ON THE PERMUTATIONAL SYMMETRY OF THE HUBBARD MODEL

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The application of the Jucys–Murphy operators, generating a maximal Abelian subalgebra in the group algebra of the symmetric group, in the process of immediate diagonalisation of the one-dimensional Hubbard Hamiltonian is demonstrated. The way of construction of appropriate projection operators of the Young orthogonal basis is pointed out, and the fact that these operators play a role of eigenvectors for Jucys–Murphy operators in the group algebra of the symmetric group is underlined. It is indicated that this operator technique is competitive to the Kostka matrix at the level of bases, which yields matrices of appropriate Clebsh–Gordan coefficients. The permutational symmetry of the lattice chain with N sites occupied by two electrons is discussed in detail.

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1. Introduction

The single-band Hubbard model [1] allows to understand many interesting phenomena of the solid state physics like ferromagnetism, antiferromagnetism, the Mott transition, *etc.* That is why many research groups from over the world are still involved in this bottomless topic. The strongly developing theory of quantum dots and generally nanostructures of properties very similar to the finite set of objects, induces to look closer on the models simulating such systems. In the present paper we consider one-dimensional Hubbard model, which is great for its possible generalizations to higher dimensions, and as one of the exactly solvable models [2, 3]. Lieb and Wu [4] solved the one-dimensional Hubbard model in the year 1968, and after them

many authors have tried to improve the theory with the book of Essler *et al.* [5] as the proper summary and supplement of their work. Although many highly sophisticated papers of tremendous importance were published in this area, the problem is still far from full understanding. In the present paper we give the application of the Jucys–Murphy operators [6, 7, 8], and the Schur–Weyl transform [9, 10, 11] in one-dimensional Hubbard model, resulting in reducing significantly the size of an effective Hamiltonian matrix through dividing it into blocks related with given permutational symmetry of the nodes of the one-dimensional lattice chain.

2. The symmetries of the model

The one-dimensional Hubbard model of a finite chain of N atoms has dynamics given by the Hubbard Hamiltonian in the following form

$$\hat{H} = -t \sum_{i \in \tilde{2}} \sum_{j \in \tilde{N}} \left(\hat{a}^{\dagger}_{ji} \hat{a}_{j+1i} + \hat{a}^{\dagger}_{j+1i} \hat{a}_{ji} \right) + U \sum_{j \in \tilde{N}} \hat{n}_{j+} \hat{n}_{j} \,, \tag{1}$$

where $\tilde{N} = \{j = 1, 2, \ldots, N\}$ denotes the set of nodes of the chain, $\tilde{2} = \{i = +, -\}$, $\hat{n}_{ji} = \hat{a}^{\dagger}_{ji} \hat{a}_{ji}$, and finally $\hat{a}^{\dagger}_{ji}$, $\hat{a}_{ji}$ are the canonical Fermi operators, that is creation and anihilation operators of electron of spin i, on the site j. The Hamiltonian (1) apart from the obvious translational symmetry ($\hat{a}_{N+1i} = \hat{a}_{1i}$) has two independent SU(2) symmetries [5, 12, 13], that is SU(2) $\times$ SU(2), involving the spin and charge degrees of freedom. There are many ways of constructing the Hilbert space $\mathcal{H}$ for N_e particles, moving along the ring with N nodes. One can start from defining the *single-node* space h_j, with the basis consisting of the vectors denoting all possible occupations of one node. Since we are dealing with fermions

$$\dim h_j = 4 \,, \qquad h_j = lc_{\mathbb{C}}\{\emptyset, +, -, \pm\} \,, \tag{2}$$

where $\emptyset$ denotes the empty node, $+$ and $-$ stand for one-node spin projection equal to $1/2$ and $-1/2$ respectively, $\pm$ denotes the double occupation of the one node by the two electrons with different spin projections, and $lc_{\mathbb{C}}A$ stands for the linear closure of a set A over the field $\mathbb{C}$. One can obtain the final space $\mathcal{H}$ of all quantum states of the system in the following way

$$\mathcal{H} = \prod_{j=1}^{N} \otimes h_j \,, \qquad \mathcal{H} = \sum_{N_e=0}^{2N} \oplus \mathcal{H}^{N_e} \,, \tag{3}$$

where $\mathcal{H}^{N_e}$ denotes the space with fixed number of electrons N_e. The set $\tilde{4} = \{\emptyset, +, -, \pm\}$ can be decomposed into two parts, where first, being the set $\tilde{2} = \{+, -\}$, is related with the first factor of the cartesian product

$\mathrm{SU}(2)\times\mathrm{SU}(2)$ of the symmetry groups of the system, reflecting the invariance of $\hat{H}$ under the spin rotation, and the second $\tilde{2}' = \{\emptyset, \pm\}$ is related with the remaining factor. Thus one has two sets of generators, $\{\hat{S}_z, \hat{S}^+, \hat{S}^-\}$ and $\{\hat{J}_z, \hat{J}^+, \hat{J}^-\}$, for spin and charge space, respectively. These generators can be written in the following forms

$$\hat{S}_z = \tfrac{1}{2}\sum_{j\in\tilde{N}}\left(\hat{a}^\dagger_{j+}\hat{a}_{j+} - \hat{a}^\dagger_{j-}\hat{a}_{j-}\right), \qquad \hat{S}_+ = \hat{S}^\dagger_- = \sum_{j\in\tilde{N}}\hat{a}^\dagger_{j+}\hat{a}_{j-}, \tag{4}$$

$$\hat{J}_z = \tfrac{1}{2}\sum_{j\in\tilde{N}}\left(\hat{a}^\dagger_{j+}\hat{a}_{j+} + \hat{a}^\dagger_{j-}\hat{a}_{j-} - 1\right), \qquad \hat{J}_+ = \hat{J}^\dagger_- = (-1)^j\sum_{j\in\tilde{N}}\hat{a}^\dagger_{j+}\hat{a}_{j-}, \tag{5}$$

and the transfer between these two sets is known as the Shiba transformation [4, 5]. The total number of particles N_e, taken as the eigenvalue of the operator $\hat{N}_e = \sum_{j\in\tilde{N}}(\hat{n}_{j+} + \hat{n}_{j-})$, together with the number of particles N_i, with given one-node spin projection $i \in \{+, -\}$, taken as the eigenvalues of the operators $\hat{N}_i = \sum_{j\in\tilde{N}}\hat{n}_{ji}$, are the good quantum numbers, this mean the conservation of the total magnetization M given as the eigenvalue of the operator $\hat{M} \equiv \hat{S}_z$. As the result, we have four good quantum numbers $(S_z = M, J_z, S, J)$ related with the operators from Eqs. (4), (5).

3. The electron configuration

The *electron configuration* $f : \tilde{N} \longrightarrow \tilde{4}$, with $\tilde{4} \equiv \{\emptyset, +, -, \pm\}$, is the N-sequence of the elements from the set $\tilde{4}$

$$|f\rangle = |f(1)f(2)\dots f(N)\rangle = |i_1 i_2 \dots i_N\rangle, \qquad i_j \in \tilde{4}, \qquad j \in \tilde{N}, \tag{6}$$

with

$$\tilde{4}^{\tilde{N}} = \left\{f : \tilde{N} \longrightarrow \tilde{4}\right\}, \qquad \mathcal{H} = lc_{\mathbb{C}}\tilde{4}^{\tilde{N}}. \tag{7}$$

The action $A : \Sigma_N \times \tilde{4}^{\tilde{N}} \longrightarrow \tilde{4}^{\tilde{N}}$ of the symmetric group Σ_N on the set $\tilde{4}^{\tilde{N}}$ provides us with orbits $\mathcal{O}_\mu$ of the group Σ_N labelled by the *weight* μ, which is a sequence of non-negative integers $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, with relation $\sum_{i\in\tilde{4}}\mu_i = N$, defined by the following equation

$$\mu_i = \left|\left\{i_j = i | j \in \tilde{N}\right\}\right|, \qquad i \in \tilde{4}. \tag{8}$$

Such an orbit constitutes the transitive representation $R^{\Sigma_N:\Sigma^\mu}$ of the group Σ_N with the stabilizer

$$\Sigma^\mu = \Sigma_{\mu_1} \times \Sigma_{\mu_2} \times \Sigma_{\mu_3} \times \Sigma_{\mu_4}, \tag{9}$$

given as the Young subgroup for the appropriate $f \in \mathcal{O}_\mu$. The *stratum* $S(\nu)$ of the action A, is labelled by the sequence ν

$$\nu = (\nu_0, \nu_1, \nu_2, \ldots, \nu_N)\,, \qquad \sum_{l=0}^{N} l\nu_l = N\,, \tag{10}$$

where

$$\nu_l = \left|\left\{\mu_i = l | i \in \tilde{4}\right\}\right|\,, \qquad l \in \tilde{N}\,, \tag{11}$$

and

$$\nu_0 = 4 - \sum_{l \in \tilde{4}} \nu_l\,. \tag{12}$$

The stratification of the set $\tilde{4}^{\tilde{N}}$ of all electron configurations under the action A of the symmetric group is the following

$$\frac{\tilde{4}^{\tilde{N}}}{A} = \bigcup_{\nu} S(\nu) \tag{13}$$

with

$$|S(\nu)| = \frac{4!}{\nu_0! \prod_{l \in \tilde{N}} \nu_l!}\,, \qquad |\mathcal{O}_\mu| = \frac{N!}{\prod_{i \in \tilde{4}} \mu_i!}\,, \qquad 4^N = \sum_{\nu} |S(\nu)| \cdot |\mathcal{O}_\mu|\,. \tag{14}$$

The stratification of the set $\tilde{4}^{\tilde{N}}$ under the action of the symmetric group Σ_N, for the chain with $N = 4$ nodes is presented in Table I.

TABLE I

The stratification of the set $\tilde{4}^{\tilde{4}}$ of all electron configurations under the action of the symmetric group Σ_4.

μ				ν					$\mid \mathcal{O}_\mu \mid$	$\mid S(\nu) \mid$	$\mid S(\nu) \mid \cdot \mid O_\mu \mid$
				0	1	2	3	4			
4	0	0	0	3	0	0	0	1	1	4	4
3	1	0	0	2	1	0	1	0	4	12	48
2	2	0	0	2	0	2	0	0	6	6	36
2	1	1	0	1	2	1	0	0	12	12	144
1	1	1	1	0	4	0	0	0	24	1	24
											$4^4 = 256$

4. Jucys–Murphy operators

The operators $\hat{M}_j$ [14, 15], defined within the symmetric group algebra $\mathbb{C}[\Sigma_N]$ as the sum of all transpositions (j, j') of the node $j \in \tilde{N}$ with preceding nodes $j' < j$, are introduced by Jucys [6, 7] and independently by Murphy [8], thus they are called Jucys–Murphy operators. These $N-1$ Hermitian operators of the form

$$\hat{M}_j = \sum_{1 \le j' < j} (j, j') \,, \qquad j = 2, 3, \ldots, N \,, \tag{15}$$

constitute a complete set of commuting operators along the general Dirac formalism of quantum mechanics [16]. The standard Young tableaux $|\lambda\, y\rangle$ [17] of the shape $\lambda \vdash N$, *i.e.* the tableaux of this shape in the alphabet $\tilde{N}$ of nodes, with strictly increasing entries in rows and columns, constitutes the common eigenvector $|\lambda\, y\rangle$ of the set of $\hat{M}_j$ operators [18, 19], that is

$$\hat{M}_j \, |\lambda\, y\rangle = m_j(y) \, |\lambda\, y\rangle \ , \tag{16}$$

with eigenvalues

$$m_j(y) = c_j(y) - r_j(y) \,, \tag{17}$$

where the pair of positive integers $(c_j(y), r_j(y))$ gives the position (the column and the row) of the number j in tableaux $|\lambda\, y\rangle$. In this, way each basis function of the irreducible representation (irrep) Δ^λ of the symmetric group Σ_N, labelled by the Young tableaux $|\lambda\, y\rangle$, can be completely determined by the sequence $\boldsymbol{m}(y) = (m_1 = 0, m_2, \ldots, m_N)$ of eigenvalues (17). The realization of each such irreducible vector within the group algebra $\mathbb{C}[\Sigma_N]$ is given via the projector operator e^λ_{yy} of the well known Young orthogonal basis [6, 7, 8, 10, 11], where the remarkable significance of the Jucys–Murphy operators is underlined.

5. The irreducible basis

Now we are going to construct the multi-electron basis in $\mathcal{H}^{N_e}$, reflecting the permutational symmetry of the N nodes of the lattice chain, using the theory of representations of the symmetric group Σ_N. This technique reduced significantly the size of the Hubbard Hamiltonian by decomposing it into blocks. We start with only one electron on the ring, and with the initial basis of $\mathcal{H}^{N_e=1}$ being the orbits $O_{\mu=(3100)}$ and $O_{\mu=(3010)}$ of the symmetric group Σ_N, with elements taken as the subsets of the set $\tilde{4}^{\tilde{N}}$ of all electron configurations. Such orbits are invariant under the action of the symmetric group Σ_N and form the carrier spaces of the transitive representations $R^{\Sigma_N : \Sigma^\mu}$, with the stabilizers Σ^μ being the Young subgroups (9), and

$\times$ denotes the Cartesian product. One can obtain the states with definite permutational symmetry of the N_e particles by taking the irreducible basis of the appropriate irreps Δ^λ of the symmetric group Σ_N, where $\lambda \vdash N$, in the tensor product of N_e transitive representations $R^{\Sigma_N:\Sigma^\mu}$, along with the appropriate decomposition

$$R^{\Sigma_N:\Sigma^\mu} \cong \sum_{\lambda \trianglerighteq \mu} K_{\lambda\,\mu}\, \Delta^\lambda\,. \tag{18}$$

The $K_{\lambda\mu}$ denotes Kostka numbers [20], and the sum runs over all partitions λ greater than, or equal to μ in the dominance order [21]. Combinatorially, $K_{\lambda\mu}$ denotes the number of all standard Young tableaux of the shape λ and weight μ. Decomposition (18) can be written at the level of bases [10,11] in the form

$$|\mu\,\lambda\,t\,y\rangle = \sum_{f\in\mathcal{O}_\mu} \begin{bmatrix} \mu & \lambda & t \\ f & y & \end{bmatrix} |f\rangle\,, \qquad y \in \mathrm{SYT}(\lambda)\,, \qquad t \in \mathrm{WT}(\lambda\tilde{n})\,, \tag{19}$$

where $\mathrm{SYT}(\lambda)$ denotes a standard basis vectors of the irreps of the symmetric group, that is the set of all standard Young tableaux of the shape λ, $\mathrm{WT}(\lambda,\tilde{n})$ denotes a standard basis vectors of the irreps of the unitary group $\mathrm{U}(n)$, that is the set of all Weyl tableaux (or semi-standard Young tableaux) of the shape λ in the alphabet $\tilde{n}$, *i.e.* with entries increasing weakly in rows, and strictly in columns. The symbols in square brackets denote an element of the Kostka matrix, with μ denoting the matrix, f its rows, and the triple $(\lambda\,t\,y)$ its columns. That is the way of finding the irreducible basis of the symmetric and the unitary groups for the case with any dimension n of single-node space h_j and any number of electrons N_e. The equally efficient way for providing such basis leads through using the Jucys–Murphy operators, what we present in details on the example of $N_e = 2$ and $N = n = 4$. In order to obtain Young orthogonal basis [6, 7, 8] of the tensor product of the appropriate transitive representations (we will omit the zeros in the weight μ)

$$\left(R^{\Sigma_N:\Sigma^{(N-1,1)}}\right)^{\otimes 2}\,, \tag{20}$$

with the decomposition into irreps Δ^λ given by the Eq. (18), we are going to create the projection operators

$$e^\lambda_{yy} = |\lambda w y\rangle\langle\lambda w y| \tag{21}$$

in the space of the tensor product $\mathcal{H}^2 = lc_{\mathbb{C}}|\lambda_1\lambda_2\lambda w y\rangle$, where w denote appropriate repetition label. Thus we have

$$e^{\lambda}_{yy} = \prod_{j=2}^{N} \prod_{\{y_{j-1}|y^{+}_{j-1}\neq y_j\}} \frac{\hat{M}_j - m_j\left(y^{+}_{j-1}\right)\hat{I}}{m_j(y) - m_j\left(y^{+}_{j-1}\right)}\,, \tag{22}$$

where $y \in \mathrm{SYT}(\lambda)$, y_j denotes the tableaux obtained from y by extracting the set $\{j+1, j+2, \dots N\}$ of numbers, y^{+}_{j-1} is created from y_{j-1} by adding to its entries the number j in a regular way, and $\hat{I}$ stands for the appropriate unit operator. Let us briefly remind that within the group algebra $\mathbb{C}[\Sigma_N]$, spanned linearly on $N!$ Young operators $e^{\lambda}_{yy'}$, $\lambda = shy = shy'$, each eigenvalue $m_j(y)$ of the Jucys–Murphy operators is realised in terms of the corresponding operator e^{λ}_{yy}. This operators satisfy

$$e^{\lambda}_{y_1y_2} e^{\lambda'}_{y'_1y'_2} = \delta_{\lambda\lambda'}\delta_{y_2y'_1} e^{\lambda}_{y_1y'_2}\,, \tag{23}$$

$$\sigma e^{\lambda}_{y_1y_2} = \sum_{y'_1\in \mathrm{SYT}(\lambda)} \Delta^{\lambda}_{y'_1y_1}(\sigma) e^{\lambda}_{y'_1y_2}\,, \qquad \sigma \in \Sigma_N\,, \tag{24}$$

and

$$\hat{M}_j e^{\lambda}_{yy} = m_j(y) e^{\lambda}_{yy}\,. \tag{25}$$

Eq. (23) implies that

$$\mathbb{C}[\Sigma_N] \cong \sum_{\lambda\vdash N} \mathrm{gl}\left(\dim \Delta^{\lambda}, \mathbb{C}\right)\,, \tag{26}$$

i.e. the group algebra $\mathbb{C}[\Sigma_N]$ is the direct sum of simple matrix algebras $\mathrm{gl}(\dim \Delta^{\lambda}, \mathbb{C})$, corresponding to each irrep Δ^{λ} of Σ_N, with the bases $e^{\lambda}_{yy'}$. Eq. (24) displays the fact that each set $\{e^{\lambda}_{y_1y_2} | y_1 \in \mathrm{SYT}(\lambda)\}$ with fixed tableau y_2 spans a carrier space of the irrep Δ^{λ} when Σ_N acts on $\mathbb{C}[\Sigma_N]$ by the left multiplication, so that y_1 and y_2 in $e^{\lambda}_{y_1y_2}$ serve as labels of the standard irreducible basis and multiplicity, respectively. Eq. (25) proves that each diagonal element e^{λ}_{yy}, $\lambda \vdash N$, $y \in \mathrm{SYT}(\lambda)$, is the eigenvector of all Jucys–Murphy operators, specified by the sequence $\boldsymbol{m}(y) = (m_1, m_2 \dots m_N)$ given by Eq. (17). At the same time, it is a representative element of the group algebra $\mathbb{C}[\Sigma_N]$.

6. The example

Tables II–IV present the decompositions of the irreducible basis for the case $N = 4$ and $N_e = 2$ onto the electron configurations f for the tensor product

$$R^{\Sigma_4:\Sigma^{(3,1)}} \otimes R^{\Sigma_4:\Sigma^{(3,1)}} = R^{\Sigma_4:\Sigma^{(3,1)}} \oplus R^{\Sigma_4:\Sigma^{\left(2,1^2\right)}} \tag{27}$$

of the transitive representations with appropriate decomposition into irreducible representations Δ^{λ} given as follows

$$R^{\Sigma_4:\Sigma^{(3,1)}} = \Delta^{\{4\}} \oplus \Delta^{\{3,1\}}\,, \tag{28}$$

and

$$\left(\Delta^{\{4\}} \oplus \Delta^{\{3,1\}}\right) \otimes \left(\Delta^{\{4\}} \oplus \Delta^{\{3,1\}}\right) = 2\Delta^{\{4\}} \oplus 3\Delta^{\{3,1\}} \oplus \Delta^{\{2^2\}} \oplus \Delta^{\{2,1^2\}}\,. \tag{29}$$

TABLE II

The irreducible basis of the representation $\Delta^{\{3,1\}}$ in the tensor product space H^2.

f	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1} \end{array}$ *sd*	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1} \end{array}$ *st*	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1} \end{array}$ *a*	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1} \end{array}$ *sd*	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1} \end{array}$ *st*	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1} \end{array}$ *a*
$\pm\,\emptyset\,\emptyset\,\emptyset$	$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{6}}$	0	0
$\emptyset\,\pm\,\emptyset\,\emptyset$	$\frac{-1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{6}}$	0	0
$\emptyset\,\emptyset\,\pm\,\emptyset$	0	0	0	$\frac{-\sqrt{(2)}}{\sqrt{3}}$	0	0
$\emptyset\,\emptyset\,\emptyset\,\pm$	0	0	0	0	0	0
$+\,-\,\emptyset\,\emptyset$	0	0	$\frac{1}{2}$	0	$\frac{1}{\sqrt{6}}$	0
$+\,\emptyset\,-\,\emptyset$	0	$\frac{1}{2\sqrt{2}}$	$\frac{1}{4}$	0	$\frac{-1}{2\sqrt{6}}$	$\frac{-\sqrt{3}}{4}$
$+\,\emptyset\,\emptyset\,-$	0	$\frac{1}{2\sqrt{2}}$	$\frac{1}{4}$	0	$\frac{1}{2\sqrt{6}}$	$\frac{-1}{4\sqrt{3}}$
$-\,+\,\emptyset\,\emptyset$	0	0	$\frac{-1}{2}$	0	$\frac{1}{\sqrt{6}}$	0
$\emptyset\,+\,-\,\emptyset$	0	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{4}$	0	$\frac{-1}{2\sqrt{6}}$	$\frac{-\sqrt{3}}{4}$
$\emptyset\,+\,\emptyset\,-$	0	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{4}$	0	$\frac{1}{2\sqrt{6}}$	$\frac{-1}{4\sqrt{3}}$
$-\,\emptyset\,+\,\emptyset$	0	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{4}$	0	$\frac{-1}{2\sqrt{6}}$	$\frac{\sqrt{3}}{4}$
$\emptyset\,-\,+\,\emptyset$	0	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{4}$	0	$\frac{-1}{2\sqrt{6}}$	$\frac{\sqrt{3}}{4}$
$\emptyset\,\emptyset\,+\,-$	0	0	0	0	$\frac{-1}{\sqrt{6}}$	$\frac{1}{2\sqrt{3}}$
$-\,\emptyset\,\emptyset\,+$	0	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{4}$	0	$\frac{1}{2\sqrt{6}}$	$\frac{1}{4\sqrt{3}}$
$\emptyset\,-\,\emptyset\,+$	0	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{4}$	0	$\frac{1}{2\sqrt{6}}$	$\frac{1}{4\sqrt{3}}$
$\emptyset\,\emptyset\,-\,+$	0	0	0	0	$\frac{-1}{\sqrt{6}}$	$\frac{-1}{2\sqrt{3}}$

The irreducible basis can be marked by additional repetition label due to the symmetry of the system of two particles. The states related with the first component of the r.h.s. of the tensor product in decomposition (27) describe the situation when two particles occupy the same atom (repetition label *sd* in Tables II and III). The transitive representation $R^{\Sigma_4:\Sigma^{(2,1^2)}}$ contains the

symmetric (repetition label *st* in Tables II and III)

$$\left(R^{\Sigma_4:\Sigma^{(2,1^2)}}\right)_{st} = \Delta^{\{4\}} \oplus \Delta^{\{3,1\}} \oplus \Delta^{\{2^2\}}\,, \tag{30}$$

and the antisymmetric (repetition label *a* in Tables II and III)

$$\left(R^{\Sigma_4:\Sigma^{(2,1^2)}}\right)_{a} = \Delta^{\{3,1\}} \oplus \Delta^{\{2,1^2\}}\,, \tag{31}$$

part of the tensor product (27).

TABLE III

The irreducible basis of the representation $\Delta^{\{3,1\}}$ and $\Delta^{\{4\}}$ in the tensor product space H^2.

f	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\,sd$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\,st$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}\,a$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\\hline\end{array}\,sd$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\\hline\end{array}\,st$
$\pm\,\emptyset\,\emptyset\,\emptyset$	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{2}$	0
$\emptyset\,\pm\,\emptyset\,\emptyset$	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{2}$	0
$\emptyset\,\emptyset\,\pm\,\emptyset$	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{2}$	0
$\emptyset\,\emptyset\,\emptyset\,\pm$	$\frac{-\sqrt{3}}{2}$	0	0	$\frac{1}{2}$	0
$+\,-\,\emptyset\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$+\,\emptyset\,-\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$+\,\emptyset\,\emptyset\,-$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$
$-\,+\,\emptyset\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,+\,-\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,+\,\emptyset\,-$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$
$-\,\emptyset\,+\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,-\,+\,\emptyset$	0	$\frac{1}{2\sqrt{3}}$	0	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,\emptyset\,+\,-$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$
$-\,\emptyset\,\emptyset\,+$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,-\,\emptyset\,+$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$
$\emptyset\,\emptyset\,-\,+$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$	0	$\frac{1}{3\sqrt{2}}$

First four rows of the Tables II–IV are labelled by electron configurations which do not form the linear combinations with the twelve left. They contribute to the symmetric part of the tensor product (27), and can be treated separately like four atoms with single-node basis given by the set $\tilde{2}' = \{\pm, \emptyset\}$ with the configurations of the weight $\mu = (1, 3)$.

TABLE IV

The irreducible basis of the representation $\Delta^{\{2^2\}}$ and $\Delta^{\{2,1^2\}}$ in the tensor product space H^2.

f	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$
$\pm\,\emptyset\,\emptyset\,\emptyset$	0	0	0	0	0
$\emptyset\,\pm\,\emptyset\,\emptyset$	0	0	0	0	0
$\emptyset\,\emptyset\,\pm\,\emptyset$	0	0	0	0	0
$\emptyset\,\emptyset\,\emptyset\,\pm$	0	0	0	0	0
$+\,-\,\emptyset\,\emptyset$	0	$\frac{1}{\sqrt{6}}$	0	$\frac{1}{2\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$
$+\,\emptyset\,-\,\emptyset$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{1}{4}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$+\,\emptyset\,\emptyset\,-$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{-1}{4}$	$\frac{-\sqrt{3}}{4}$	0
$-\,+\,\emptyset\,\emptyset$	0	$\frac{1}{\sqrt{6}}$	0	$\frac{-1}{2\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\emptyset\,+\,-\,\emptyset$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{1}{4}$	$\frac{-1}{4\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$
$\emptyset\,+\,\emptyset\,-$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{-1}{4}$	$\frac{\sqrt{3}}{4}$	0
$-\,\emptyset\,+\,\emptyset$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{-1}{4}$	$\frac{-1}{4\sqrt{3}}$	$\frac{-1}{\sqrt{6}}$
$\emptyset\,-\,+\,\emptyset$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{-1}{4}$	$\frac{1}{4\sqrt{3}}$	$\frac{1}{\sqrt{6}}$
$\emptyset\,\emptyset\,+\,-$	0	$\frac{1}{\sqrt{6}}$	$\frac{1}{2}$	0	0
$-\,\emptyset\,\emptyset\,+$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{1}{4}$	$\frac{\sqrt{3}}{4}$	0
$\emptyset\,-\,\emptyset\,+$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{6}}$	$\frac{1}{4}$	$\frac{-\sqrt{3}}{4}$	0
$\emptyset\,\emptyset\,-\,+$	0	$\frac{1}{\sqrt{6}}$	$\frac{-1}{2}$	0	0

7. Final remarks and conclusions

We have provided the efficient way of construction of the multi-electron, orthonormal basis appropriate for the one-dimensional Hubbard model, using the Young idempotents expressed by the Jucys–Murphy operators. We have found this method as being competitive to the Kostka matrix at the level of bases. This approach reduces the size of the Hubbard Hamiltonian, decomposing it into blocks which consist of some irreps of the permutational symmetry group Σ_N.

The matrix of the Hamiltonian given by the Eq. (1) for the case of general N, $N_e = 2$ with $N_+ = N_- = 1$, in the irreducible basis of the symmetric group Σ_N, is quasidiagonal as expected. For example, of $N = 4$, (basis given by the Tables II–IV), that is for the pair (S_z, J_z) equal to $(0, -1)$, it turns out to have four blocks on the diagonal. The first block is determined by the states of the permutational symmetry given by the Young tableaux of the shapes $\lambda = \{4\}$ with the repetition labels sd and st, and $\lambda = \{2^2\}$.

The symmetry of the second block is given by the shape $\lambda = \{31\}$ with the repetition labels *sd* and *st*, the third by $\lambda = \{31\}$ too, but with the anti-symmetric ones, and the fourth by $\lambda = \{21^2\}$. The case with two electrons of the same spin projection, that is $N = 4$, $N_e = 2 = N_+$, $(S_z, J_z) = (1, -1)$ (or $N_e = 2 = N_-$, $(S_z, J_z) = (-1, -1)$), is reducible to blocks on the diagonal by using vectors from Tables II and III with the repetition label *st* only. The first block is determined by the states of the permutational symmetry given by the Young tableaux of the shapes $\lambda = \{4\}$ with the repetition label *st*, and $\lambda = \{2^2\}$, the symmetry of the second block is given by the shape $\lambda = \{31\}$ with the label *st* too.

REFERENCES

[1] J. Hubbard, *Proc. R. Soc.* **A276**, 238 (1963).

[2] B. Sutherland, *Beautiful Models, 70 Years of Exactly Solved Quantum Many-Body Problems*, World Scientific Publishing Co. Pte. Ltd., Singapore 2005.

[3] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London 1982.

[4] E.H. Lieb, F. Wu, *Phys. Rev. Lett.* **20**, 1445 (1968).

[5] F.H.L. Essler *et al.*, *The One-Dimensional Hubbard Model*, Cambridge University Press, 2005.

[6] A.A. Jucys, *Lietuvos Fizikos Rinkinys* **6**, 163 (1966).

[7] A.A. Jucys, *Lietuvos Fizikos Rinkinys* **11**, 5 (1971).

[8] G.E. Murphy, *J. Algebra* **81**, 258 (1983).

[9] D. Bacon, I.L. Chuang, A.W. Harrow, *Phys. Rev. Lett.* **97**, 170502 (2006).

[10] B. Lulek, T. Lulek, P. Jakubczyk, D. Jakubczyk, *Physica B* **404**, 115 (2009).

[11] P. Jakubczyk, D. Jakubczyk, T. Lulek, *Phys. Status Solidi* **B246**, 1029 (2009).

[12] V.E. Korepin, F.H.L. Essler, *NATO ASI Series B: Physics* **343**, 71 (1993).

[13] M. Noga, *Czech. J. Phys.* **43**, 1223 (1993).

[14] A. Okounkov, A. Vershik, *Selecta Mathematica* **2**, 581 (1996).

[15] A. Lascoux, in: *Symmetry and Structural Properties of Condensed Matter*, Eds. T. Lulek, B. Lulek, A. Wal, World Sci., Singapore 2000, p. 94.

[16] P.A.M. Dirac *The Principles of Quantum Mechanics*, Clarendon Press, Oxford 1930.

[17] B. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer-Verlag, New York 2001.

[18] Cheng-Jiu Zhu, Jin-Quan Chen, *J. Math. Phys.* **24**, 2266 (1983).

[19] Dian-Min Tong, Cheng-Jiu Zhu, Zhong-Qi Ma, *J. Math. Phys.* **33**, 2660 (1992).

[20] C. Kostka, *Crelles Journal* **93**, 89 (1882).

[21] A. Kerber, *Algebraic Combinatorics via Finite Group Action* Wissenschaftsverlag, Manheim 1991.