TRACE OPERATORS FOR THE STATE LABELLING PROBLEM IN THE EXCEPTIONAL LIE ALGEBRA F_4

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The Casimir operators of the exceptional Lie algebra F_4 are constructed using the method of trace operators. Taking into account a decomposition of matrices related to the embedding $F_4 \supset \mathfrak{so}(9)$, subgroup scalars for the corresponding state labelling problem are determined as traces of powers of the components, enabling us to propose an orthonormal basis of states for each generic irreducible representation (IRREP) of F_4. The basis of eigenstates for the IRREP [1000] of F_4 is explicitly given.

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1. Introduction

In contrast to the classical groups, which emerged naturally in the quantum mechanical formalism and in nuclear physics, exceptional Lie groups entered physics relatively late, with the pioneering work of Racah on the classification of f^n electron configurations and the computation of matrix elements in the spin-orbit interaction [1, 2]. The deep relation between G_2 and the rotation group SO(7), in connection with their representation theory and their underlying branching rules, motivated a closer analysis of the exceptional groups as symmetry groups in physical phenomena. In this context, besides the role of G_2 in atomic spectroscopy (of rare earths) [3], the E_i-series comprising the exceptional groups E_6, E_7 and E_8 has become an important object in high energy physics [4], notably in the context of Grand Unified Theories of the fundamental interactions [5].

The remaining exceptional group F_4, although less known and exploited [6], also has played an indirect role in various physical problems, like the classification of the states of N-electron configurations [6], the study of the atomic f-shell [7], or even recent interpretations of the Standard Model [8].

The main interest of F_4 in these problems resides in the fact that it contains various groups relevant to the specific physical problem under analysis, like the chain $F_4 \supset G_2 \times \mathrm{SO}_I(3)$ enlarging Racah's ansatz [2]. Another suggesting chain involving this group is for example $E_6 \supset F_4 \times G_2 \supset \mathrm{SO}(9) \times G_2$, which essentially makes reference to the reduction $F_4 \supset \mathrm{SO}(9)$. Enlarging the latter chain we obtain the embedding $F_4 \supset \mathrm{SO}(9) \supset \mathrm{SO}(8)$, a case that has been applied to study the branching rules used in [7] for the so-called "quark" model of atomic spectroscopy by means of the triality of SO(8).

The purpose of this work is twofold: on the one hand we reformulate the problem of the Casimir operators of F_4 by means of the trace method developed by Gruber and O'Raifeartaigh [9], but using a basis of F_4 obtained from expansion of a basis of the maximal subgroup SO(9) of F_4. This will enable us to simultaneously construct the invariants of F_4, SO(9) and to derive subgroup scalars for the labelling problem (MLP) corresponding to the embedding $\mathrm{SO}(9) \subset F_4$. Explicitly, these subgroup scalars will emerge as the traces of products of successive powers of the components. Once a complete set of commuting labelling operators is constructed, we are able to construct an orthonormal basis of states for each generic irreducible representation of F_4. As an illustration, the basis of eigenstates is explicitly computed for the 26-dimensional fundamental representation of F_4.

2. Casimir operators of F_4 as trace operators

Using only the structure tensor of the Lie algebra, and generalizing the approach by Casimir [10], Racah developed a method to construct invariant operators of arbitrary order that commute with all the generators of a semisimple Lie algebra [1]. His construction essentially made use of the properties of the adjoint representation of Lie algebras. However, it was observed that this construction does not, in general, lead to a complete set of independent operators. A significant advance in this direction was obtained by Gruber and O'Raifeartaigh [9], who considered themselves a generalization of the Casimir invariants based on the characters of products of generators in irreducible representations of (compact) semisimple Lie groups. For the special case of the adjoint representation, their result essentially recovered the already known constructions of the Casimir operators [1]. Moreover, they showed that in order to extract a compete set of invariants, the representations to be considered are essentially the fundamental representations of the Lie algebra $\mathfrak{s}$. In this way, a systematical way to describe the Casimir operators of semisimple Lie algebras of arbitrary rank was given.

For completeness, we briefly recall here the main features of the procedure. Let $[X_i, X_j] = C_{ij}^k X_k$ be the basis of a (compact) semisimple Lie algebra $\mathfrak{s}$ of dimension n and rank l. It is well known [1] that operators of the shape

$$F_p = C_{\iota\sigma_2}^{\sigma_1} C_{j\sigma_3}^{\sigma_2} C_{k\sigma_4}^{\sigma_3} \dots C_{\rho\sigma_1}^{\sigma_p} X_i X_j X_k \dots X_\rho$$

constitute an order p invariant of $\mathfrak{s}$, *i.e.*, $[F_p, X] = 0$ for any $X \in \mathfrak{s}$. Using the adjoint representation $\mathrm{ad}\,(\mathfrak{s})$, the latter operator can be rewritten as

$$F_p = \mathrm{Tr}\left(\mathrm{ad}\,(X_i)\,\mathrm{ad}\,(X_j)\,\mathrm{ad}\,(X_k) \dots \mathrm{ad}\,(X_\rho)\right) X_i X_j X_k \dots X_\rho\,. \qquad (1)$$

As observed, the operators (1) will usually not provide a set of l independent (Casimir) operators, thus are insufficient to construct the invariants of $\mathfrak{s}$. Gruber and O'Raifeartaigh [9] reformulated the results of Racah in terms of tensor products of the l fundamental representations of $\mathfrak{s}$ and showed that for any (compact) semisimple Lie algebra $\mathfrak{s}$ we can always find an appropriate (fundamental) representation Γ and an associated polynomial matrix

$$M = \sum_{j=1}^{n} \left(X_j \otimes \Gamma\,(X_j)\right)\,, \qquad (2)$$

such that from the quantities

$$I_p = \mathrm{Tr}\,(M^p)\,, \qquad p \geq 2 \qquad (3)$$

we can extract exactly l independent invariants of the Lie algebra, which correspond naturally to the l Casimir operators of $\mathfrak{s}$ [9]. Explicit expressions of the polynomial matrix M for the series of classical Lie algebras were given in that work. In this manner, the Casimir operators appear naturally as trace operators, inheriting an intrinsic meaning in terms of representations, and clarifying their significance in the labelling of multiplets. For the corresponding non-compact real forms of these Lie algebras, the Casimir operators can be also constructed along the previous method, with slight variations in the choice of the representation and the basis[1].

In the following, we will apply the Gruber–O'Raifeartaigh procedure to the exceptional Lie algebra F_4. As far as the author is aware, this has not been done previously for this group, although alternative constructions have been considered [18]. However, we will introduce a further reaching variation into the ansatz. We are not only interested in computing the four Casimir operators of F_4 as trace operators, but also in obtaining the labelling operators for the reduction chain $F_4 \supset \mathfrak{so}\,(9)$. In order to do this, we will have to consider a basis of F_4 expanded from a basis of $\mathfrak{so}\,(9)$, *i.e.*, we need the branching rules of representations [12,13,14]. In particular, the

[1] The trace method, relying on the structure of semisimple Lie algebras, cannot in general be generalized to non-semisimple Lie algebras. In particular, for solvable Lie algebras it is of no use, as the IRREPs are all one-dimensional.

embedding $F_4 \supset \mathfrak{so}(9)$ implies that the adjoint representation of F_4 must decompose as follows as a sum of $\mathfrak{so}(9)$ representations[2]

$$[1000] \supset (0100) + (0001) \ , \tag{4}$$

where (0100) is the adjoint representation of $\mathfrak{so}(9)$ and (0001) corresponds to the spinor representation. This means, in particular, that we can find a basis $\mathcal{B} = \{H_i, E_j, F_k, S_l, T_l\}$ of F_4, where $i = 1, \ldots, 4$, $j = 1, \ldots, 16$, $l = 1, \ldots, 8$, such that $\{H_i, E_j, F_k\}$ constitutes a basis of $\mathfrak{so}(9)$ and $\{T_l, S_l\}$ is a basis of the spinor representation (0001) of $\mathfrak{so}(9)$[3]. The generators $H_1, \ldots, H_4$ correspond to the Cartan subalgebra of F_4, which is simultaneously the Cartan subalgebra of the subalgebra $\mathfrak{so}(9)$.

As the exceptional algebra F_4 possesses four primitive Casimir operators of degrees $2, 6, 8$ and 12 respectively [11][4], we must find an appropriate (fundamental) representation Γ and a polynomial matrix of type (2) such that the traces of the second, sixth, eighth and twelfth powers are functionally independent invariants. To this extent, we consider the fundamental representation $\Gamma = [1000]$ of dimension 26. As the basis of F_4 has been obtained using the branching rule (4), the polynomial matrix of (2) will decompose as a sum

$$M = U + R \,, \tag{5}$$

where the matrices are defined as

$$\begin{aligned} U &= \sum_{i=1}^{4} \left(H_i \otimes \Gamma\left(H_i\right)\right) + \sum_{k=1}^{16} \left(E_k \otimes \Gamma\left(E_k\right)\right) + \sum_{k=1}^{16} \left(F_k \otimes \Gamma\left(F_k\right)\right) \ , \\ R &= \sum_{k=1}^{8} \left(S_k \otimes \Gamma\left(S_k\right)\right) + \sum_{k=1}^{8} \left(T_k \otimes \Gamma\left(T_k\right)\right) \ . \end{aligned} \tag{6}$$

The structure constants of F_4 over the preceding basis $\mathcal{B}$ is easily obtained from the commutation relations of the (scalar) matrices $\Gamma(H_i)$, $\Gamma(E_k)$, $\Gamma(F_k)$, $\Gamma(S_k)$ and $\Gamma(T_k)$.

Denoting $j = \exp(i\pi/3)$, the two matrices U and R are explicitly given by:

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[2] In the following, we will always use the notations for the labelling of representations as done in [13].

[3] Without loss of generality we can further chose the generators E_j, F_j, S_j, T_j such that $F_j = E_j^{\dagger}$, $T_j = S_j^{\dagger}$.

[4] By primitive, we mean that the Casimir operators cannot be obtained as polynomials of lower order invariants.

$$
U = \begin{pmatrix}
H_1 & 0 & 0 & 0 & 0 & 0 & E_5 & 0 & -E_7 & 0 & -E_9 & 0 & \frac{E_1}{\sqrt{2}} & \frac{-E_1}{\sqrt{2}} & 0 & -E_{10} & 0 & E_8 & 0 & E_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & K_1 & \frac{E_4}{\sqrt{2}} & \frac{-E_3}{\sqrt{2}} & -E_{16} & \frac{E_2}{\sqrt{2}} & 0 & E_{14} & 0 & E_{12} & 0 & 0 & 0 & 0 & \frac{E_1}{\sqrt{2}} & 0 & E_{10} & 0 & -E_8 & 0 & 0 & -E_6 & 0 & 0 & 0 & 0 \\
0 & \frac{F_4}{\sqrt{2}} & K_2 & E_{15} & \frac{E_3}{\sqrt{2}} & -E_{13} & 0 & \frac{-E_2}{\sqrt{2}} & 0 & 0 & 0 & E_{12} & 0 & 0 & E_9 & 0 & \frac{E_1}{\sqrt{2}} & 0 & 0 & 0 & E_8 & 0 & E_6 & 0 & 0 & 0 \\
0 & \frac{-F_3}{\sqrt{2}} & F_{15} & K_3 & \frac{E_4}{\sqrt{2}} & E_{11} & 0 & 0 & 0 & \frac{-E_2}{\sqrt{2}} & 0 & -E_{14} & 0 & 0 & -E_7 & 0 & 0 & 0 & \frac{-E_1}{\sqrt{2}} & 0 & -E_{10} & 0 & 0 & -E_6 & 0 & 0 \\
0 & -F_{16} & \frac{F_3}{\sqrt{2}} & \frac{F_4}{\sqrt{2}} & K_4 & 0 & 0 & E_{11} & 0 & E_{13} & 0 & \frac{E_2}{\sqrt{2}} & 0 & 0 & 0 & 0 & E_7 & 0 & -E_9 & 0 & \frac{-E_1}{\sqrt{2}} & 0 & 0 & 0 & E_6 & 0 \\
0 & \frac{F_2}{\sqrt{2}} & -F_{13} & F_{11} & 0 & K_5 & 0 & \frac{E_4}{\sqrt{2}} & 0 & \frac{-E_3}{\sqrt{2}} & 0 & -E_{16} & 0 & 0 & -E_5 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{E_1}{\sqrt{2}} & E_{10} & -E_8 & 0 & 0 \\
F_5 & 0 & 0 & 0 & 0 & 0 & H_2 & 0 & E_{11} & 0 & E_{13} & 0 & \frac{E_2}{\sqrt{2}} & \frac{-E_2}{\sqrt{2}} & 0 & E_{14} & 0 & -E_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -E_6 \\
0 & F_{14} & \frac{-F_2}{\sqrt{2}} & 0 & F_{11} & \frac{F_4}{\sqrt{2}} & 0 & K_6 & 0 & E_{15} & 0 & \frac{E_3}{\sqrt{2}} & 0 & 0 & 0 & 0 & E_5 & 0 & 0 & 0 & 0 & E_9 & \frac{E_1}{\sqrt{2}} & 0 & E_8 & 0 \\
-F_7 & 0 & 0 & 0 & 0 & 0 & F_{11} & 0 & H_3 & 0 & E_{15} & 0 & \frac{E_3}{\sqrt{2}} & \frac{-E_3}{\sqrt{2}} & 0 & E_{16} & 0 & 0 & 0 & E_{12} & 0 & 0 & 0 & 0 & 0 & -E_8 \\
0 & F_{12} & 0 & \frac{-F_2}{\sqrt{2}} & F_{13} & \frac{-F_3}{\sqrt{2}} & 0 & F_{15} & 0 & K_7 & 0 & \frac{E_4}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & -E_5 & 0 & 0 & -E_7 & 0 & \frac{-E_1}{\sqrt{2}} & -E_{10} & 0 \\
-F_9 & 0 & 0 & 0 & 0 & 0 & F_{13} & 0 & F_{15} & 0 & H_4 & 0 & \frac{E_4}{\sqrt{2}} & \frac{-E_4}{\sqrt{2}} & 0 & 0 & 0 & -E_{16} & 0 & -E_{14} & 0 & 0 & 0 & 0 & 0 & E_{10} \\
0 & 0 & F_{12} & -F_{14} & \frac{F_2}{\sqrt{2}} & -F_{16} & 0 & \frac{F_3}{\sqrt{2}} & 0 & \frac{F_4}{\sqrt{2}} & 0 & K_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & E_5 & 0 & E_7 & -E_9 & \frac{-E_1}{\sqrt{2}} & 0 \\
\frac{F_1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{F_2}{\sqrt{2}} & 0 & \frac{F_3}{\sqrt{2}} & 0 & \frac{F_4}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{E_4}{\sqrt{2}} & 0 & \frac{E_3}{\sqrt{2}} & 0 & \frac{E_2}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{E_1}{\sqrt{2}} \\
\frac{-F_1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{-F_2}{\sqrt{2}} & 0 & \frac{-F_3}{\sqrt{2}} & 0 & \frac{-F_4}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{-E_4}{\sqrt{2}} & 0 & \frac{-E_3}{\sqrt{2}} & 0 & \frac{-E_2}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{-E_1}{\sqrt{2}} \\
0 & \frac{F_1}{\sqrt{2}} & F_9 & -F_7 & 0 & -F_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_9 & 0 & \frac{-E_4}{\sqrt{2}} & 0 & \frac{-E_3}{\sqrt{2}} & 0 & E_{16} & \frac{-E_2}{\sqrt{2}} & E_{14} & -E_{12} & 0 & 0 \\
-F_{10} & 0 & 0 & 0 & 0 & 0 & F_{14} & 0 & F_{16} & 0 & 0 & 0 & \frac{F_4}{\sqrt{2}} & \frac{-F_4}{\sqrt{2}} & 0 & -H_4 & 0 & -E_{15} & 0 & -E_{13} & 0 & 0 & 0 & 0 & 0 & E_9 \\
0 & F_{10} & \frac{F_1}{\sqrt{2}} & 0 & F_7 & 0 & 0 & F_5 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-F_4}{\sqrt{2}} & 0 & K_{10} & 0 & -E_{15} & 0 & \frac{E_3}{\sqrt{2}} & -E_{13} & \frac{E_2}{\sqrt{2}} & 0 & -E_{12} & 0 \\
F_8 & 0 & 0 & 0 & 0 & 0 & -F_{12} & 0 & 0 & 0 & -F_{16} & 0 & \frac{F_3}{\sqrt{2}} & \frac{-F_3}{\sqrt{2}} & 0 & -F_{15} & 0 & -H_3 & 0 & -E_{11} & 0 & 0 & 0 & 0 & 0 & E_7 \\
0 & -F_8 & 0 & \frac{-F_1}{\sqrt{2}} & -F_9 & 0 & 0 & 0 & 0 & -F_5 & 0 & 0 & 0 & 0 & \frac{-F_3}{\sqrt{2}} & 0 & -F_{15} & 0 & K_{11} & 0 & \frac{-E_4}{\sqrt{2}} & -E_{11} & 0 & \frac{E_2}{\sqrt{2}} & -E_{14} & 0 \\
F_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & F_{12} & 0 & -F_{14} & 0 & \frac{F_2}{\sqrt{2}} & \frac{-F_2}{\sqrt{2}} & 0 & -F_{13} & 0 & -F_{11} & 0 & -H_2 & 0 & 0 & 0 & 0 & 0 & -E_5 \\
0 & 0 & F_8 & -F_{10} & \frac{-F_1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & F_5 & 0 & 0 & F_{16} & 0 & \frac{F_3}{\sqrt{2}} & 0 & \frac{-F_4}{\sqrt{2}} & 0 & K_{12} & 0 & -E_{11} & E_{13} & \frac{-E_2}{\sqrt{2}} & 0 \\
0 & -F_6 & 0 & 0 & 0 & \frac{F_1}{\sqrt{2}} & 0 & F_9 & 0 & -F_7 & 0 & 0 & 0 & 0 & \frac{-F_2}{\sqrt{2}} & 0 & -F_{13} & 0 & -F_{11} & 0 & 0 & K_{13} & \frac{-E_4}{\sqrt{2}} & \frac{-E_3}{\sqrt{2}} & E_{16} & 0 \\
0 & 0 & F_6 & 0 & 0 & F_{10} & 0 & \frac{F_1}{\sqrt{2}} & 0 & 0 & 0 & F_7 & 0 & 0 & F_{14} & 0 & \frac{F_2}{\sqrt{2}} & 0 & 0 & 0 & -F_{11} & \frac{-F_4}{\sqrt{2}} & K_{14} & -E_{15} & \frac{E_3}{\sqrt{2}} & 0 \\
0 & 0 & 0 & -F_6 & 0 & -F_8 & 0 & 0 & 0 & \frac{-F_1}{\sqrt{2}} & 0 & -F_9 & 0 & 0 & -F_{12} & 0 & 0 & 0 & \frac{F_2}{\sqrt{2}} & 0 & F_{13} & \frac{-F_3}{\sqrt{2}} & -F_{15} & K_{15} & \frac{-E_4}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 0 & F_6 & 0 & 0 & F_8 & 0 & -F_{10} & 0 & \frac{-F_1}{\sqrt{2}} & 0 & 0 & 0 & 0 & -F_{12} & 0 & -F_{14} & 0 & \frac{-F_2}{\sqrt{2}} & F_{16} & \frac{F_3}{\sqrt{2}} & \frac{-F_4}{\sqrt{2}} & K_{16} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -F_6 & 0 & -F_8 & 0 & F_{10} & 0 & \frac{F_1}{\sqrt{2}} & \frac{-F_1}{\sqrt{2}} & 0 & F_9 & 0 & F_7 & 0 & -F_5 & 0 & 0 & 0 & 0 & 0 & -H_1
\end{pmatrix}
$$

$$
R=\frac{1}{\sqrt{2}}\begin{pmatrix}
0 & T_8 & -T_7 & T_6 & T_5 & -T_4 & 0 & -T_3 & 0 & -T_2 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_8 & 0 & 0 & 0 & 0 & 0 & -T_5 & 0 & T_3 & 0 & T_2 & 0 & -j^2T_1 & jT_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-S_7 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & -T_4 & 0 & 0 & 0 & jT_2 & -j^2T_2 & 0 & -T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_6 & 0 & 0 & 0 & 0 & 0 & T_7 & 0 & 0 & 0 & -T_4 & 0 & -jT_3 & j^2T_3 & 0 & 0 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
S_5 & 0 & 0 & 0 & 0 & 0 & T_8 & 0 & 0 & 0 & 0 & 0 & -j^2T_4 & jT_4 & 0 & -T_3 & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-S_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_7 & 0 & -T_6 & 0 & -jT_5 & -j^2T_5 & 0 & 0 & 0 & 0 & 0 & -T_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -S_5 & S_6 & S_7 & S_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4 & 0 & -T_3 & 0 & T_2 & 0 & T_1 & 0 & 0 & 0 & 0 & 0 \\
-S_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_8 & 0 & 0 & 0 & -j^2T_6 & jT_6 & 0 & -T_5 & 0 & 0 & 0 & -T_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S_3 & -S_4 & 0 & 0 & S_7 & 0 & S_8 & 0 & 0 & 0 & 0 & 0 & 0 & T_6 & 0 & -T_5 & 0 & 0 & 0 & 0 & -T_2 & -T_1 & 0 & 0 & 0 \\
-S_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_8 & 0 & -j^2T_7 & jT_7 & 0 & 0 & 0 & T_5 & 0 & T_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S_2 & 0 & -S_4 & 0 & -S_6 & 0 & 0 & 0 & S_8 & 0 & 0 & 0 & 0 & T_7 & 0 & 0 & 0 & T_5 & 0 & 0 & T_3 & 0 & T_1 & 0 & 0 \\
S_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & jT_8 & -j^2T_8 & 0 & -T_7 & 0 & -T_6 & 0 & -T_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -jS_1 & j^2S_2 & -j^2S_3 & -jS_4 & -j^2S_5 & 0 & -jS_6 & 0 & -jS_7 & 0 & j^2S_8 & 0 & 0 & j^2T_8 & 0 & -jT_7 & 0 & -jT_6 & 0 & -j^2T_5 & -jT_4 & -j^2T_3 & j^2T_2 & -jT_1 & 0 \\
0 & j^2S_1 & -jS_2 & jS_3 & j^2S_4 & -jS_5 & 0 & j^2S_6 & 0 & j^2S_7 & 0 & -jS_8 & 0 & 0 & -jT_8 & 0 & j^2T_7 & 0 & j^2T_6 & 0 & -jT_5 & j^2T_4 & jT_3 & -jT_2 & j^2T_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_4 & 0 & S_6 & 0 & S_7 & 0 & j\,S_8 & -j^2S_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_1 \\
0 & 0 & -S_1 & 0 & -S_3 & 0 & 0 & -S_5 & 0 & 0 & 0 & -S_7 & 0 & 0 & 0 & 0 & -T_8 & 0 & 0 & 0 & T_6 & 0 & T_4 & 0 & -T_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -S_3 & 0 & -S_5 & 0 & 0 & 0 & -j^2S_7 & jS_7 & 0 & -S_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 \\
0 & 0 & 0 & S_1 & S_2 & 0 & 0 & 0 & 0 & S_5 & 0 & -S_6 & 0 & 0 & 0 & 0 & 0 & 0 & -T_8 & 0 & -T_7 & 0 & 0 & T_4 & -T_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_2 & 0 & 0 & 0 & S_5 & 0 & -j^2S_6 & jS_6 & 0 & 0 & 0 & -S_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_3 \\
0 & 0 & 0 & 0 & 0 & -S_1 & 0 & -S_2 & 0 & S_3 & 0 & -S_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_8 & -T_7 & -T_6 & T_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 & 0 & 0 & 0 & 0 & -jS_5 & -j^2S_5 & 0 & S_6 & 0 & -S_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S_2 & 0 & S_3 & 0 & -j^2S_4 & jS_4 & 0 & 0 & 0 & 0 & 0 & -S_8 & 0 & 0 & 0 & 0 & 0 & -T_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S_1 & 0 & 0 & 0 & -jS_3 & j^2S_3 & 0 & S_4 & 0 & 0 & 0 & -S_7 & 0 & 0 & 0 & 0 & 0 & -T_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1 & 0 & jS_2 & -j^2S_2 & 0 & 0 & 0 & S_4 & 0 & -S_6 & 0 & 0 & 0 & 0 & 0 & T_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -j^2S_1 & jS_1 & 0 & -S_2 & 0 & -S_3 & 0 & S_5 & 0 & 0 & 0 & 0 & 0 & -T_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -S_1 & 0 & S_2 & 0 & S_3 & 0 & S_4 & -S_5 & -S_6 & S_7 & -S_8 & 0
\end{pmatrix}
$$

The K_i denote the following linear combinations of the Cartan generators H_i:

$$
\begin{aligned}
K_1 &= \tfrac{(H_1+H_2+H_3+H_4)}{2}\,, & K_2 &= \tfrac{(H_1+H_2+H_3-H_4)}{2}\,,\\
K_3 &= \tfrac{(H_1+H_2-H_3+H_4)}{2}\,, & K_4 &= \tfrac{(H_1+H_2-H_3-H_4)}{2}\,,\\
K_5 &= \tfrac{(H_1-H_2+H_3+H_4)}{2}\,, & K_6 &= \tfrac{(H_1-H_2+H_3-H_4)}{2}\,,\\
K_7 &= \tfrac{(H_1-H_2-H_3+H_4)}{2}\,, & K_8 &= \tfrac{(H_1-H_2-H_3-H_4)}{2}\,,\\
K_9 &= \tfrac{(-H_1+H_2+H_3+H_4)}{2}\,, & K_{10} &= \tfrac{(-H_1+H_2+H_3-H_4)}{2}\,,\\
K_{11} &= \tfrac{(-H_1+H_2-H_3+H_4)}{2}\,, & K_{12} &= \tfrac{(-H_1+H_2-H_3-H_4)}{2}\,,\\
K_{13} &= \tfrac{(-H_1-H_2+H_3+H_4)}{2}\,, & K_{14} &= \tfrac{(-H_1-H_2+H_3-H_4)}{2}\,,\\
K_{15} &= \tfrac{(-H_1-H_2-H_3+H_4)}{2}\,, & K_{16} &= -\tfrac{(H_1+H_2+H_3+H_4)}{2}\,.
\end{aligned}
$$

From the construction of M we clearly have that $\mathrm{Tr}\,[M] = 0$, as both summands are zero trace matrices. We claim that the traces of M^k for $k = 2, 6, 8, 12$ provide the primitive Casimir operators of F_4.

In order to evaluate the traces of higher powers of M, we make use of the identity

$$\mathrm{Tr}\left[M^k\right] = \mathrm{Tr}\left[(U+R)^k\right]\,. \tag{7}$$

Expanding the sum in the right-hand side of (7) we arrive at sums of products of the type

$$U^{a_1}R^{b_1}\dots U^{a_s}R^{b_s}\,, \tag{8}$$

where $a_i, b_j \geq 0$ are integers such that $\sum_{i=1}^{s} a_i + b_i = k$. The traces of such terms can be further simplified using the elementary properties of traces

$$
\begin{aligned}
\mathrm{Tr}\left[U^{a_1}R^{b_1}\dots U^{a_s}R^{b_s}\right] &= \mathrm{Tr}\left[RU^{a_1}R^{b_1}\dots U^{a_s}R^{b_s-1}\right] = \dots\\
\dots = \mathrm{Tr}\left[R^{b_s-1}U^{a_1}R^{b_1}\dots U^{a_s}R\right] &= \mathrm{Tr}\left[R^{b_s}U^{a_1}R^{b_1}\dots U^{a_s}\right]\,.
\end{aligned} \tag{9}
$$

The matrices U and R can be further decomposed, which will allow us to analyse their properties more closely. More precisely, we decompose the matrices as follows: $U = D + N_1 + N_2$ and $R = P_1 + P_2$, where D is the diagonal matrix having the H_i, K_j as entries, N_1, P_1 are the upper triangular matrices having as entries the E_i and T_j respectively, and N_2, P_2 are the lower triangular having as entries the F_i and S_j respectively[5]. Using this decomposition and the fact that N_1, N_2, P_1, P_2 are nilpotent [15], it is straightforward to verify that

$$\mathrm{Tr}\left[U^k\right] = \mathrm{Tr}\left[R^k\right] = 0 \tag{10}$$

[5] Observe that this decomposition is consistent with the fact that F_i and S_j are the Hermitean conjugates of E_i and T_j respectively.

for any odd value of k. Further, as the matrices N_1, N_2 and P_1, P_2 are related by Hermitean conjugation, and that commutators of nilpotent matrices are nilpotent [15], a cumbersome but routine computation shows that

$$\mathrm{Tr}\left[U^{a_1}R^{b_1}\dots U^{a_s}R^{b_s}\right]=0\,, \tag{11}$$

whenever either $\sum_{i=1}^{s} a_i = 2p+1$ or $\sum_{i=1}^{s} b_i = 2p+1$ holds for some $p \geq 0$. From (11) we can, in particular, deduce that for odd values of k we obtain the identity $\mathrm{Tr}\left[M^k\right]=0$. Therefore, only the even powers M^{2p} for $1 \leq p \leq 6$ have to be considered. This fact is in agreement with the non-existence of odd order Casimir operators for F_4.

Using (9) and (11) we considerably simplify the computation of the traces of even powers of M. According to (7) and (11), for $k=2$ the trace reduces to

$$\mathrm{Tr}\left[M^2\right]=\mathrm{Tr}\left[U^2\right]+\mathrm{Tr}\left[R^2\right]\,. \tag{12}$$

In terms of the generators, this operator has the expression

$$\mathrm{Tr}\left[M^2\right]=\left(H_1^2+H_2^2+H_3^2+H_4^2\right)+\sum_{i=1}^{16}\left(E_iF_i+F_iS_i\right)+\sum_{j=1}^{8}\left(S_jT_j+T_jS_j\right)\,. \tag{13}$$

It is straightforward to verify that it commutes with all generators of F_4, thus it can be taken as the quadratic Casimir operator in its symmetric representative. As expected, for the fourth power of M the trace is dependent on (13)

$$\mathrm{Tr}\left[M^4\right]=\frac{1}{12}\Big(\mathrm{Tr}\left[M^2\right]\Big)^2. \tag{14}$$

For the sixth and eight powers, equation (7) provides the decomposition

$$\begin{aligned}
\mathrm{Tr}\left[M^6\right]=&\,\mathrm{Tr}\left[U^6\right]+6\mathrm{Tr}\left[U^4R^2\right]+6\mathrm{Tr}\left[U^3RUR\right]+3\mathrm{Tr}\left[\left(U^2R\right)^2\right]\\
&+6\mathrm{Tr}\left[U^2R^4\right]+6\mathrm{Tr}\left[URUR^3\right]+3\mathrm{Tr}\left[(UR)^2\right]+\mathrm{Tr}\left[R^6\right]\,,\\
\mathrm{Tr}\left[M^8\right]=&\,\mathrm{Tr}\left[U^8\right]+8\mathrm{Tr}\left[\left(UR^3\right)^2\right]+8\mathrm{Tr}\left[U^5RUR\right]+8\mathrm{Tr}\left[U^4RU^2R\right]\\
&+4\mathrm{Tr}\left[\left(U^3R\right)^2\right]+8\mathrm{Tr}\left[U^4R^4\right]+8\mathrm{Tr}\left[U^3RUR^3\right]+\mathrm{Tr}\left[R^8\right]\\
&+8\mathrm{Tr}\left[U^3R^3UR\right]+8\mathrm{Tr}\left[U^2RU^2R^3\right]+8\mathrm{Tr}\left[U^2\left(RU\right)^2R^2\right]\\
&+8\mathrm{Tr}\left[U^2RUR^2UR\right]+4\mathrm{Tr}\left[U^2R^2\left(UR\right)^2\right]+2\mathrm{Tr}\left[(UR)^4\right]\\
&+8\mathrm{Tr}\left[U^2R^6\right]+8\mathrm{Tr}\left[URUR^5\right]+8\mathrm{Tr}\left[UR^2UR^4\right]+8\mathrm{Tr}\left[U^6R^2\right]\,.
\end{aligned} \tag{15}$$

To show that $\mathrm{Tr}\left[M^6\right]$ and $\mathrm{Tr}\left[M^8\right]$ are, respectively, a sixth and eight order Casimir operator of F_4, we can make use of the analytical approach to the invariant problem [16, 17]. Actually, in both cases it suffices to show that

$$\left[E_i, \mathrm{Tr}\left[M^k\right]\right] = \left[F_i, \mathrm{Tr}\left[M^k\right]\right] = \left[T_j, \mathrm{Tr}\left[M^k\right]\right] = 0\,, \tag{16}$$

for $i = 1, \dots, 4$, $j = 1, \dots, 8$ and $k = 6, 8$, as the remaining relations follow automatically from the commutators of the generators[6]. Equation (15) also shows that $\mathrm{Tr}\left[M^2\right]$, $\mathrm{Tr}\left[M^6\right]$ and $\mathrm{Tr}\left[M^8\right]$ are independent operators. This is a consequence of the independence of the operators $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$ and $\mathrm{Tr}\left[U^8\right]$, which can be easily verified.

As any invariant of F_4 having order ten is a polynomial of lower degree Casimir operators [11], it should be expected that the trace of M^{10} is a function of the preceding traces. In fact, taking into account that $\mathrm{Tr}\left[M^4\right]$ is a power of $\mathrm{Tr}\left[M^2\right]$, we arrive at the following dependence relation

$$\mathrm{Tr}\left[M^{10}\right] = \tfrac{3}{8}\mathrm{Tr}\left[M^8\right]\mathrm{Tr}\left[M^2\right] - \tfrac{7}{144}\mathrm{Tr}\left[M^6\right]\mathrm{Tr}\left[M^2\right]^2 + \tfrac{7}{41472}\mathrm{Tr}\left[M^2\right]^5\,. \tag{17}$$

It remains to compute the trace of M^{12}. Using (9) and (11) leads to the expression

$$\mathrm{Tr}\left[M^{12}\right] = \mathrm{Tr}\left[U^{12}\right] + \Theta^{[10,2]} + \Theta^{[8,4]} + \Theta^{[6,6]} + \Theta^{[4,8]} + \Theta^{[2,10]} + \mathrm{Tr}\left[R^{12}\right]\,, \tag{18}$$

where[7]

$$\Theta^{[a,b]} = \sum_{\substack{a_1+\cdots+a_s=a \\ b_1+\cdots+b_s=b}} \mathrm{Tr}\left[U^{a_1}R^{b_1}\dots U^{a_s}R^{b_s}\right]\,. \tag{19}$$

As before, it suffices to show that $\mathrm{Tr}\left[M^{12}\right]$ commutes with E_i, F_i and T_j for $i = 1, \dots, 4$; $j = 1, \dots, 8$. Further, its independence from the lower order traces follows again from the functional independence of the operators $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$, $\mathrm{Tr}\left[U^8\right]$ and $\mathrm{Tr}\left[U^{12}\right]$. With this construction we have therefore shown the following result:

Proposition 1 *The primitive Casimir operators of the exceptional Lie algebra F_4 are given by the trace operators $\mathrm{Tr}\left[M^2\right]$, $\mathrm{Tr}\left[M^6\right]$, $\mathrm{Tr}\left[M^8\right]$ and $\mathrm{Tr}\left[M^{12}\right]$ associated to the polynomial matrix M.*

---

[6] This is a consequence of the fact that for differential operators X, Y and scalar fields f the identity $X(f) = Y(f) = 0$ implies that $[X, Y](f) = 0$.

[7] We avoid the explicit expression in terms of the traces, as the sum involves more than 140 component traces.

In this context, it should be mentioned that in [18] the Casimir operators of F_4 were obtained using the chain $\mathrm{SO}(9) \subset F_4 \subset \mathrm{SO}(26)$, by means of a projection method of SO(26)-generators into F_4. Both the basis and the procedure differ from that chosen here, as we have expanded the Casimir operators of the maximal subalgebra $\mathfrak{so}(9)$ by means of trace operators, in contrast to the projection technique. This also implies that the particular representatives for the invariants obtained here differ from those in [18].

As we mentioned above, the operators $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$, $\mathrm{Tr}\left[U^8\right]$ and $\mathrm{Tr}\left[U^{12}\right]$ are independent. Actually, it follows at once from the decomposition that they commute with the generators of $\mathfrak{so}(9)$, and therefore constitute Casimir operators of this subalgebra. As the primitive invariants of $\mathfrak{so}\,(9)$ have degrees $d = 2, 4, 6, 8$ [11], there should exist an order four operator C_4 that jointly with the traces $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$ and $\mathrm{Tr}\left[U^8\right]$ provides a polynomial dependence relation for $\mathrm{Tr}\left[U^{12}\right]$. With some heavy computations it can be shown that such an operator can be extracted from the equation

$$\begin{aligned}\mathrm{Tr}\left[U^{12}\right] = {} & \tfrac{59}{576}\mathrm{Tr}\left[U^8\right]\mathrm{Tr}\left[U^2\right]^2 - \tfrac{49}{2592}\mathrm{Tr}\left[U^6\right]\mathrm{Tr}\left[U^2\right]^3 + \tfrac{17}{248832}\mathrm{Tr}\left[U^2\right]^2 \\ & \times\tfrac{1}{18}\mathrm{Tr}\left[U^6\right]^2 + \left(\tfrac{1}{4}\mathrm{Tr}\left[U^6\right]\mathrm{Tr}\left[U^2\right] - \tfrac{9}{8}\mathrm{Tr}\left[U^8\right]\right)C_4 \\ & -\tfrac{5}{64}\mathrm{Tr}\left[U^2\right]^2 C_4^2 + \tfrac{15}{8}C_4^3\,. \end{aligned} \tag{20}$$

The reason that C_4 does not appear as a trace lies in the fact that we have constructed the matrix M in such manner that the traces of successive powers reproduce the Casimir operators of F_4, and that the latter Lie algebra does not have an order four invariant independent from its quadratic one.

3. Applications to the labelling problem $\mathrm{SO}(9) \subset F_4$

The trace method used to obtain the Casimir operators of F_4 has a remarkable advantage to the explicit expressions of the invariants. Since we have used generators of F_4 in a $\mathfrak{so}(9)$-basis, we naturally recover the embedding $F_4 \supset \mathfrak{so}(9)$ and the corresponding labelling problem. It will turn out that the traces of operators of the type (8) will provide subgroup scalars for this embedding, *i.e.*, operators in the generators of F_4 that commute with all generators of $\mathfrak{so}(9)$. As known, the l eigenvalues of the Casimir operators of a semisimple Lie algebra $\mathfrak{s}$ of rank l label irreducible representations [1], while the Cartan subalgebra can be used identify states within a multiplet. Nonetheless, these operators are often not enough to completely separate multiplicities. The total number of internal labels required is

$$i = \tfrac{1}{2}(\dim\mathfrak{s} - l)\,. \tag{21}$$

An analogous situation holds when a subalgebra $\mathfrak{s}'$ of rank l' is used to label the basis states of irreducible representations of $\mathfrak{s}$. The subgroup provides $\frac{1}{2}(\dim\mathfrak{s}'+l')-l_0$ labels, where l_0 is the number of invariants of $\mathfrak{s}$ that depend only on generators of the subalgebra $\mathfrak{s}'$ [16]. To distinguish elements within a (generic) IRREP of $\mathfrak{s}$, we need to find

$$n=\frac{1}{2}\left(\dim\mathfrak{s}-l-\dim\mathfrak{s}'-l'\right)+l_0 \tag{22}$$

additional operators, called missing label operators or subgroup scalars. These operators, belonging to the enveloping algebra of $\mathfrak{s}$, commute with all generators of the subalgebra $\mathfrak{s}'$. The total number of available operators is $m=2n$ [16, 19]. For $n>1$, the labelling operators must moreover commute with each other.

According to (22), for $F_4\supset\mathfrak{so}(9)$ we need to find $n=\frac{1}{2}\,(52-4-36-4)=4$ subgroup scalars that commute with each other and the Casimir operators of F_4 and $\mathfrak{so}(9)$.

Starting from equations (15) and (18), we group together all traces operators $\mathrm{Tr}[U^{a_1}R^{b_1}\dots U^{a_s}R^{b_s}]$ having the same degree in the $\mathfrak{so}(9)$ and spinor representation generators. The operators $\mathrm{Tr}\left[U^k\right]$ are discarded, since we have seen that they are the Casimir operators of $\mathfrak{so}(9)$. Ignoring for the moment the traces $\mathrm{Tr}\left[R^k\right]$, we are led to the following operators

$$\begin{aligned}
\Theta^{[4,2]} &= 6\mathrm{Tr}\left[U^4R^2\right]+6\mathrm{Tr}\left[U^3RUR\right]+3\mathrm{Tr}\left[\left(U^2R\right)^2\right],\\
\Theta^{[2,4]} &= 6\mathrm{Tr}\left[U^2R^4\right]+6\mathrm{Tr}\left[URUR^3\right]+3\mathrm{Tr}\left[(UR)^2\right],\\
\Theta^{[6,2]} &= 8\mathrm{Tr}\left[U^6R^2\right]+8\mathrm{Tr}\left[U^5RUR\right]+8\mathrm{Tr}\left[U^4RU^2R\right]+4\mathrm{Tr}\left[\left(U^3R\right)^2\right],\\
\Theta^{[4,4]} &= 8\mathrm{Tr}\left[U^4R^4\right]+8\mathrm{Tr}\left[U^3RUR^3\right]+8\mathrm{Tr}\left[U^3R^2UR^2\right]+8\mathrm{Tr}\left[U^3R^3UR\right]\\
&\quad +8\mathrm{Tr}\left[U^2RU^2R^3\right]+8\mathrm{Tr}\left[U^2RURUR^2\right]+4\mathrm{Tr}\left[U^2R^2\left(UR\right)^2\right]\\
&\quad +8\mathrm{Tr}\left[U^2RUR^2UR\right]+2\mathrm{Tr}\left[(UR)^4\right],\\
\Theta^{[2,6]} &= 8\mathrm{Tr}\left[U^2R^6\right]+8\mathrm{Tr}\left[URUR^5\right]+8\mathrm{Tr}\left[UR^2UR^4\right]+8\mathrm{Tr}\left[\left(UR^3\right)^2\right],
\end{aligned}$$

as well as the degree twelve operators from (19). Observe that the bi-index $[p,q]$ of $\Theta^{[p,q]}$ denotes the degree in the $\mathfrak{so}(9)$ and representation generators respectively. Using analytical methods, it has been shown in general that operators of this kind arising from decomposed Casimir operators always provide subgroup scalars (see [20] and references therein). Therefore, the $\Theta^{[p,q]}$ constitute subgroup scalars for the reduction $\mathfrak{so}(9)\subset F_4$.

In order to be useful for the labelling of representations, we need $n=4$ subgroup scalars $\Theta^{[p,q]}$, which must commute with each other to avoid interaction. The problem of finding adequate combinations of labelling operators for general reduction of groups is still an unsolved one, although some criteria and methods have been developed to avoid direct computation [17, 21]. Even for degenerate representations[8], which usually require less labels, no general method has been developed yet.

However, for the case under analysis, the decomposition (15) and (18) provides some useful information to chose commuting labelling operators. To this extent, we recall how the commutator of two subgroup scalars of the type $\Theta^{[p,q]}$ decomposes as a sum of subgroup scalars

$$\left[\Theta^{[p,q]},\Theta^{[r,s]}\right]=\Theta^{[p+r-1,q+s]}+\Theta^{[p+r,q+s-1]}+\Theta^{[p+r+1,q+s-2]}. \tag{23}$$

In [20], the preceding formula (23) was used to analyse the precise structure of subgroup scalars. Two criteria for the commutativity of labelling operators were extracted, that we recall briefly:

Criterion A: If C_p decomposes as $C_p=\lambda\,\Theta^{[p,0]}+\Theta^{[p-\alpha,\alpha]}+\Theta^{[p-\beta,\beta]}+\Theta^{[p-\beta-2,\beta+2]}$ with $|\beta-\alpha|\leq 2$ and $\lambda=0,1$, then

$$\begin{aligned}\left[\Theta^{[p-\alpha,\alpha]},\Theta^{[p-\beta,\beta]}\right] &= \left[\Theta^{[p-\alpha,\alpha]},\Theta^{[p-\beta-2,\beta+2]}\right]\\ &= \left[\Theta^{[p-\beta,\beta]},\Theta^{[p-\beta-2,\beta+2]}\right]=0\,.\end{aligned} \tag{24}$$

In this criterion, operators of the type $\Theta^{[p,0]}$ simply correspond to Casimir invariants of the subalgebra.

Criterion B: Let $C_p=\Theta^{[p-\alpha,\alpha]}+\Theta^{[p-\beta,\beta]}+\Theta^{[p-\gamma,\gamma]}$ $(0\neq\alpha<\beta<\gamma)$ be a Casimir operator of $\mathfrak{s}$ with $\gamma-\alpha\geq 3$. If $\Theta^{[r,s]}$ is a subgroup scalar of $\mathfrak{s}'\subset\mathfrak{s}$ such that $\left[\Theta^{[r,s]},\Theta^{[p-\beta,\beta]}\right]=0$, then $\left[\Theta^{[r,s]},\Theta^{[p-\alpha,\alpha]}\right]=0$ and $\left[\Theta^{[r,s]},\Theta^{[p-\gamma,\gamma]}\right]=0$.

By (13), for any generator X of F_4 we have the identity

$$\left[X,\mathrm{Tr}\left[M^2\right]\right]=\left[X,\mathrm{Tr}\left[U^2\right]+\mathrm{Tr}\left[R^2\right]\right]=0\,, \tag{25}$$

as the Casimir operators commute with all generators. In particular, if $\Theta^{[p,q]}$ is a subgroup scalar of degree p in the $\mathfrak{so}\,(9)$ generators and q in the generators of the spinor representation, the previous identity implies that

$$\left[\Theta^{[p,q]},\mathrm{Tr}\left[M^2\right]\right]=\left[\Theta^{[p,q]},\mathrm{Tr}\left[R^2\right]\right]=0\,. \tag{26}$$

[8] According to [19], we call an IRREP degenerate if one or more of the Dynkin labels vanishes.

Now, as $\mathrm{Tr}\left[R^6\right] = \frac{1}{96}\mathrm{Tr}\left[R^2\right]^3$, $\mathrm{Tr}\left[R^8\right] = \frac{11}{6912}\mathrm{Tr}\left[R^2\right]^4$ and $\mathrm{Tr}\left[R^{12}\right] = \frac{19}{442368}\mathrm{Tr}\left[R^2\right]^6$, we obtain that for any subgroup scalar $\Theta^{[p,q]}$ the following identity holds[9]

$$\left[\Theta^{[p,q]}, \mathrm{Tr}\left[R^{2r}\right]\right] = 0\,, \qquad r = 1, 3, 4, 6\,. \tag{27}$$

It follows at once from Criterion A that

$$\left[\Theta^{[4,2]}, \mathrm{Tr}\left[M^6\right]\right] = \left[\Theta^{[4,2]}, \Theta^{[4,2]} + \Theta^{[2,4]}\right] = \left[\Theta^{[4,2]}, \Theta^{[2,4]}\right] = 0\,.$$

Similarly, taking into account the decomposition of the order eight Casimir operator $\mathrm{Tr}\left[M^8\right]$, and discarding the terms depending only on $\mathfrak{so}(9)$ or the representation generators, we obtain that

$$\left[\Theta^{[4,4]}, \mathrm{Tr}\left[U^8\right] + \Theta^{[6,2]} + \Theta^{[4,4]} + \Theta^{[2,6]} + \mathrm{Tr}\left[R^8\right]\right] = 0\,.$$

Since $\left[\Theta^{[4,4]}, \mathrm{Tr}\left[U^8\right]\right] = \left[\Theta^{[4,4]}, \mathrm{Tr}\left[R^8\right]\right] = 0$ by the preceding observations, we again conclude by Criterion A that

$$\left[\Theta^{[4,4]}, \Theta^{[6,2]}\right] = \left[\Theta^{[4,4]}, \Theta^{[2,6]}\right] = \left[\Theta^{[6,2]}, \Theta^{[2,6]}\right] = 0\,.$$

For $\mathrm{Tr}\left[M^{12}\right]$ the criterion is not directly applicable to the isolated subgroup scalars, although we obtain that

$$\left[\Theta^{[6,6]}, \Theta^{[10,2]} + \Theta^{[8,4]}\right] = \left[\Theta^{[6,6]}, \Theta^{[4,8]} + \Theta^{[2,10]}\right] = 0\,.$$

The commutators $\left[\Theta^{[6,6]}, \Theta^{[10,2]}\right] = \left[\Theta^{[6,6]}, \Theta^{[4,8]}\right] = 0$ must be computed directly. As we only need four (independent) labelling operators, we chose the simplest possible ones. Taking *e.g.*, $\left\{\Theta^{[4,2]}, \Theta^{[6,2]}, \Theta^{[4,4]}, \Theta^{[2,6]}\right\}$, it remains to see that $\left[\Theta^{[4,2]}, \Theta^{[4,4]}\right] = 0$ to conclude, by means of Criterion B, that

$$\left[\Theta^{[4,2]}, \Theta^{[6,2]}\right] = \left[\Theta^{[4,2]}, \Theta^{[2,6]}\right] = 0\,.$$

A lengthy direct computation shows that $\left[\Theta^{[4,2]}, \Theta^{[4,4]}\right] = 0$ actually holds. As a consequence, the four operators $\left\{\Theta^{[4,2]}, \Theta^{[6,2]}, \Theta^{[4,4]}, \Theta^{[2,6]}\right\}$ can be taken as labelling operators for the reduction $F_4 \supset \mathfrak{so}\,(9)$.

By (21), we need $f = 20$ internal labels to specify states in a IRREP of F_4. The trace method based on the polynomial matrix M has allowed us to obtain 16 of these 20 labels, divided into the four following types:

[9] The dependence of the previous traces on $\mathrm{Tr}\left[R^2\right]$ is the reason for which we left them out from the construction of labelling operators.

- the four Casimir operators $\mathrm{Tr}\left[M^2\right]$, $\mathrm{Tr}\left[M^6\right]$, $\mathrm{Tr}\left[M^8\right]$, $\mathrm{Tr}\left[M^{12}\right]$ of F_4,
- the four Casimir operators $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$, $\mathrm{Tr}\left[U^8\right]$, $\mathrm{Tr}\left[U^{12}\right]$ of $\mathfrak{so}(9)$,
- the four missing label operators $\Theta^{[2,4]}$, $\Theta^{[6,2]}$, $\Theta^{[4,4]}$ and $\Theta^{[2,6]}$,
- the four internal operators $H_1, \ldots, H_4$ of the Cartan subalgebra.

In general, for a generic irreducible representation we still need four internal subgroup operators, that can be obtained easily by the classical procedures, like the Gel'fand–Tseitlin method [11]. Observe that these internal labels automatically commute with the Casimir and labelling operators, as they are formed by generators of $\mathfrak{so}(9)$. As known, the four eigenvalues $\rho_1, \ldots, \rho_4$ of the Casimir operators $\mathrm{Tr}\left[M^2\right]$, $\mathrm{Tr}\left[M^6\right]$, $\mathrm{Tr}\left[M^8\right]$ and $\mathrm{Tr}\left[M^{12}\right]$ specify the IRREP $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ of F_4, while the remaining operators will be used to separate degeneracies and states [19, 20]. Now denote by:

- $\lambda_1, \ldots, \lambda_4$: eigenvalues of $\mathrm{Tr}\left[U^2\right]$, $\mathrm{Tr}\left[U^6\right]$, $\mathrm{Tr}\left[U^8\right]$ and $\mathrm{Tr}\left[U^{12}\right]$[10],
- $\xi_1, \ldots, \xi_4$: eigenvalues of $\Theta^{[2,4]}$, $\Theta^{[6,2]}$, $\Theta^{[4,4]}$ and $\Theta^{[2,6]}$,
- $\varphi_1, \ldots, \varphi_8$: eigenvalues of the internal operators $H_1, \ldots, H_4, \Phi_1, \ldots, \Phi_4$.

As these 20 operators commute with each other, and they are diagonalizable, we can always find a basis where they are all simultaneously diagonalizable [20]. As a consequence, the states of $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ are determined by a basis of eigenstates of the form

$$\left|\lambda_1, \ldots, \lambda_4;\ \xi_1, \ldots, \xi_4;\ \varphi_1, \ldots, \varphi_8\right\rangle . \tag{28}$$

This basis is valid for generic irreducible representations. For degenerate representations (or some multiplicity free reductions), some of the previous labels may be redundant [19], and the basis of eigenstates must be accordingly modified to avoid dependence of the operators. However, these special cases must be analysed case by case, as there are nowadays no general methods to conclude the dependence of labelling operators whenever we deal with degenerate IRREPs. Although some special cases have been analysed in detail [19], the state labelling for degenerate multiplets remains incomplete.

As an illustration of the procedure we consider the fundamental representation [1000] considered for the construction of the labelling operators. By the branching rule (4), there is no degeneracy, *i.e.*, the representations of $\mathfrak{so}(9)$ appearing in the decomposition all have multiplicity one. In this situation, according to [19], it is expected that some of the labelling operators will be dependent. Actually, for this case, the operators $\Theta^{[6,2]}, \Theta^{[4,4]}$ and $\Theta^{[2,6]}$ are all scalar multiples of $\Theta^{[2,4]}$, and thus redundant for the labelling of states. Only one labelling operator is needed, that we chose $\Theta^{[2,4]}$ for simplicity. As the representation is a fundamental one, only four of the eight

[10] Due to the dependence relation (20), we can also use the eigenvalue of C_4' instead of that of $\mathrm{Tr}\left[U^{12}\right]$.

internal labels are needed. Therefore, a basis of eigenstates is determined by

$$|\lambda_1,\lambda_2,\lambda_3,\lambda_4;\xi_1;\varphi_1,\varphi_2,\varphi_3,\varphi_4\rangle \ , \tag{29}$$

with the λ_i the eigenvalues of the $\mathfrak{so}\,(9)$ Casimir operators C'_{2i}, ξ_1 that of $\Theta^{[2,4]}$ and φ_i the eigenvalues of the Cartan generators, which suffice as internal labels. The eigenvalues of the operators are given in Table I.

TABLE I

Basis of eigenstates for $\Gamma=[1000]^\dagger$.

	H_1	H_2	H_3	H_4	C'_2	C'_4	C'_6	C'_8	$\Theta^{[2,4]}$
$\lvert v_1\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_2\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_3\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_4\rangle$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_5\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_6\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_7\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_8\rangle$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_9\rangle$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{10}\rangle$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{11}\rangle$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{12}\rangle$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{13}\rangle$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{14}\rangle$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{15}\rangle$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{16}\rangle$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	9	$\frac{537}{2}$	10539	5821461	$\frac{1753}{80}$
$\lvert v_{17}\rangle$	1	0	0	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{18}\rangle$	0	1	0	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{19}\rangle$	0	0	1	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{20}\rangle$	0	0	0	1	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{21}\rangle$	-1	0	0	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{22}\rangle$	0	-1	0	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{23}\rangle$	0	0	-1	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{24}\rangle$	0	0	0	-1	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{25}\rangle$	0	0	0	0	12	232	14568	932072	$\frac{349}{30}$
$\lvert v_{26}\rangle$	0	0	0	0	0	0	0	0	$\frac{39}{4}$

†As the eigenvalues of the Casimir operators of F_4 coincide for all states, we skip them from the orthonormal basis.

4. Conclusions

Using a convenient reformulation of the Gruber–O'Raifeartaigh method [9], we have extended the ansatz of [18] in order to obtain a trace formula that allows to express the Casimir operators of the exceptional Lie algebra F_4 as sums of the Casimir operators of the maximal subalgebra $\mathfrak{so}(9)$ and subgroup scalars for the reduction $F_4 \supset \mathfrak{so}(9)$. Basing on a decomposition of the Casimir operators induced by the embedding of the algebras, the labelling operators arise as traces of powers of the components of the decomposed matrix. Moreover, using some criteria on the commutativity of subgroup scalars, developed in [20], we were able to establish a set of commuting labelling operators for representations of F_4. A possible orthonormal basis of states constructed along these lines has been proposed, and for the IRREP [1000] of F_4, the basis of eigenstates has been explicitly constructed. Bases of this type can be useful in the context of branching rules and their applications to the atomic f-shell [22]. In fact, as the reduction considered here can be easily extended to the relevant chain $F_4 \supset \mathrm{SO}(9) \supset \mathrm{SO}(8)$ [7], the labelling operators obtained in the previous section can be further decomposed into SO(8) scalars, which may provide additional labels that recall atomic properties. The main difficulty at this stage is merely computational, due to the enormous number of terms of the labelling operators and the high dimension of F_4-representations exhibiting some degeneracy. Whether this approach is useful for the analysis of vanishing matrix elements [3,22] is still an unanswered question that deserves a more detailed study.

Another open question concerns the possibility of extending this procedure for the remaining exceptional Lie algebras of the E_i-series. The main constraints in this problem are the quite large dimension of the fundamental representations and the high degrees of the corresponding Casimir operators, which imply severe computational difficulties in the manipulation of traces of powers of matrices of type (2). A solution in this direction would also provide candidates for the state labelling problem, as well as provide additional structural tools for their applications to the unified gauge theories [4,5].

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