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SPATIAL SOLITONS IN NONLINEAR SCHRÖDINGER EQUATION WITH VARIABLE NONLINEARITY AND QUADRATIC EXTERNAL POTENTIAL

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An improved self-similar transformation is used to construct exact solutions of the nonlinear Schrödinger equation with variable nonlinearity and quadratic external potential, which both depend on the distance of propagation and the transverse spatial coordinate. By means of analytical and numerical methods we reveal the main features of the spatial solitons found. We focus on the most important optical examples, where the applied optical field is a function of both linearly or periodically varying distance and spatial coordinate. In the case of periodically varying nonlinearity, the variations of confining external potential are found to be sign-reversible (periodically attractive and repulsive) and thus supporting the soliton management.

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1. Introduction

Spatial optical solitons are self-trapped light beams supported by the balance between diffraction and nonlinearities of various types [1, 2, 3], which in most cases are produced by the refractive index modification in the material. Depending on the type of nonlinearity, nonlinear (NL) media may

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support either bright or dark solitons [1]. While bright solitons are just finite-size beams formed in media with self-focusing nonlinearity, dark solitons are more complex objects, representing an intensity dip in an otherwise constant background with nontrivial phase profile [4]. Spatial solitons have potential for use in all-optical data-processing schemes designed for switching, pattern recognition, and parallel information storage [1, 4]. Nonlinear Schrödinger (NLS) equation possesses practical interest, since it appears in many branches of physics, such as NL optics, nuclear physics, and Bose–Einstein condensates (BECs). In NL optics it describes the propagation of pulses in optical fibers [5]. In BECs it models the condensate wavefunction [6, 7, 8], once the two-body interactions are taken into account. The three-body interactions between cold atoms, in the form of Efimov resonances, have also been observed in an ultracold gas of cesium atoms [6].

In NL optics, the construction of exact solutions for various NLS equations is one of the most important and essential tasks. With the help of exact solutions one can better understand the phenomena modeled by these equations, such as the propagation and stability of optical solitons. In recent years, many powerful methods for constructing exact analytical solutions have been proposed, such as the inverse scattering transform [9,10], the Hirota's binary operator approach [11,12], the Bäcklund transformation [13], and the truncated Painlevé expansion [14]. With the advent of symbolic computation algorithms, the methods of direct algebraic manipulation have become feasible. New powerful methods of solution have been invented, such as the homogeneous balance principle [15], the F-expansion technique [16], the hyperbolic tangent expansion method [17], the generalized Riccati equation method [18], and the Jacobi elliptic function (JEF) expansion method [19], among other. Very recently, for the NLS equation with variable coefficients depending on the propagation distance, we have obtained exact solutions by the self-similar method [20, 21, 22] and the homogeneous balance and F-expansion technique [23, 24, 25, 26].

In this paper we study spatial solitary waves in a generic Kerr medium with variable nonlinearity and quadratic external potential. They describe the propagation of optical pulses when the nonlinearity coefficient and the external potential are functions of both the propagation distance and the spatial coordinate. We solve the corresponding NLS equation for the dynamics of localized waves using the similarity transformation, which is suitable for finding analytical solutions of such NL optical systems. We discover a variety of localized solutions which describe, for example, physically important applications of amplification and compression of pulses in NL optics. For periodically varying nonlinearity, sign-reversible variations of the confining external potential are found, which are useful for soliton management.

The paper is organized as follows. Section 2 introduces the NLS equation with variable nonlinearity and quadratic external potential, and presents the solution method for the problem. Section 3 analyses different types of spatial solitons obtained. Section 4 gives the conclusions.

2. The solution method

Optical wave propagation in Kerr medium with variable nonlinearity and external potential in $(1+1)$ dimensions $[(1+1)\mathrm{D}]$ is governed by the scaled NLS equation of the form

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+g(z,x)|u|^2u+V(z,x)u=0\,, \tag{1}$$

where $u(z,x)$ is the complex envelope of the optical field, z is the dimensionless coordinate along the propagation direction, and x is the transverse spatial variable. The function $g(z,x)$ is a variable nonlinearity coefficient and $V(z,x)$ is the external potential; both of them are assumed to be functions of the propagation distance and the spatial coordinate. When the functions g and V depend only on the longitudinal variable z, Eq. (1) turns into the NLS equation used in [23]. If $z\to t$, Eq. (1) becomes the generalized Gross–Pitaevskii (GP) equation with time-dependent coefficients; this equation is extensively used in BECs [26,27,28,29]. Pérez-García *et al.* [27] constructed exact breathing solutions of the generalized GP equation with the help of a similarity transformation to the standard NLS equation. Serkin *et al.* [29] obtained nonautonomous solitons in the linear and harmonic oscillator potentials. Kruglov *et al.* [30] obtained exact self-similar solutions to the generalized NLS equation with distributed coefficients. Kundu [31] demonstrated that the nonautonomous NLE equation in $(1+1)$D indeed is equivalent to the standard autonomous NLS equation.

Here, we consider the generalized NLS equation with the general quadratic potential of the form $V(z,x)=\frac{1}{2}\Omega^2(z)x^2+\alpha(z)x+\mu(z)$. Our first goal is to transform Eq. (1) into the standard NLS equation

$$i\frac{\partial U}{\partial z}+\frac{1}{2}\frac{\partial^2 U}{\partial \theta^2}+G|U|^2U=0\,, \tag{2}$$

where $U=U(\theta)$, $\theta\equiv\theta(z,x)$ is a new variable to be determined, and G is a constant nonlinearity parameter. Equation (2) is the standard $(1+1)$D NLS equation, possessing the well-known exact first-order and second-order bright $(G=1)$ soliton solutions [32]

$$U_1(z,\theta)=\lambda\,\mathrm{sech}(\lambda\theta)e^{\frac{1}{2}i\lambda^2 z}\,, \tag{3a}$$

$$U_2(z,\theta)=\frac{2}{\Delta}\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right)\left[\lambda_1\cosh(\lambda,\theta)e^{\frac{1}{2}i\lambda_1^2 z}+\lambda_2\cosh(\lambda_2\theta)e^{\frac{1}{2}i\lambda_2^2 z}\right]\,, \tag{3b}$$

where $\Delta = \cosh[(\lambda_1+\lambda_2)\theta]+(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2})^2\cosh[(\lambda_1-\lambda_2)\theta]+\frac{4\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}\cos(\frac{\lambda_1^2-\lambda_2^2}{2})z$. Here λ, λ_1, λ_2 ($\lambda_1 \neq \lambda_2$) are the amplitudes, and $\Delta \neq 0$.

To connect the solutions of Eq. (1) with those of Eq. (2), we use the similarity transformation [27, 28, 29, 30]

$$u(z,x) = \rho(z,x)e^{i\varphi(z,x)}U[\theta(z,x)]\,, \tag{4}$$

where $\rho(z,x)$ and $\varphi(z,x)$ are the real (non-negative) amplitude and the phase, to be determined. Here we require $U(\theta)$ to satisfy Eq. (2) and $u(z,x)$ to be the solution of Eq. (1). The substitution of Eq. (4) into Eq. (1) and the requirement that the real and imaginary parts be separately equal to zero, with the use of Eq. (2) leads to the following set of equations for ρ, θ and φ

$$\left(\rho^2\right)_z + \left(\rho^2\varphi_x\right)_x = 0\,, \tag{5a}$$

$$\varphi_x = -\frac{\theta_z}{\theta_x}\,, \tag{5b}$$

$$\tfrac{1}{2}\left(\rho^2\theta_x\right)_x = 0\,, \tag{5c}$$

$$g\rho^2 = G\theta_x^2\,, \tag{5d}$$

$$V(z,x) = \varphi_z - \tfrac{1}{2}\frac{\rho_{xx}}{\rho} + E\theta_x^2 + \tfrac{1}{2}\varphi_x^2\,, \tag{5e}$$

where E is the eigenvalue of the NL Eq. (2), which corresponds to the chemical potential in the Bose–Einstein condensate framework or to the propagation constant in the NL optics framework. The subscripts here denote the partial derivatives. From Eq. (5c) we have

$$\rho^2(z,x) = \frac{\lambda^2(z)}{\theta_x(z,x)}\,, \tag{6}$$

where $\lambda(z)$ is an arbitrary function of the propagation distance z. The substitution of Eqs. (6) and (5b) into Eq. (5a) gives

$$\frac{\theta_{zx}\theta_x - \theta_z\theta_{xx}}{\theta_x^2} = \frac{\lambda_z}{\lambda}\,. \tag{7a}$$

Taking the derivative of Eq. (5b) with respect to x, one gets

$$\frac{\theta_{zx}\theta_x - \theta_z\theta_{xx}}{\theta_x^2} = -\varphi_{xx}\,. \tag{7b}$$

From Eq. (7a) we find

$$\left(\frac{\theta_z}{\theta_x}\right)_x = \frac{\lambda_z}{\lambda}\,. \tag{8}$$

After integrating Eq. (8), one obtains

$$\theta(z,x) = F(\xi)\,, \qquad \xi = \lambda(z)x + \delta(z)\,, \tag{9}$$

where $\delta(z)$ is an arbitrary function of z. Comparing Eq. (7a) with Eq. (7b) and using Eq. (5b), we arrive at the following result

$$\varphi(z,x) = -\frac{\lambda_z}{2\lambda}x^2 - \frac{\delta_z}{\lambda}x + \gamma(z)\,, \tag{10}$$

where $\gamma(z)$ is also an arbitrary function of z. Thus, from Eqs. (4), (6), (9), and (10) we get the following exact analytical solution of Eq. (1)

$$u(z,x) = \sqrt{\frac{\lambda}{\partial F(\xi)/\partial \xi}}U[\theta(z.x)]e^{i\left[\frac{\lambda_z}{2\lambda}x^2 - \frac{\delta_z}{\lambda}x + y(z)\right]}\,, \tag{11}$$

where U is the soliton solution of the standard NLS Eq. (2), determined by Eqs. (3); other parameters in Eq. (11) satisfy the following equations, which are found from Eqs. (5d), (5e), (6) and (10)

$$g(z,x) = G\frac{\lambda^4}{\rho^6}\,, \tag{12a}$$

$$V = -\frac{\rho_{xx}}{2\rho} + E\frac{\lambda^4}{\rho^4} + \gamma_z + \frac{\left(2\lambda_z^2 - \lambda\lambda_{zz}\right)x^2 + 2\left(2\lambda_z\delta_z - \lambda\delta_{zz}\right)x + \delta_z}{2\lambda^2}\,. \tag{12b}$$

Therefore, by choosing appropriately the functions $\lambda(z)$, $\delta(z)$, and $\gamma(z)$, we can generate the functions $\rho(z,x)$, $g(z,x)$, and $V(z,x)$, for which the solutions of Eq. (1) can be obtained from those of Eq. (11) via Eq. (3).

We now discuss the optical intensity $I = |u|^2$ distributions and exploit them to exhibit interesting non-trivial behavior of the analytical solution of Eq. (1). We want to point out that the procedure requires a suitable choice of $\theta(z,x)$, $\rho(z,x)$ and the functions $g(z,x)$ and $V(z,x)$, to determine the solution given by Eq. (11) subsequently.

3. Discussion

To illustrate some interesting examples, we focus on the specific nonlinearities. First, we presume that the nonlinearity coefficient is given explicitly by $g = G\lambda(z)$, which corresponds to the choice $F = \xi = \lambda(z)x + \delta(z)$. Substituting the nonlinearity coefficient $g(z,x)$ into Eq. (12a) yields $\rho^2(z,x) = \lambda(z)$. In the following, we consider a generic case for the set of parameters:

$\delta(z) = 0$, $E = 0$, $\gamma = 0$ and $\lambda(z) = 1 + \lambda_0 \sin(\omega_0 z)$, where $\lambda_0 \in (-1, 1)$ and $0 \neq \omega_0 \in R$. From Eq. (12b) one finds that the potential can be cast as

$$V(z, x) = \frac{2\lambda_z^2 - \lambda\lambda_{zz}}{2\lambda^2} x^2 .$$

Thus, the solution of Eq. (1) is obtained from Eq. (11). For this choice, namely, we consider the harmonic potential case.

Typical behavior of these solitons is depicted in Fig. 1 for the first-order soliton and in Fig. 2 for the second-order soliton. We find that these solitons take the shape of a breather. We calculate some physical quantities characterizing the breathers, such as the breather amplitude and the period. For the first-order soliton, the amplitude of peaks depends on the propagation distance as

$$|u_1|^2 = (1 + \lambda_0 \sin \omega_0 z)^2 \mathrm{sech}^2 \left[(1 + \lambda_0 \sin \omega_0 z)^2 x \right]$$

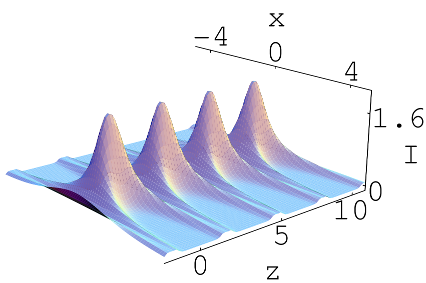

Fig. 1. (Color online) Breathing oscillations of order-one soliton, after choosing the soliton management parameters as $\omega_0 = 2$, and (left) $\lambda_0 = 0.8$; (right) $\lambda_0 = 0.2$.

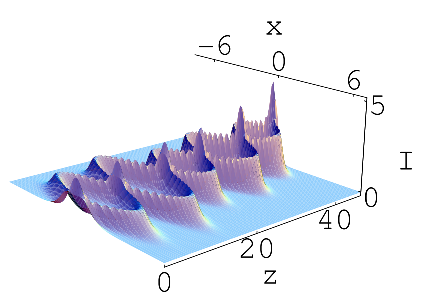

Fig. 2. (Color online) Second-order soliton interactions. The soliton management parameters are: $\omega_0 = 2$, and (left) $\lambda_{10} = 0.2$, $\lambda_{20} = 0.8$; (right) $\lambda_{10} = 0.3$, $\lambda_{20} = 0.7$.

and the breather period is $2\pi/\omega_0$. For the second-order soliton, the corresponding expressions are as follows [32]

$$
\begin{aligned}
|u_2|^2 &= \frac{4A^2}{C^2}\left[B+\sum_{j=1}^{2}\left(1+\lambda_{j0}\sin\omega_0 z\right)^2\cosh^2\left(1+\lambda_{j0}\sin\omega_0 z\right)x\right].\\
T_{\mathrm{sol}} &= \frac{\pi}{\left(\lambda_{10}^2-\lambda_{20}^2\right)\cos\omega_0 z}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{2+\left(\lambda_{10}+\lambda_{20}\right)\sin\omega_0 z}{\left(\lambda_{10}+\lambda_{20}\right)\sin\omega_0 z}\,,\\
B &= 2\cos\frac{\left(\lambda_{10}^2-\lambda_{20}^2\right)}{2}z\prod_{j=1}^{2}\left(1+\lambda_{j0}\sin\omega_0 z\right)\cosh(1+\lambda_{j0}\sin\omega_0 z)x\,,
\end{aligned}
$$

and

$$
\begin{aligned}
C = {} & \cosh\left[\left(\lambda_{10}+\lambda_{20}\right)x\sin\omega_0 z\right]+\left(\frac{\lambda_{10}+\lambda_{20}}{\lambda_{10}-\lambda_{20}}\right)^2\\
& \times\cosh\left[\left(\lambda_{10}-\lambda_{20}\right)x\sin\omega_0 z\right]+\frac{4\lambda_{10}\lambda_{20}}{\left(\lambda_{10}-\lambda_{20}\right)^2}\cos\left(\frac{\lambda_{10}^2-\lambda_{20}^2}{2}z\right).
\end{aligned}
$$

Order-two spatial soliton has been observed in the pioneering experiments in fibers [33]. In the general case of a system with distance-dependent nonlinearity, the soliton period becomes dependent on the propagation distance. The intensity profiles of the breathers build up a complex landscape of peaks and valleys, and reach increasing peaks as the distance is increased. Since the problem is in $(1+1)$D, no collapse is expected. Consequently, variations of the confining harmonic potential are found to be periodically attractive and repulsive, to support the stable soliton management in this example [33].

Next, we assume that the nonlinearity coefficient is constant, $g = G$; this corresponds to the choice $F = \xi = \lambda(z)x + \delta(z)$. Other parameters are chosen as: $\lambda(z) = \lambda_0 = \text{const.}$, $\delta(z) = \lambda_0\cos(\omega_0 z)$, and $E = 1$, $\gamma = 0$, where $\omega_0 \neq 0$. With this choice, from Eq. (12) we find:

$$
V(z,x) = \omega_0^2\cos^2(\omega_0 z)x + \lambda_0^{4/3} - \lambda_0\omega_0\sin(\omega_0 z)\,.
$$

The potential is a linear function of x. In Figs. 3 and 4 the first-order and the second-order bright solitary intensity profiles $I = |u|^2$ are presented, as functions of x and z. Similar to the case above, the amplitude of peaks and the period can also be written down easily. As one can see, with the periodic change of the linear potential parameter, the first- and the second-order solitons display a zig-zag propagation. Thus, stable soliton propagation can be achieved by manipulating the linear potential field. An interesting collision behavior is noted for the two peaks of the second-order soliton. After the first collision, the soliton reemerges as two distinct wave packets, which then collide repeatedly.

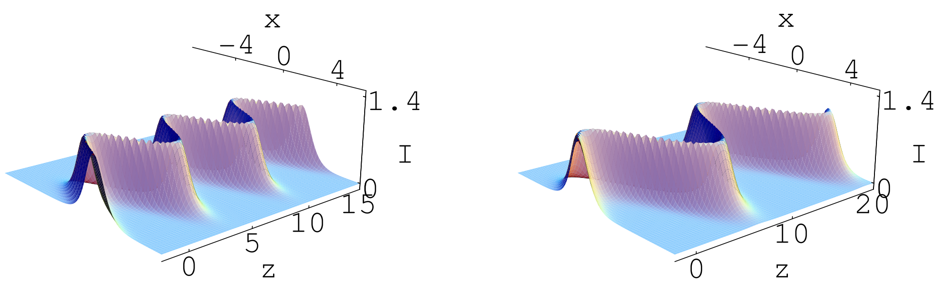

Fig. 3. (Color online) Periodic oscillations of the order-one soliton, after choosing the soliton management parameters, $\lambda_0 = 1$ and (left) $\omega_0 = 1$; (right) $\omega_0 = 0.5$.

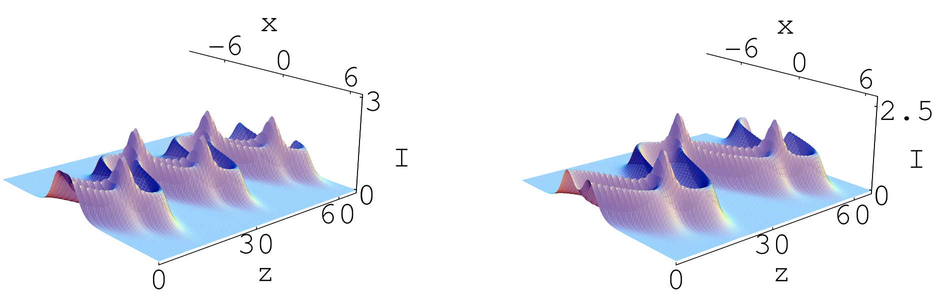

Fig. 4. (Color online) Periodic interactions within the second-order soliton, with zig-zag propagation. The soliton management parameters are $\omega_0 = 0.25$ and (left) $\lambda_{10} = 0.2$, $\lambda_{20} = 0.8$; (right) $\lambda_{10} = 0.3$, $\lambda_{20} = 0.7$.

To confirm the validity of solution (11) and test the stability of those solitons, we perform a numerical study and compare the analytical solution with the numerical simulation. Figure 5 shows the comparison of the exact solution from Eq. (11) with the results of the numerical simulation of Eq. (1), obtained by utilizing the split-step beam propagation method [24] and adding 10% white noise to the initial condition. Numerical simulation of the solitary wave from Fig. 1, right is performed, and we set initial conditions given by the analytical solution (11) at $z = 0$, with the step-length $\Delta z = 0.02$. It is seen that the analytical solution is consistent with the numerical simulation. It is evident that the noise does not cause the instability of the solitary wave and that the solitary wave indeed is stable.

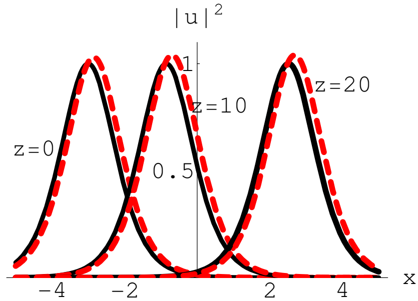

Fig. 5. (Color online) Comparison of the analytical solution with the numerical simulation at different propagation distances. Solid line is the analytical solution from Eq. (11), dashed line is the numerical simulation of Eq. (1).

4. Conclusions

In summary, we have studied exact solutions of the nonlinear Schrödinger equation with a variable nonlinearity and a quadratic external potential, both depending on the propagation distance and the transverse spatial coordinate. A simple procedure is presented for controlling the behavior of solitons, in which one may select the parameters of nonlinearity and external potential, to control the propagation behavior of solitons. We have considered two interesting examples, the harmonic and the potential linear in x. In the case of periodically varying nonlinearity coefficient, the soliton is found to be periodically attractive for the second-order soliton.

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