Eq. (74) is known for an arbitrary space-time (in particular a stationary electro-vac space-time), the sigma-model form requiring both time independence and source freedom.

In order to understand the physical meaning of this result, it is interesting to examine the potential structure of the vacuum theory from the gravito-electromagnetic standpoint. For a stationary gravitational field, the $\binom{\alpha}{0}$ component of the Einstein equations, given by Eq. (58), reads

$$\mathrm{curl}\boldsymbol{H}^{(\mathrm{g})} = -\boldsymbol{\nabla}\left(\ln\sqrt{h}\right) \times \boldsymbol{H}^{(\mathrm{g})} - 16\pi G_{\mathrm{N}}\boldsymbol{j}^{(\mathrm{m})}\,. \qquad (76)$$

As we have emphasized earlier, the density and the flow of gravitational energy are not localizable in space, and therefore it is possible to absorb the gravitational Poynting vector $\boldsymbol{E}^{(\mathrm{g})} \times \boldsymbol{H}^{(\mathrm{g})}/\sqrt{h}$ into the left-hand side of Eq. (76), which we rewrite as

$$\mathrm{curl}\left(\sqrt{h}\boldsymbol{H}^{(\mathrm{g})}\right) = -16\pi G_{\mathrm{N}}\sqrt{h}\boldsymbol{j}^{(\mathrm{m})} . \tag{77}$$

Thus, in vacuo the stationary gravito-magnetic intensity satisfies the equation

$$\mathrm{curl}\left(\sqrt{h}\boldsymbol{H}^{(\mathrm{g})}\right) = 0 , \tag{78}$$

which can be re-expressed in terms of the scalar potential $\varPsi$ defined by

$$\sqrt{h}\boldsymbol{H}^{(\mathrm{g})} = \nabla\varPsi . \tag{79}$$

As usually formulated, the Einstein gravitational theory contains no gravito-magnetic monopoles (see also the next section), consistent with the definition (33) of $\boldsymbol{B}^{(\mathrm{g})}$ in terms of the gravito-vector potential $\boldsymbol{A}^{(\mathrm{g})} \equiv \sqrt{h}\,\boldsymbol{\gamma}$. From the second of Eqs. (56), we obtain the relationship between the scalar and vector gravito-potentials, *viz.*

$$\mathrm{curl}\boldsymbol{\gamma} = h^{-3/2}\nabla\varPsi . \tag{80}$$

Using the scaling laws (72), we can rewrite Eq. (80) in the conformal space $\tilde{\gamma}_{\alpha\beta}$ as

$$\widetilde{\mathrm{curl}}\boldsymbol{\gamma} = h^{-2}\tilde{\nabla}\varPsi , \tag{81}$$

in agreement with Eqs. (73). Thus, $\varPsi$ is essentially the gravito-magnetostatic potential.

The auxiliary vector potential $\boldsymbol{\gamma}$ can now be eliminated to yield equations for the two scalar potentials $\varPhi$ and $\varPsi$, via the equations for $\mathrm{div}\boldsymbol{D}^{(\mathrm{g})}$ and $\mathrm{div}\boldsymbol{B}^{(\mathrm{g})}$. In a stationary gravitational field, for the first of Eqs. (56), after substitution for $\boldsymbol{E}^{(\mathrm{g})}$ from Eqs. (30) and (33) and for $\boldsymbol{H}^{(\mathrm{g})}$ from Eq. (79), we have

$$\boldsymbol{D}^{(\mathrm{g})} = -\frac{1}{\sqrt{h}}\left(\nabla\sqrt{h} + \boldsymbol{\gamma} \times \nabla\varPsi\right) , \tag{82}$$

while Eq. (61) in the stationary vacuum case yields

$$\mathrm{div}\boldsymbol{D}^{(\mathrm{g})} = -\frac{1}{\sqrt{h}}\boldsymbol{D}^{(\mathrm{g})} \cdot \nabla\sqrt{h} - \frac{1}{2h^2}\left(\nabla\varPsi\right)^2 . \tag{83}$$

As with Eq. (76) for $\mathrm{curl}\boldsymbol{H}^{(\mathrm{g})}$, the first term on the right-hand side can be absorbed into the left-hand side to give

$$\mathrm{div}\left(\sqrt{h}\boldsymbol{D}^{(\mathrm{g})}\right) = -\frac{1}{2h^{3/2}}\left(\nabla\varPsi\right)^2 . \tag{84}$$

Substituting for $\sqrt{h}\boldsymbol{D}^{(\mathrm{g})}$ from Eq. (82), we have

$$\nabla^2\sqrt{h} + \mathrm{div}\left(\boldsymbol{\gamma}\times\nabla\Psi\right) = \frac{1}{2h^{3/2}}\left(\nabla\Psi\right)^2 . \tag{85}$$

The second term on the left-hand side of Eq. (85) can be expanded to yield

$$\mathrm{div}\left(\boldsymbol{\gamma}\times\nabla\Psi\right) = -\boldsymbol{\gamma}\cdot\mathrm{curl}\left(\nabla\Psi\right) + \nabla\Psi\cdot\mathrm{curl}\boldsymbol{\gamma} = h^{-3/2}\left(\nabla\Psi\right)^2 , \tag{86}$$

as the result of which Eq. (85) reads

$$\nabla^2\sqrt{h} + \frac{1}{2h^{3/2}}\left(\nabla\Psi\right)^2 = 0 , \tag{87}$$

which is easily shown to be equivalent to the real part of Eq. (74).

Note, when $\Psi = 0$, that is for an irrotational space-time with $\boldsymbol{\gamma} = 0$, that Eq. (87) reduces to the linear Laplace equation for $\sqrt{h}$, formulated in the space $\gamma_{\alpha\beta}$,

$$\nabla^2\sqrt{h} = 0 , \tag{88}$$

whereas the non-linear sigma model yields the equivalent non-linear equation formulated in the space $\tilde{\gamma}_{\alpha\beta}$,

$$\tilde{\nabla}^2\Phi - \Phi^{-1}\left(\tilde{\nabla}\Phi\right)^2 = 0 . \tag{89}$$

Next, consider the equation

$$\mathrm{div}\boldsymbol{B}^{(\mathrm{g})} = 0 , \tag{90}$$

which is an identity in the Einstein gravitational theory. By successive application of Eq. (80), we have

$$\nabla^2\Psi = \mathrm{div}\left(h^{3/2}\mathrm{curl}\boldsymbol{\gamma}\right) = \frac{3}{\sqrt{h}}\nabla\sqrt{h}\cdot\nabla\Psi , \tag{91}$$

which is readily shown to be equivalent to the imaginary part of Eq. (74).

Thus, Eq. (74) is a restatement of the stationary vacuum Eqs. (83) and (90) in terms of the potentials $\Phi \equiv h$ and Ψ.

6. The gravito-electromagnetic four-vector potential $A_i^{(\mathrm{g})}$

The part of the action functional (13) that deals with the interaction of a test particle with the electromagnetic field is $\int eA_i dx^i$, and therefore, given a gravitational four-vector potential $A_i^{(\mathrm{g})}$ and a 'gravitational charge' ε, we are led to ask what rôle the action integral $\int \epsilon A_i^{(\mathrm{g})} dx^i$ plays. In pursuance of this enquiry, we try rewriting the action functional (13) as

$$S = -\int \left(mds + \varepsilon A_i^{(\mathrm{g})} dx^i \right) + \int \varepsilon A_i^{(\mathrm{g})} dx^i + \int eA_i dx^i \,. \tag{92}$$

Obviously, the homogeneity condition is still satisfied, so that expression (92) implies a rearranged Lagrangian, which, substituting expressions (19) and (30) into expression (16), can be written as

$$L = L_\mathrm{p} + L_\mathrm{g} + L_\mathrm{e} \,, \tag{93}$$

where

$$L_\mathrm{p} = m\Gamma(v)v^2 \,, \qquad L_\mathrm{g} = \varepsilon \left(\boldsymbol{A}_0^{(\mathrm{g})} + \boldsymbol{A}^{(\mathrm{g})} \cdot \boldsymbol{v} \right) , \qquad L_\mathrm{e} = e \left(A_0 + \boldsymbol{A} \cdot \boldsymbol{v} \right) \,. \tag{94}$$

Notice that the expressions for L_g and L_e are formally identical, with ε, $A_i^{(\mathrm{g})}$ in the first replacing e, A_i, respectively, in the second. The terms L_p and L_g in the Lagrangian provide separate descriptions of, respectively, the response of the particle to the gravitational and electromagnetic fields, and its interaction with the gravitational field.

The contribution L_g shows that the potential $A_i^{(\mathrm{g})}$ has a meaning in its own right. Our understanding of non-local aspects of quantum mechanics, in particular the Aharonov–Bohm effect, implies a physical significance for the electromagnetic four-potential A_i.

The classical gravitational analogue of this effect was discussed by Ford and Vilenkin[18]. Starting from the observation that the rôles of gravitational potential and field strength are played by the metric g_{ij} and the Riemann–Christoffel curvature tensor R_{ijkl}, respectively, it was then shown that a test particle can experience non-local effects due to the curvature at the classical level, even if it moves solely in a region where $R_{ijkl} = 0$, due to topological influences.

As emphasized in Ref. [18], the Aharonov–Bohm effect proper is a quantum-mechanical effect, dependent on $\hbar$. Frolov *et al.* [19] analysed the behaviour of a spin-zero field ζ of rest-mass m, obeying the Klein–Gordon equation

$$\Box\zeta + m^2\zeta = 0 \,, \tag{95}$$

in the weak gravitational field of a slowly rotating massive body, showing that the energy spectrum and wave function of the particle depend upon the angular momentum of the rotating body, even if the quantum system traverses only a flat region of space-time. This effect is a gravitational analogue of the Aharonov–Bohm effect, in which the gravito-magnetic flux depends not on m, but on the total energy of the particle ε, so that $\Phi^{(\mathrm{g})} = 2\pi/\varepsilon$, and is therefore not universal, whereas the quantum of magnetic flux is $\Phi^{(\mathrm{m})} = 2\pi/e$. This gravitational application thus confirms that ϵ does indeed play the rôle of 'gravitational charge' analogous to electrical charge, precisely as expected from Eqs. (94).

Regarding $\boldsymbol{\gamma}$ and $\boldsymbol{v}$ as small quantities to be retained only up to second order, we see that there is only one term in L proportional to $\varepsilon \boldsymbol{A}^{(\mathrm{g})} \cdot \boldsymbol{v}$. The corresponding action can be rewritten as the line integral

$$S_0 = \oint \varepsilon \boldsymbol{A}^{(\mathrm{g})} \cdot d\boldsymbol{x} \,. \tag{96}$$

Expression (96) is the basis of the speculation by Zee[20, 21] that gravitomagnetic monopoles may exist, analogously to the magnetic monopoles of electromagnetism. For in the post-Newtonian limit that $(1 - h)$ is also a small quantity, we have

$$S_0 \approx m \oint \boldsymbol{A}^{(\mathrm{g})} \cdot d\boldsymbol{x} = m \int \boldsymbol{B}^{(\mathrm{g})} \cdot d\boldsymbol{\Sigma} = m \int \mathrm{div} \boldsymbol{B}^{(\mathrm{g})} dV \,, \tag{97}$$

and if we further assume, following Refs. [20, 21], that the second of Eqs. (33) is modified in such a way that

$$\mathrm{div} \boldsymbol{B}^{(\mathrm{g})} = \mu \delta^3(\boldsymbol{x}) \,, \tag{98}$$

where μ is the strength of the gravito-magnetic monopole, then it follows that mass is quantized in units of $2\pi\hbar/\mu$ to ensure continuity of the quantum-mechanical wave function,

$$\Psi_0 \approx e^{iS_0/\hbar} \approx e^{2n\pi i} \,, \tag{99}$$

This is the argument originally used by Dirac[22] in the case of magnetic monopoles.

It seems, however, that this argument breaks down in the presence of a stronger gravitational field, because the constant charge e of electromagnetism is replaced by the non-universal energy ε, rather then by the constant m. This presumably reflects the non-covariant nature of Eq. (98) and forces us to set $\mu = 0$.

7. The stationary non-vacuum gravitational field

In the presence of matter, the non-linear sigma-model structure of the gravitational field described in Sec. 5 no longer holds. Thus, if there is a matter current $\boldsymbol{j}^{(\mathrm{m})}$, then the potential Ψ cannot be defined, but the quantity $\sqrt{h}\boldsymbol{H}^{(\mathrm{g})}$ retains a meaning — in place of the electromagnetic Ampère law (47), which in a stationary gravitational field reads

$$\mathrm{curl}\boldsymbol{H} = 4\pi\boldsymbol{j}\,, \tag{100}$$

where $\boldsymbol{j} = \rho\boldsymbol{v}$, we have Eq. (77), in which the effect of the gravitational-energy current is associated entirely with the matter source $\boldsymbol{j}^{(\mathrm{m})}$.

We have seen in Sec. 1 that the total energy of a matter distribution that is non-zero only within a finite bounded region Ω can be expressed as the integral (9) over the matter source alone, and it is therefore of interest to obtain the differential gravito-electromagnetic form of this equation.

In Sec. 4, we found that the quantity $\mathrm{div}\boldsymbol{D}^{(\mathrm{g})}$ is given in the time-independent case by the sum of Eqs. (57) and (60). To generate the matter source density $(T^0_0 - T^\alpha_\alpha)$ identified by Nordström [3], however, we have instead to form the sum [Eq. (57) $+\frac{1}{2}$ Eq. (60)], which yields

$$\begin{aligned}\mathrm{div}\left[h^{-\frac{1}{2}}\boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\right] \;=\; & h^{-1}\boldsymbol{E}^{(\mathrm{g})2} + \tfrac{1}{2}h^{-\frac{1}{2}}\boldsymbol{\gamma}\cdot\boldsymbol{E}^{(\mathrm{g})}\times\boldsymbol{H}^{(\mathrm{g})} \\ & -4\pi G_{\mathrm{N}}\left(T^0_0 - T^\alpha_\alpha\right).\end{aligned} \tag{101}$$

Multiplying through by $h^{1/2}$, we find, as expected, that the non-linear gravitational-energy terms on the right-hand side of Eq. (101) can be completely absorbed into the divergence, to yield

$$\mathrm{div}\left(\boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}\sqrt{h}\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\right) = -4\pi G_{\mathrm{N}}\sqrt{h}\left(T^0_0 - T^\alpha_\alpha\right). \tag{102}$$

Spatial integration of $(-\sqrt{\gamma}/4\pi G_{\mathrm{N}})$ times Eq. (102) yields

$$-\frac{1}{4\pi G_{\mathrm{N}}}\int d^3x\frac{\partial\left[\sqrt{\gamma}\left(\boldsymbol{E}^{(\mathrm{g})} - \frac{1}{2}\sqrt{h}\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\right)\right]^\alpha}{\partial x^\alpha} = \int\sqrt{-g}\left(T^0_0 - T^\alpha_\alpha\right) = M_{\mathrm{N}}\,, \tag{103}$$

the left-hand side of which can be converted into a surface integral by the application of Gauss's theorem, so that the energy is given by

$$M_{\mathrm{N}} = -\frac{1}{4\pi G_{\mathrm{N}}}\int d\boldsymbol{\Sigma}\cdot\left(\boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}\sqrt{h}\boldsymbol{\gamma}\times\boldsymbol{H}^{(\mathrm{g})}\right). \tag{104}$$

Note, firstly, that the right-hand side of Eq. (104) can be evaluated over *any* two-surface that completely encloses the matter distribution. If we allow this surface to go to infinity, where the terms in $\boldsymbol{\gamma}$ are ignorable, we have

$$M_{\mathrm{N}} = \frac{1}{G_{\mathrm{N}}}\lim_{r\to\infty}\left(\frac{r^2\partial\sqrt{h}}{\partial r}\right), \tag{105}$$

in agreement with the standard asymptotic mass-energy formula

$$h = 1 - 2G_{\mathrm{N}}M_{\mathrm{N}}/r + \mathcal{O}\left(1/r^2\right) + \ldots\,. \tag{106}$$

Secondly, the integrand of expression (104) can be expanded, substituting for $\boldsymbol{H}^{(\mathrm{g})}$ from Eqs. (33) and (56), as

$$\boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}\sqrt{h}\boldsymbol{\gamma} \times \boldsymbol{H}^{(\mathrm{g})}) = \boldsymbol{E}^{(\mathrm{g})} - \tfrac{1}{2}h^{\frac{3}{2}}\boldsymbol{\gamma} \times \mathrm{curl}\boldsymbol{\gamma}\,. \tag{107}$$

Expression (107) is linear in $\boldsymbol{E}^{(\mathrm{g})}$, but quadratic in $\boldsymbol{\gamma}$, which implies that a linearly perturbative analysis of any finite system will yield a change in energy gravito-electrically, but no change gravito-magnetically. It is therefore interesting that this is exactly what was found by Sorge[23, 24] in his microscopic analysis of the Casimir effect from the gravito-electromagnetic point of view.

8. Coordinate conditions and the gravitational displacement current

We have already mentioned above in Sec. 4 the fact that no term corresponding to Maxwell's displacement current $\partial \boldsymbol{D}/\partial x^0$ enters the Einstein equations as initially written. Such a term can be brought causally into the gravitational equations, however, if we impose coordinate conditions of a suitable kind.

Consider, in particular, the de Donder[25]–Lanczos[26] (see also Ref. [27]) coordinate conditions, called "harmonic" by Fock[28]. This system of coordinates is defined by the equivalent conditions on the metric derivatives and the Christoffel connexion,

$$\left(\sqrt{-g}g^{ij}\right)_{,j} \equiv -\Gamma^{i}_{jk}g^{jk} = 0\,. \tag{108}$$

This identity is given by Eq. [14] of Note 2 on p. 220 and its vanishing by Eq. (117), on p. 110 of Ref. [25], being a generalization of the linear approximation introduced by Einstein[29] to formulate the propagation of gravitational waves.

Let us first examine the $\left(^{0}\right)$ component of Eq. (108). Upon expansion, and after substitution from Eqs. (30), this equation takes the form

$$\mathrm{div}\boldsymbol{A}^{(\mathrm{g})} - \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^0}\left[\sqrt{\gamma}\left(\frac{1}{\phi^{(\mathrm{g})}} - \gamma^2\phi^{(\mathrm{g})}\right)\right] = 0\,, \tag{109}$$

where $(\boldsymbol{A}^{(\mathrm{g})})^{\alpha} = \gamma^{\alpha\beta}A^{(\mathrm{g})}_{\alpha} = -\sqrt{h}g^{0\alpha}$, which generalizes the familiar Lorenz gauge condition of electromagnetic theory in flat space-time,

$$\mathrm{div}\boldsymbol{A} + \frac{\partial\phi}{\partial x^0} = 0\,. \tag{110}$$

In the stationary limit, Eq. (109) reduces without approximation to the gravitational Coulomb gauge

$$\mathrm{div}\boldsymbol{A}^{(\mathrm{g})} = 0\,. \tag{111}$$

Next, consider the three spatial components $\left({}^{\alpha}\right)$ of Eq. (108), which, on multiplication by $\sqrt{h}$, can be expanded as

$$\frac{E^{(\mathrm{g})\alpha}}{\sqrt{h}} = \frac{1}{\sqrt{\gamma}}\frac{\partial\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial x^{\beta}} - \frac{\gamma_{\beta}}{\sqrt{\gamma}}\frac{\partial\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial x^{0}}\,. \tag{112}$$

While Eqs. (108) are not four-dimensionally covariant, note that Eq. (112) can be written in the chronometrically invariant form

$${}^{*}E^{(\mathrm{g})\alpha} = \frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial x^{\beta}}\,, \tag{113}$$

where ${}^{*}\boldsymbol{E}^{(\mathrm{g})}$ is the chronometrically invariant gravito-electric field defined by Eq. (A.10) of the Appendix, which also shows that these three conditions fix the reference system.

Operating on Eq. (113) with ${}^{*}\partial\sqrt{\gamma}/\sqrt{\gamma}\partial t$, then applying the first of the commutation relations (A.9), we obtain the equation

$$\frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}\,{}^{*}E^{(\mathrm{g})\alpha}\right)}{\partial t} = \frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial^{2}\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial t\partial x^{\beta}} = \frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial^{2}\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial x^{\beta}\partial t} - {}^{*}E^{(\mathrm{g})}_{\beta}\frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}\gamma^{\alpha\beta}\right)}{\partial t}\,, \tag{114}$$

that is

$$\frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}\,{}^{*}E^{(\mathrm{g})\alpha}\right)}{\partial t} = \frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left[\sqrt{\gamma}\left(\gamma^{\alpha\beta}K - 2K^{\alpha\beta}\right)\right]}{\partial x^{\beta}} - {}^{*}E^{(\mathrm{g})}_{\beta}\left(\gamma^{\alpha\beta}K - 2K^{\alpha\beta}\right)\,, \tag{115}$$

where $K^{\alpha\beta}$ and K are defined by Eqs. (A.16).

For the sake of simplicity, assume that the spatial metric changes isotropically with time, that is to say

$$K_{\alpha\beta} = \tfrac{1}{3}K\gamma_{\alpha\beta}\,, \tag{116}$$

in which case Eq. (115) reads

$$\frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}\,{}^{*}E^{(\mathrm{g})\alpha}\right)}{\partial t} = \frac{1}{\sqrt{\gamma}}\frac{{}^{*}\partial\left(\sqrt{\gamma}K^{\alpha\beta}\right)}{\partial x^{\beta}} - {}^{*}E^{(\mathrm{g})}_{\beta}K^{\alpha\beta}\,. \tag{117}$$

Now the chronometrically invariant components $h^{-1}\binom{}{0\,0}$ and $h^{-1/2}\binom{\alpha}{0}$ of the Einstein equations (55), expressed in the form

$$R_{ij} = \kappa^2 \left(T_{ij} - \tfrac{1}{2} T g_{ij}\right) , \tag{118}$$

are [9]

$$\frac{1}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma}\,{}^*E^{(\mathrm{g})\alpha}\right)}{\partial x^\alpha} = {}^*E^{(\mathrm{g})}_\alpha\,{}^*E^{(\mathrm{g})\alpha} + \tfrac{1}{4}\,{}^*H^{(\mathrm{g})}_{\alpha\beta}\,{}^*H^{(\mathrm{g})\alpha\beta} - K_{\alpha\beta}K^{\alpha\beta} - \frac{{}^*\partial K}{\partial t} - 4\pi G_{\mathrm{N}}\left(\rho^{(\mathrm{m})} + \sigma\right) \tag{119}$$

and

$$\frac{1}{\sqrt{\gamma}} \frac{{}^*\partial(\sqrt{\gamma}\,{}^*H^{(\mathrm{g})\alpha\beta})}{\partial x^\beta} = 2\,{}^*E^{(\mathrm{g})}_\beta\,{}^*H^{(\mathrm{g})\alpha\beta} + 2\left(\gamma^{\alpha\beta}K - K^{\alpha\beta}\right)_{\overset{*}{:}\beta} - 16\pi G_{\mathrm{N}}\,{}^*j^{(\mathrm{m})\alpha} , \tag{120}$$

respectively, where the chronometrically invariant covariant derivative is denoted by $\overset{*}{:}$ and ${}^*j^{(\mathrm{m})\alpha} = T^\alpha_0/\sqrt{h}$, which generalize Eqs. (57) and (58) to the time-dependent gravitational field.

Under the assumption (116), Eqs. (119) and (120) simplify to

$$\frac{1}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma}\,{}^*E^{(\mathrm{g})\alpha}\right)}{\partial x^\alpha} = {}^*E^{(\mathrm{g})}_\alpha\,{}^*E^{(\mathrm{g})\alpha} + \tfrac{1}{4}\,{}^*H^{(\mathrm{g})}_{\alpha\beta}\,{}^*H^{(\mathrm{g})\alpha\beta} - \tfrac{1}{3}K^2 - \frac{{}^*\partial K}{\partial t} - 4\pi G_{\mathrm{N}}\left(\rho^{(\mathrm{m})} + \sigma\right) \tag{121}$$

and

$$\frac{1}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma}\,{}^*H^{(\mathrm{g})\alpha\beta}\right)}{\partial x^\beta} = 2\,{}^*E^{(\mathrm{g})}_\beta\,{}^*H^{(\mathrm{g})\alpha\beta} + \frac{4}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma} K^{\alpha\beta}\right)}{\partial x^\beta} - 16\pi G_{\mathrm{N}}\,{}^*j^{(\mathrm{m})\alpha} , \tag{122}$$

respectively. Substitution from (four times) Eq. (117) into Eq. (122) then yields the equation

$$\frac{1}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma}\,{}^*H^{(\mathrm{g})\alpha\beta}\right)}{\partial x^\beta} - \frac{4}{\sqrt{\gamma}} \frac{{}^*\partial\left(\sqrt{\gamma}\,{}^*E^{(\mathrm{g})\alpha}\right)}{\partial t} = 2{}^*E^{(\mathrm{g})}_\beta\left({}^*H^{(\mathrm{g})\alpha\beta} + 2K^{\alpha\beta}\right) - 16\pi G_{\mathrm{N}}\,{}^*j^{(\mathrm{m})\alpha} , \tag{123}$$

in which a gravitational displacement-current term appears.

Suppose, further, the following two restrictions: firstly, that the gravitational field is weak, in the sense that quadratic terms in ${}^*\boldsymbol{E}^{(\mathrm{g})}$, ${}^*\boldsymbol{H}^{(\mathrm{g})}$ and K are ignorable, together with derivatives of K with respect to time (but not space); and secondly, that the stresses are small, so that σ may be ignored by comparison with $\rho^{(\mathrm{m})}$. Then the correspondingly simplified forms of Eqs. (121) and (123) are

$$\frac{1}{\sqrt{\gamma}}\frac{{}^*\partial\left(\sqrt{\gamma}\;{}^*E^{(\mathrm{g})\alpha}\right)}{\partial x^\alpha} = -4\pi G_{\mathrm{N}}\rho^{(\mathrm{m})} \tag{124}$$

and

$$\frac{1}{\sqrt{\gamma}}\frac{{}^*\partial\left(\sqrt{\gamma}\;{}^*H^{(g)\alpha\beta}\right)}{\partial x^\beta} - \frac{4}{\sqrt{\gamma}}\frac{{}^*\partial\left(\sqrt{\gamma}\;{}^*E^{(g)\alpha}\right)}{\partial t} = -16\pi G_{\mathrm{N}}\;{}^*j^{(\mathrm{m})\alpha}\,, \tag{125}$$

respectively.

Eqs. (124) and (125) constitute a dynamical system. To within the present approximations, they imply the continuity equation

$${}^*\mathrm{div}^*\boldsymbol{j}^{(\mathrm{m})} + \frac{1}{\sqrt{\gamma}}\frac{{}^*\partial\left(\sqrt{\gamma}\rho^{(\mathrm{m})}\right)}{\partial t} = 0\,, \tag{126}$$

that is in accord with the appropriate limit of the $\binom{}{0}$ component of Eq. (1) from which we began. Written in terms of chronometrically invariant quantities, this is Eq. (11) of Ref. [9], namely

$${}^*\mathrm{Div}\;{}^*\boldsymbol{j}^{(\mathrm{m})} + \frac{1}{\sqrt{\gamma}}\frac{{}^*\partial\left(\sqrt{\gamma}\rho^{(\mathrm{m})}\right)}{\partial t} - {}^*\boldsymbol{E}^{(\mathrm{g})}\cdot{}^*\boldsymbol{j}^{(\mathrm{m})} + K_{\alpha\beta}\sigma^{\alpha\beta} = 0\,, \tag{127}$$

where the chronometrically invariant physical divergence is defined as[9]

$${}^*\mathrm{Div}\boldsymbol{X} = {}^*\mathrm{div}\boldsymbol{X} - {}^*\boldsymbol{E}^{(\mathrm{g})}\cdot\boldsymbol{X} \tag{128}$$

for an arbitrary vector $\boldsymbol{X}$. Note, in general, by comparison with Eq. (126), that Eq. (127) contains additional terms which describe the interaction between the gravitational field, represented by $\boldsymbol{E}^{(\mathrm{g})}$ and $K_{\alpha\beta}$, and the matter sources ${}^*\boldsymbol{j}^{(\mathrm{m})}$ and $\sigma^{\alpha\beta}$, respectively.

9. The superstring vacuum

The foregoing analysis has been chiefly concerned with the classical Einstein theory, starting from Eq. (1), which describes the continuity of energy and momentum and for a stationary gravitational field can be reformulated via the energy-momentum pseudo-tensor t_i^j as the differential energy conservation law (3). After integration over the three-space, this results in the Nordström global energy (9), which can be re-expressed in terms of the gravito-electromagnetic fields $\boldsymbol{E}^{(\mathrm{g})}$ and $\boldsymbol{H}^{(\mathrm{g})}$ as the two-dimensional surface integral (104).

Eq. (9) is an integral over the matter source distribution alone, implying an *a priori* multifold degeneracy due to the fact that different energy-momentum tensors T_{ij} which give rise to the same integral of the densitized Nordström energy-density $\sqrt{-g}\rho_{\mathrm{N}}$ yield the same total energy M_{N}. This feature of the theory results in particular in a classical degeneracy of the vacuum, for consider a perfect-fluid source characterized by energy density ρ and pressure p, linked through the relativistic equation of state

$$p = (n-1)\rho\,, \tag{129}$$

where n is the adiabatic index. The energy-momentum tensor is

$$T_{ij} = \rho\left[n u_i u_j - (n-1)g_{ij}\right]\,, \tag{130}$$

from which we find that

$$\rho_{\mathrm{N}} = (3n-2)\,\rho\,. \tag{131}$$

As discussed previously[30, 31], the equation

$$\rho_{\mathrm{N}} = 0 \tag{132}$$

possesses the two solutions

$$\rho = 0\,, \qquad \rho \neq 0\,, \qquad n = 2/3\,, \tag{133}$$

the first of which includes most importantly Minkowski space, while the second defines matter with the stringy equation of state.

Thus, if the fundamental theory of the Universe is a string theory, for self-consistency the heterotic superstring theory of Gross *et al.* [32–34], the question arises as to the precise status of this string vacuum state, which was argued by Dabholkar *et al.* [35] to be the most basic classical solution to the low-energy field-theory limit, and which was shown in Ref. [31] to be the chronometrically invariant vacuum state, in which, in the stationary case, a three-dimensional de Sitter space is embedded in four-dimensional space-time.

Application of the Nordström expression (9) presupposes a stationary space-time, and therefore we consider this limit of the Friedmann line-element

$$ds^2 = dt^2 - a^2(t)d\boldsymbol{x}^2\,, \tag{134}$$

where t is comoving time and $a(t) \equiv a_0 e^{\alpha(t)}$ is the radius function of the three-space $d\boldsymbol{x}^2$, in the non-trivial case of a closed Universe. The Friedmann and Raychaudhuri equations then read

$$\dot{\alpha}^2 = \frac{1}{3}\kappa^2\rho_0 - \frac{1}{a_0^2} = 0\,, \qquad \ddot{\alpha} = \tfrac{1}{2}n\kappa^2\rho_0 - \frac{1}{a_0^2} = 0\,, \tag{135}$$

respectively, where $^\bullet \equiv d/dt$ and ρ_0 is the constant energy density, which can only be non-zero if $n = 2/3$, yielding the stringy solution (133) with

$$\rho_0 = 3/\kappa^2 a_0^2\,. \tag{136}$$

From Ref. [31] we find that this matter can be simulated by the string field ξ, for which the energy density and pressure are defined by

$$\rho_\xi = -3p_\xi = \tfrac{1}{2}\left[(\nabla\xi)^2\right]^{-1}\,. \tag{137}$$

The effective Lagrangian is therefore

$$L = -\frac{R}{2\kappa^2} - \tfrac{1}{6}\left[(\nabla\xi)^2\right]^{-1}\,, \tag{138}$$

yielding the energy-momentum tensor

$$T_{ij} = \tfrac{1}{3}\rho_\xi\left(2u_i u_j + g_{ij}\right) = \tfrac{1}{3}\left[(\nabla\xi)^2\right]^{-2}\left[\xi_{,i}\xi_{,j} + \tfrac{1}{2}(\nabla\xi)^2 g_{ij}\right] \tag{139}$$

and the non-linear equation of motion

$$\partial_j\left(\frac{\partial\mathcal{L}}{\partial\xi_{,j}}\right) = \left[(\nabla\xi)^2\right]^{-2}\left\{\Box\xi - 4\left[(\nabla\xi)^2\right]^{-1}\xi_{,i,j}\xi^{,i}\xi^{,j}\right\} = 0 \tag{140}$$

(see Eqs. (81)–(84) of Ref. [31]).

In the stationary string metric (135), the solution to Eq. (140) is

$$\xi = \xi_0 t\,, \qquad \nabla\xi = (\xi_0, 0)\,, \qquad \Box\xi = \xi_{,i,j} = 0\,, \tag{141}$$

where $\xi_0 = \kappa_0 a_0/\sqrt{6}$, substitution of which into expressions (138) and (139), setting $\rho_\xi = \rho_0$, results in the Lagrangian

$$L = \tfrac{2}{3}\rho_0\,. \tag{142}$$

The finite three-volume is $\int d^3x\sqrt{\gamma} = 2\pi^2 a_0^3$, and therefore the action is

$$\mathcal{S} = \frac{4\sqrt{3}\pi^2}{\kappa^3\sqrt{\rho_0}}t\,. \tag{143}$$

Classically, the two vacua (133) appear to be equivalent from the energetic standpoint, regarding both the spatial integral (9) and the mass function defined from the gravito-electric field through Eq. (106). This raises the question why strings are totally absent in the vacuum state that we actually perceive, especially since the superstring world sheet is supersymmetric if, and only if, one or other of the solutions (133) holds [30].

The situation is reminiscent of the more familiar paradox of why the cosmological constant Λ vanishes, suggesting an answer along similar lines. Hawking[36, 37] pointed out that the Euclidean action of the de Sitter instanton generated by a positive semi-definite Λ is

$$\mathcal{S}_{\mathrm{E}} = \int d^4x\sqrt{-g}\left(\frac{R+2\Lambda}{2\kappa^2}\right) = -\frac{3\pi}{G_{\mathrm{N}}\Lambda}\,. \tag{144}$$

In the path-integral approach to quantum gravity, the wave function is given by

$$\Psi \approx \Psi_0 \exp\left(i\mathcal{S}\right) \approx \Psi_0 \exp\left(-\mathcal{S}_{\mathrm{E}}\right), \tag{145}$$

where the Euclideanization is brought about by Wick rotation of the time coordinate,

$$t \to -i\tau\,. \tag{146}$$

This is so designed that matter kinetic terms are positive in both $\mathcal{S}$ and $\mathcal{S}_{\mathrm{E}}$, since

$$i\mathcal{S} = i\int dtd^3x\sqrt{-g}\left(\frac{\dot{\zeta}^2}{2}+\dots\right) \to -\int d\tau d^3x\sqrt{-g}\left(\frac{\zeta'^2}{2}+\dots\right) \equiv -\mathcal{S}_{\mathrm{E}}\,, \tag{147}$$

where ζ is a typical matter field, $^{\bullet} \equiv d/dt$ and $' \equiv d/d\tau$.

Assuming that $\Psi\Psi^*$ can be interpreted as a probability density, it then follows that the most probable configuration is obtained by maximizing $(-\mathcal{S}_{\mathrm{E}})$, and is therefore given by

$$\Lambda = 0\,, \tag{148}$$

as we have also discussed in Ref. [38].

Now let us apply this idea to the string vacuum solution (135), (136). Performing the Wick rotation (146) in expression (143), we obtain the Euclidean string action

$$\mathcal{S}_{\mathrm{E}} = \frac{-4\sqrt{3}\pi^2\tau}{\kappa^3\sqrt{\rho_0}}\,. \tag{149}$$

Since t, and hence also τ, are bounded by the age of the Universe, and therefore finite, the exponent $(-\mathcal{S}_{\mathrm{E}})$ in expression (145) is maximized by setting

$$\rho_0 = 0\,, \tag{150}$$

in analogy to expression (148), thus explaining why the true vacuum state of the Universe is devoid of strings, being Minkowski space.

The interpretation of $\Psi\Psi^*$ as a probability density is justified by the inclusion in the superstring effective Lagrangian of quadratic higher-derivative gravitational terms $\mathcal{R}^2$, obtained by reduction of the ten-dimensional quartic terms $\widehat{\mathcal{R}}^4$. The Wheeler–DeWitt equation[39, 40] for the wave function of the Universe Ψ then takes the form of a Schrödinger equation[41, 42], from which a conserved probability current can be defined in the Friedmann space-time (134) or mini-superspace approximation.

Appendix A

Chronometric invariance

We shall first prove Theorem I, due to Zel'manov[9], that given a world tensor $Y^{\alpha\beta\gamma\ldots}_{000\ldots}$ of rank m with all the superscripts different from zero and all the subscripts, numbering n, equal to zero, then the quantities ${}^*Z^{\alpha\beta\gamma\ldots}_{000\ldots} \equiv h^{-\frac{n}{2}} Y^{\alpha\beta\gamma\ldots}_{000\ldots}$ form a three-dimensional contravariant tensor of rank $(m-n)$ which is chronometrically invariant.

Applying the transformations (22) to $Y^{\alpha\beta\gamma\ldots}_{000\ldots}$, in the new coordinate system we have

$$Y'^{\alpha\beta\gamma\ldots}_{000\ldots} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}} \frac{\partial x'^{\beta}}{\partial x^{\sigma}} \frac{\partial x'^{\gamma}}{\partial x^{\tau}} \cdots \left(\frac{\partial x^0}{\partial x'^0}\right)^n Y^{\rho\sigma\tau\ldots}_{000\ldots}\,, \tag{A.1}$$

or, since

$$\frac{\partial x^0}{\partial x'^0} = \left(\frac{g'_{00}}{g_{00}}\right)^{1/2}, \tag{A.2}$$

$$\left(g'_{00}\right)^{-n/2} Y'^{\alpha\beta\gamma\ldots}_{000\ldots} = \frac{\partial x'^{\alpha}}{\partial x^{\rho}} \frac{\partial x'^{\beta}}{\partial x^{\sigma}} \frac{\partial x'^{\gamma}}{\partial x^{\tau}} \cdots \left(g_{00}\right)^{-n/2} Y^{\rho\sigma\tau\ldots}_{000\ldots}\,. \tag{A.3}$$

Thus, the quantities

$${}^*Z^{\alpha\beta\gamma\ldots}_{000\ldots} \equiv h^{-n/2} Y^{\alpha\beta\gamma\ldots}_{000\ldots} \tag{A.4}$$

transform according to the law for three-dimensional tensors, which proves the theorem, and are denoted with an asterisk to indicate chronometric invariance.

In particular, the contravariant components of the chronometrically invariant three-metric, are ${}^*\gamma^{\alpha\beta} = \gamma^{\alpha\beta} = -g^{\alpha\beta}$, the covariant components satisfying ${}^*\gamma^{\alpha\sigma}\,{}^*\gamma_{\sigma\beta} = \delta^\alpha_\beta$. Thus, ${}^*\gamma_{\alpha\beta}$ is the physical three-metric $\gamma_{\alpha\beta}$, defined from the line element in the form (14) first introduced by von Weyssenhoff[43, 44], who also identified the group of coordinate transformations (22) under which $\gamma_{\alpha\beta}$ remains invariant (see p. 66 of Ref. [44]).

Chronometrically invariant differential operators can be established by consideration of the effect of the transformations (22) upon normal partial differential operators. Thus $\partial/\partial x^0$ and $\partial/\partial x^\alpha$ transform as

$$\frac{\partial}{\partial x^0} = \frac{\partial x'^0}{\partial x^0}\frac{\partial}{\partial x'^0} = \left(\frac{g_{00}}{g'_{00}}\right)^{\frac{1}{2}}\frac{\partial}{\partial x'^0} \tag{A.5}$$

and

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial}{\partial x'^\beta} + \frac{\partial x'^0}{\partial x^\alpha}\frac{\partial}{\partial x'^0}, \tag{A.6}$$

respectively. In order to deal with expression (A.6), we form the transformation of $\gamma_\alpha \equiv -g_{0\alpha}/g_{00}$,

$$\gamma_\alpha = \frac{\partial x'^\beta}{\partial x^\alpha}\frac{\partial x^0}{\partial x'^0}\gamma'_\beta - \frac{\partial x'^0}{\partial x^\alpha}\frac{\partial x^0}{\partial x'^0}. \tag{A.7}$$

From the relations (A.5)–(A.7), we find that

$$\frac{{}^*\partial}{\partial t} = h^{-1/2}\frac{\partial}{\partial x^0}, \qquad \frac{{}^*\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha} + \gamma_\alpha\frac{\partial}{\partial x^0}. \tag{A.8}$$

These chronometrically invariant operators do not generally commute with one another. Rather, their commutators define two important quantities. We have

$$\frac{{}^*\partial^2}{\partial x^\alpha\partial t} - \frac{{}^*\partial^2}{\partial t\partial x^\alpha} = {}^*E^{(\mathrm{g})}_\alpha\frac{{}^*\partial}{\partial t}, \qquad \frac{{}^*\partial^2}{\partial x^\alpha\partial x^\beta} - \frac{{}^*\partial^2}{\partial x^\beta\partial x^\alpha} = {}^*H^{(\mathrm{g})}_{\alpha\beta}\frac{{}^*\partial}{\partial t}, \tag{A.9}$$

where

$${}^*E^{(\mathrm{g})}_\alpha = \frac{-{}^*\partial\left(\ln\sqrt{h}\right)}{\partial x^\alpha} - \frac{\partial}{\partial x^0}\gamma_\alpha = \frac{1}{\sqrt{h}}E^{(\mathrm{g})}_\alpha \tag{A.10}$$

and

$${}^*H^{(\mathrm{g})}_{\alpha\beta} = \sqrt{h}\left(\frac{{}^*\partial}{\partial x^\alpha}\gamma_\beta - \frac{{}^*\partial}{\partial x^\beta}\gamma_\alpha\right) = \frac{1}{\sqrt{h}}H^{(\mathrm{g})}_{\alpha\beta} = \frac{1}{\sqrt{h}}\epsilon_{\alpha\beta\gamma}H^{(g)\gamma} \tag{A.11}$$

are the chronometrically invariant three-tensors formed from the gravito-electric field $\boldsymbol{E}^{(\mathrm{g})}$ defined in Eq. (33) and the gravito-magnetic field intensity $\boldsymbol{H}^{(\mathrm{g})}$ defined in Eq. (56), respectively.

Here it is interesting to recall that the law of force on a test particle, Eq. (35), involves the gravito-electromagnetic fields $\boldsymbol{E}^{(\mathrm{g})}$, $\boldsymbol{B}^{(\mathrm{g})}$ contained in the *covariant* tensor $F_{ij}^{(\mathrm{g})}$, which obeys the first gravito-Maxwellian system of equations

$$F_{[ij;k]}^{(\mathrm{g})} = 0\,, \tag{A.12}$$

while the Einstein equations (55) are formulated primarily in terms of the fields $\boldsymbol{D}^{(\mathrm{g})}$, $\boldsymbol{H}^{(\mathrm{g})}$ contained in the *contravariant* tensor $f^{(\mathrm{g})\mathrm{ij}}$, which, in the stationary case, obeys the analogue of the second Maxwellian system Eq. (45) for the source, namely Eq. (70).

For the commutators (A.9) involve one field from each of these two pairs, that is the two fields $\boldsymbol{E}^{(\mathrm{g})}$, $\boldsymbol{H}^{(\mathrm{g})}$. The factor of $1/\sqrt{h}$ occurring in Eqs. (A.10) and (A.11) can be understood from Theorem I, setting $n = 1$ in Eq. (A.4). The contravariant components of the gravito-electric field $\boldsymbol{E}^{(\mathrm{g})}$ are

$$E^{(\mathrm{g})\alpha} = \gamma^{\alpha\beta} E_{\beta}^{(\mathrm{g})} = -\gamma^{\alpha\beta} F_{0\beta}^{(\mathrm{g})} = F_{0}^{(g)\alpha}\,, \tag{A.13}$$

leading to Eq. (A.10), and the gravito-magnetic intensity $\boldsymbol{H}^{(\mathrm{g})}$ can be expressed via the dual field strength

$$\tilde{F}_{ij}^{(\mathrm{g})} = \tfrac{1}{2}\eta_{ijkl} F^{(g)kl}\,, \tag{A.14}$$

where $\eta_{ijkl} = \sqrt{-g}\,\delta_{ijkl}$, as

$$H_{\alpha}^{(\mathrm{g})} = \tfrac{1}{2}\,\epsilon_{\alpha\beta\gamma}\,\sqrt{h}\,F^{(\mathrm{g})\beta\gamma} = \tfrac{1}{2}\,\eta_{0\alpha kl}\,F^{(\mathrm{g})\mathrm{kl}} = \tilde{F}_{0\alpha}^{(\mathrm{g})}\,, \tag{A.15}$$

leading to Eq. (A.11).

Note, geometrically, that the space-time metric cross-terms $g_{0\alpha}$ can only be made to vanish everywhere by means of the transformations (22) if $\boldsymbol{H}^{(\mathrm{g})} = 0$, that is if $H_{\alpha\beta}^{(\mathrm{g})} = 0$, which is the condition obtained by von Weyssenhoff[44] (see also p. 284 of Ref. [6]).

Finally, there is a third chronometrically invariant field which enters the equations of gravitation when they are time dependent. This is the chronometrically invariant time derivative of the three-metric, defined such that

$$K_{\alpha\beta} = \tfrac{1}{2}\frac{{}^{*}\partial}{\partial t}\gamma_{\alpha\beta}\,, \qquad K^{\alpha\beta} = -\tfrac{1}{2}\frac{{}^{*}\partial}{\partial t}\gamma^{\alpha\beta}\,, \qquad K \equiv \gamma^{\alpha\beta}K_{\alpha\beta} = \frac{{}^{*}\partial\left(\ln\sqrt{\gamma}\right)}{\partial t}\,. \tag{A.16}$$

This paper was written at the University of Cambridge, Cambridge, England.

Appendix B

*

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