DRESSING METHOD FOR A GENERALIZED FOCUSING NLS EQUATION VIA LOCAL RIEMANN–HILBERT PROBLEM*

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A generalized focusing NLS equation is studied by the dressing method via local Riemann–Hilbert problem. The associated RH problem with zeros is solved by means of regularization. The explicit solutions, including one-soliton, two-soliton solution and breather solution, are obtained.

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1. Introduction

In this paper, the generalized (self-)focusing NLS (GFNLS) equation

$$iu_t + u_{xx} + 2\varepsilon|u|^2u + 4\beta^2|u|^4u - 4i\varepsilon\beta\left(|u|^2\right)_x u = 0 \tag{1}$$

will be studied by dressing method via local Riemann–Hilbert (RH) problem [1, 2, 3, 4, 5, 6, 7, 8, 9], where $\varepsilon > 0$ and β is a real constant. Equation (1) can be transformed to the classical NLS equation [10, 11, 12, 13, 14] which describes the self-modulation of one-dimensional waves in a nonlinear dispersive medium. It is well known that the solution of focusing NLS equation with rapidly decreasing type [10, 12] can be transformed into the solution of the defocusing NLS equation with finite density type [3, 15, 16]. It is readily verified that the solution of GFNLS equation becomes that of generalized defocusing NLS equation ($\varepsilon < 0$) under the transformation

$$x \to ix\,, \qquad t \to -t\,, \qquad \varepsilon \to -\varepsilon\,, \qquad \beta \to i\beta\,.$$

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The GFNLS equation (1) is the compatibility condition between two linear matrix equations [17, 18]

$$\psi_x = U\psi\,, \qquad \psi_t = V\psi\,, \tag{2}$$

where

$$U = -ik\sigma_3 - i\beta Q^2\sigma_3 + Q\,, \qquad Q = i\begin{pmatrix} 0 & u \\ \varepsilon\bar{u} & 0 \end{pmatrix}\,, \tag{3}$$

and

$$V = -2ik^2\sigma_3 + 2kQ + i\sigma_3 Q_x - 2\beta Q^3 + \beta[Q_x, Q] + \left(4i\beta^2 Q^4 - iQ^2\right)\sigma_3\,. \tag{4}$$

In fact, the compatibility condition of the system (2), that is

$$U_t - V_x + [U, V] = 0\,, \tag{5}$$

gives the nonlinear evolution equation

$$Q_t - i\sigma_3 Q_{xx} + 2i\sigma_3 Q^3 - 4i\beta^2\sigma_3 Q^5 + 4\beta\left(Q^2\right)_x Q = 0\,, \tag{6}$$

and a conservation law

$$-i\beta\left(Q^2\right)_t = \left(\beta[Q_x, Q]\sigma_3 + 4i\beta^2 Q^4\right)_x\,. \tag{7}$$

In the dressing procedure, the zero seed solution is chosen to define the Jost matrix solutions which are interconnected by the scattering matrix. The analytic properties are used to factorize the scattering matrix, from which the associated local matrix RH problem is constructed. The reconstruction formula for the potential is obtained by the normalization which is not canonical. The reconstruction formula about the solution of RH problem is transformed into the expression about the so-called dressing factor [19] (or soliton matrix [20, 21]) which is obtained by means of regularization from the RH problem with zeros [22]. The dressing factor is the expression of (x,t)-dependent vector parameters which can be derived by the spectral equations. In the special cases about the number and form of the zeros the RH problem, the explicit solutions, including one-soliton, two-soliton solution and breather solution, are obtained.

2. Jost solutions

It will be more convenient to write the spectral equation in terms of the matrix $J = \psi E^{-1}$, where $E = \exp(-ikx\sigma_3)$ is a solution of the spectral equation $\psi_x = U\psi$ for zero potential. Hence, the matrix-valued function $J(x,k)$ satisfies

$$J_x = -ik[\sigma_3, J] + \tilde{U}J\,, \qquad \tilde{U} = Q - i\beta Q^2\sigma_3\,. \tag{8}$$

For the spectral equation (8), the standard approach is to define the so-called Jost solutions $J_\pm(x,k)$ by the asymptotic conditions

$$J_\pm(x,k) \to e^{\alpha_\pm(x)\sigma_3}\,, \qquad x \to \pm\infty\,, \tag{9}$$

where

$$\alpha_\pm(x) = i\varepsilon\beta \int\limits_{\pm\infty}^{x} |u(\xi)|^2 d\xi\,. \tag{10}$$

Since $\mathrm{tr}\tilde{U} = 0$, these boundary conditions imply that $\det J_\pm = 1$ for all x.

Being solutions of the first-order differential equation, the Jost functions $J_\pm$ are not mutually independent. In fact, they are interconnected by the scattering matrix $S(k)$,

$$J_- = J_+ESE^{-1}\,, \qquad S(k) = \begin{pmatrix} a(k) & -\varepsilon\bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \qquad \det S(k) = 1\,. \tag{11}$$

The involutive condition of $U(x,k)$ in (2) is $U^\dagger(x,\bar{k}) = -HU(x,k)H$, $H = \mathrm{diag}(1,\varepsilon)$, and hence $\psi^\dagger(x,\bar{k}) = H\psi^{-1}(x,k)H$, which implies that the Jost solutions obey the involutive condition

$$J_\pm^\dagger\left(x,\bar{k}\right) = HJ_\pm^{-1}(x,k)H\,. \tag{12}$$

As a result, the scattering matrix $S(k)$ satisfies the same involution $S^\dagger(\bar{k}) = HS^{-1}(k)H$.

In the following, we will investigate the analytic properties of the Jost solutions. For convenience, we introduce the notations:

- $[M]_j$, $(j = 1,2)$ denote the j-th column vector of matrix M;
- $[M]^j$ denote the j-th row vector.

It is noted that $[J_\pm]_j$ satisfy the following linear differential equations

$$\begin{aligned} \partial_x[J_\pm]_1 - 2ik\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}[J_\pm]_1 &= \tilde{U}[J_\pm]_1\,, \\ \partial_x[J_\pm]_2 + 2ik\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}[J_\pm]_2 &= \tilde{U}[J_\pm]_2\,, \end{aligned} \tag{13}$$

which are equivalent to linear integral equations

$$\begin{aligned} [J_\pm]_1 &= \begin{pmatrix} e^{-\alpha_\pm(x)} \\ 0 \end{pmatrix} + \int\limits_{\pm\infty}^{x} G_\mp(x-\xi,k)\tilde{U}[J_\pm]_1 d\xi\,, \\ [J_\pm]_2 &= \begin{pmatrix} 0 \\ e^{\alpha_\pm(x)} \end{pmatrix} + \int\limits_{\pm\infty}^{x} \hat{G}_\pm(x-\xi,k)\tilde{U}[J_\pm]_2 d\xi\,, \end{aligned} \tag{14}$$

where

$$\begin{aligned} G_{\mp}(x-\xi,k) &= \begin{pmatrix} 1 & 0 \\ 0 & e^{2ik(x-\xi)} \end{pmatrix}, \qquad \mathrm{Im}k \lessgtr 0\,, \\ \hat{G}_{\pm}(x-\xi,k) &= \begin{pmatrix} e^{-2ik(x-\xi)} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{Im}k \gtrless 0\,. \end{aligned} \tag{15}$$

We see that $[J_-]_1$, $[J_+]_2$ are analytic in the upper half k plane, and we define the matrix function $\Phi_+(x,k)$ as

$$\Phi_+(x,k) = \left([J_-]_1, [J_+]_2\right)\,, \tag{16}$$

which also satisfies the spectral problem (8), and is analytic in the upper half k plane. From equation (11), we know that $[J_-]_1 = a(k)[J_+]_1 + b(k)\exp(2ikx)[J_+]_2$. Then $\Phi_+(x,k)$ can be expressed in terms of the Jost solution $J_+(x,k)$ and the elements of the scattering matrix as following

$$\Phi_+(x,k) = J_+(x,k)ES_+(k)E^{-1}\,, \qquad S_+(k) = \begin{pmatrix} a(k) & 0 \\ b(k) & 1 \end{pmatrix}. \tag{17}$$

Similarly,

$$\Phi_+(x,k) = J_-(x,k)ES_-(k)E^{-1}\,, \qquad S_-(k) = \begin{pmatrix} 1 & \varepsilon\bar{b}(k) \\ 0 & a(k) \end{pmatrix}. \tag{18}$$

In order to obtain the analytic function in the lower half plane, we define such a function $\Phi_-^{-1}(x,k)$ by means of the involution (12) as

$$\Phi_-^{-1}(x,k) = H\Phi_+^{\dagger}\left(x,\bar{k}\right)H\,. \tag{19}$$

It is remarked that $[J_+]_1$, $[J_-]_2$, in (14), are analytic in the lower half plane, and analytic function in the lower half plane being relative to Φ_+ should be defined by $[J_+]_1$, $[J_-]_2$. While the reason we introduce the definition (19) instead of $\Phi_- = ([J_+]_1, [J_-]_2)$ is that the former is more convenient to the Riemann–Hilbert problem in the next section. It is noted that the row vectors of $J_\pm^{-1}$ are relative to the column ones of $J_\pm$ as

$$\left[J_\pm^{-1}\right]^1 = -\left(\sigma[J_\pm]_2\right)^{\mathrm{T}}\,, \qquad \left[J_\pm^{-1}\right]^2 = \left(\sigma[J_\pm]_1\right)^{\mathrm{T}}\,, \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So, Φ_-^{-1} can be expressed as

$$\Phi_-^{-1}(x,k) = \begin{pmatrix} \left[J_-^{-1}\right]^1 \\ \left[J_+^{-1}\right]^2 \end{pmatrix} = \begin{pmatrix} -\left(\sigma[J_-]_2\right)^{\mathrm{T}} \\ \left(\sigma[J_+]_1\right)^{\mathrm{T}} \end{pmatrix}.$$

On the real axis, we have, by virtue of (17) and (18),

$$\Phi_-^{-1}(x,k) = HES_\pm^\dagger E^{-1} J_\pm^\dagger H\,. \tag{20}$$

In order to reconstruct the potential, we write the asymptotic expansion for $\Phi_+(x,k)$,

$$\Phi_+(x,k) = e^{-i\gamma(x)\sigma_3} + \frac{1}{k}\Phi^{(1)}(x) + O\left(\frac{1}{k^2}\right)\,, \qquad k\to\infty\,, \tag{21}$$

where $\gamma(x)$ satisfies the following equation about $\Phi^{(1)}(x)$ as

$$\gamma_x(x)I = \beta\left[\sigma_3, \Phi^{(1)}(x)\right] H \left[\sigma_3, {\Phi^{(1)}}^\dagger(x)\right] H\,. \tag{22}$$

On use of the involution (19), it is readily verified that the function $\Phi_+^{(1)}(x)$ in (21) satisfies the following involutive condition

$${\Phi^{(1)}}^\dagger(x) = -e^{i\gamma(x)\sigma_3} H\Phi^{(1)}(x)He^{i\gamma(x)\sigma_3}\,. \tag{23}$$

Substituting the asymptotic expansion (21) into the spectral equation (8), and considering the $O(1)$ terms, we find $\gamma_x = \beta Q^2$ and

$$Q = i\left[\sigma_3, \Phi^{(1)}\right] e^{i\gamma\sigma_3}\,. \tag{24}$$

It is noted that the expression of Q, in virtue of (23), can be rewritten as

$$Q = -ie^{-i\gamma\sigma_3} H\left[\sigma_3, {\Phi^{(1)}}^\dagger\right] H\,.$$

Then the equation $\gamma_x = \beta Q^2$ coincides with the definition (22) by collecting the above two expressions of Q. Equations (24) and (22) give the reconstruction formulae of potential. Hence, in order to solve the GFNLS equation, we should find the analytic solution $\Phi_+(x,k)$.

3. The Local Matrix Riemann–Hilbert problem

The Local Matrix RH problem can be obtained by the product $\Phi_-^{-1}(x,k)\Phi_+(x,k)$ on the real axis (Im$k=0$), where $\Phi_-^{-1}(x,k)$ and $\Phi_+(x,k)$ are defined by (20) and (17) (or (18)). The RH problem has the form

$$\Phi_-^{-1}(x,k)\Phi_+(x,k) = EG_0(k)E^{-1}\,, \qquad G_0 = S_+^\dagger S_+ = \begin{pmatrix} 1 & \varepsilon\bar{b} \\ b & 1\end{pmatrix}\,, \tag{25}$$

and the normalization of the RH problem is

$$\Phi_+(x,k) = e^{-i\gamma(x)\sigma_3} + \frac{1}{k}\Phi^{(1)}(x) + O\left(\frac{1}{k^2}\right), \qquad k \to \infty, \tag{26}$$

where $\gamma(x)$ satisfies equation (22).

The RH problem (25) is characterized by the so-called RH data which are categorized into discrete data (eigenvalue k_j and eigenvectors $|j\rangle$) and continuous data (the matrix element $b(k)$). Solitons correspond to the discrete data of RH problem with zeros of the scattering coefficients $a(k)$ and $\bar{a}(k)$. The equations (17) (or (18)) and (20) imply that the determinants of the matrices Φ_+ and Φ_-^{-1} are given by $a(k)$ and $\bar{a}(k)$, respectively. We give the assumption that $a(k)$ and $\bar{a}(k)$ have simple zeros at the points k_j and $\bar{k}_l$ in their domains of analyticity, which imply that $\det\phi_+(k_j) = 0$, $\mathrm{Im}k > 0$, $j = 1,2,\ldots,N$ and $\det\Phi_-^{-1}(\bar{k}_l) = 0$, $\mathrm{Im}k < 0$, $l = 1,2,\ldots,N$, here the equal number N of the zeros in both domains are guaranteed by the involution (19).

We will solve the RH problem with zeros (25) by means of its regularization. This procedure consists in extracting rational factors from Φ_+ which are responsible for the existence of zeros. If $\det\Phi_+(k_j) = 0$, then at the point k_j there exists an eigenvector $|v_j\rangle$ in the kernel of matrix $\Phi_+(k_j)$, *i.e.* $\Phi_+(k_j)|v_j\rangle = 0$. Let us introduce a rational matrix function

$$\chi_j^{-1} = I + \frac{k_j - \bar{k}_j}{k - k_j}P_j, \qquad P_j = \frac{|v_j\rangle\langle v_j|}{\langle v_j|v_j\rangle}, \tag{27}$$

where P_j is the rank 1 projector, $P_j^2 = P_j$, and $\langle v_j| = |v_j\rangle^\dagger$. It can be shown that $\det\chi_j^{-1} = (k-\bar{k}_j)(k-k_j)^{-1}$. Since $\det\Phi_+(k) \sim (k-k_j)$ near the point k_j, we evidently have $\det(\Phi_+\chi_j^{-1}) \neq 0$ at the point k_j. In this way, we succeeded in regularizing the RH problem at the point k_j. Similarly, zero $\bar{k}_l$ of the matrix function Φ_-^{-1} is regularized by the rational function

$$\chi_l = I - \frac{k_l - \bar{k}_l}{k - \bar{k}_l}P_l, \tag{28}$$

and the matrix $\chi_l\Phi_-^{-1}$ has no zero in the point $\bar{k}_l$. The regularization of all the other zeros is performed similarly and eventually we obtain the following representation for the analytic solutions

$$\Phi_\pm = \phi_\pm\Gamma, \qquad \Gamma = \chi_N\chi_{N-1}\ldots\chi_1, \tag{29}$$

where the rational matrix function $\Gamma(x,k)$ accumulates all zeros of the RH problem, while the matrix functions $\phi_\pm$ solve the regular RH problem

$$\phi_-^{-1}(x,k)\phi_+(x,k) = \Gamma(x,k)EG_0(k)E^{-1}\Gamma^{-1}(x,k). \tag{30}$$

In order to obtain the soliton solution of the GFNLS equation, we need only consider the discrete data, that is $b(k) = 0$ or $G_0(k) = I$. In this case, by posing $\phi_\pm = e^{-i\gamma(x)\sigma_3}$, then it follows from (29) that the asymptotic expansion for Γ is written as

$$\Gamma(x,k) = I + \frac{1}{k}\Gamma^{(1)}(x) + O\left(\frac{1}{k^2}\right), \qquad k \to \infty . \tag{31}$$

For practical purposes, it is more convenient to decompose the product (29) into simple fractions [22, 23]. In general, the rational matrix function $\Gamma(k)$ and its inverse can be decomposed into terms of two sets of the vectors $|m_j\rangle$ and $|n_j\rangle$

$$\begin{aligned} \Gamma(k) &= \prod_{l=0}^{N-1}\left(I - \frac{k_{N-l} - \bar{k}_{N-l}}{k - \bar{k}_{N-l}} P_{N-l}\right) = I - \sum_{l=1}^{N} \frac{k_l - \bar{k}_l}{k - \bar{k}_l} |m_l\rangle\langle n_l| , \\ \Gamma^{-1}(k) &= \prod_{j=1}^{N}\left(I + \frac{k_j - \bar{k}_j}{k - k_j} P_j\right) = I + \sum_{j=1}^{N} \frac{k_j - \bar{k}_j}{k - k_j} |n_j\rangle\langle m_j| . \end{aligned} \tag{32}$$

It can be shown that the $2N$ vectors $|m_j\rangle$ and $|n_j\rangle$ are not independent, but satisfy the following equation

$$|n_j\rangle = \sum_{l=1}^{N} |m_l\rangle\langle n_l| \frac{k_l - \bar{k}_l}{k_j - \bar{k}_l} |n_j\rangle . \tag{33}$$

Let us introduce $N \times N$ matrices

$$\begin{aligned} \mathcal{M} &= (|m_1\rangle, |m_2\rangle, \ldots, |m_N\rangle) , & \mathcal{N} = (|n_1\rangle, |n_2\rangle, \ldots, |n_N\rangle) , \\ D &= (D_{lj}) = \left(\langle n_l| \frac{1}{k_j - \bar{k}_l} |n_j\rangle\right) , & K = \operatorname{diag}\left(\ldots, k_l - \bar{k}_l, \ldots\right) . \end{aligned}$$

Then (33) can be written as $\mathcal{N} = \mathcal{M}KD$ or $\mathcal{M}K = \mathcal{N}D^{-1}$. In components,

$$\left(k_l - \bar{k}_l\right) |m_l\rangle = \sum_{j=1}^{N} \left(D^{-1}\right)_{jl} |n_j\rangle .$$

Substituting it into (32) and introducing more convenient notation $|j\rangle \equiv |n_j\rangle$, we obtain the desired formulae

$$\Gamma(k) = I - \sum_{j,l=1}^{N} \frac{1}{k - \bar{k}_l} |j\rangle \left(D^{-1}\right)_{jl} \langle l| , \qquad D_{lj} = \frac{\langle l|j\rangle}{k_j - \bar{k}_l} , \tag{34}$$

and

$$\Gamma^{-1}(k) = I + \sum_{j,l} \frac{1}{k - k_j} |j\rangle \left(D^{-1}\right)_{jl} \langle l| \,. \tag{35}$$

Let us remember the reconstruction formula (24), which is now written as

$$Q = ie^{-i\gamma(x)\sigma_3} \left[\sigma_3, \Gamma^{(1)}(x)\right] e^{i\gamma(x)\sigma_3} \,, \tag{36}$$

where

$$\gamma_x I = \beta \left[\sigma_3, \Gamma^{(1)}(x)\right] H \left[\sigma_3, \Gamma^{(1)^\dagger}(x)\right] H \,, \qquad \Gamma^{(1)^\dagger} = -H\Gamma^{(1)}H \,. \tag{37}$$

As a result, we will be able to find solutions of the GFNLS equation, provided that we can calculate explicitly the matrix Γ or the vector $|j\rangle$. To this end, let us differentiate the equation $\Phi_+(k_j)|j\rangle = 0$ in x. Since Φ_+ satisfies the spectral equation (8), then we have the linear x-dependent equation about $|j\rangle$

$$|j\rangle_x = -ik_j\sigma_3|j\rangle \,. \tag{38}$$

In the same manner, we can find the evolutionary equation

$$|j\rangle_t = -2ik_j^2\sigma_3|j\rangle \,. \tag{39}$$

Hence, the vector $|j\rangle$ has the form explicitly as

$$|j\rangle = \exp\left\{-i\left(k_j x + 2k_j^2 t\right)\sigma_3\right\}|j_0\rangle \,, \tag{40}$$

where $|j_0\rangle$ is a vector integration constant.

4. Soliton solutions

From the reconstruction (36) and (37), we know that the potential $u(x,t)$ takes the following expression

$$u(x,t) = 2e^{-2i\gamma}\Gamma^{(1)}_{12} \,, \tag{41}$$

where

$$\gamma = -4\varepsilon\beta\partial_x^{-1}\left(\left|\Gamma^{(1)}_{12}\right|^2\right) \,, \tag{42}$$

and $\Gamma^{(1)}_{12}$ can be obtained from (34). Specially, for $N = 1$, we let $k_1 = \xi + i\eta$, then the one-soliton solution of GFNLS equation takes the form

$$u(x,t) = 2\eta e^{i\tilde{\varphi}} \operatorname{sech} z \,, \tag{43}$$

where $e^{a+ib} = |1_0\rangle_1 / |1_0\rangle_2$ and

$$\begin{aligned} z &= 2\eta(x+4\xi t)+a\,, & \varphi = -2\xi x - 4\left(\xi^2-\eta^2\right)t + b\,, \\ \gamma &= -4\varepsilon\beta\eta^2\partial_x^{-1}\left(|\text{sech}\, z|^2\right)\,, & \tilde{\varphi} = \varphi - 2\gamma - \frac{\pi}{2}\,. \end{aligned}$$

The two-soliton solution can be obtained from (41) in the case of $N=2$. From the asymptotic expansion of (34), we have

$$\Gamma^{(1)} = (\det D)^{-1}\left[-D_{22}|1\rangle\langle 1| + D_{21}|2\rangle\langle 1| + D_{12}|1\rangle\langle 2| - D_{11}|2\rangle\langle 2|\right]\,, \quad (44)$$

where the matrix D takes the form

$$D = \begin{pmatrix} \frac{\langle 1|1\rangle}{k_1-\bar{k}_1} & \frac{\langle 1|2\rangle}{k_2-\bar{k}_1} \\ \frac{\langle 2|1\rangle}{k_1-\bar{k}_2} & \frac{\langle 2|2\rangle}{k_2-\bar{k}_2} \end{pmatrix}\,,$$

and the vectors $|j\rangle, (j=1,2)$ have the form

$$|j\rangle = e^{(a_j+ib_j)/2}\begin{pmatrix} e^{(z_j+i\varphi_j)/2} \\ e^{-(z_j+i\varphi_j)/2} \end{pmatrix}\,,$$

where $k_j = \xi_j + i\eta_j,\ e^{a_j+ib_j} = |j_0\rangle_1/|j_0\rangle_2$ and

$$z_j = 2\eta_j(x+4\xi_j t)+a_j\,, \qquad \varphi_j = -2\xi_j x - 4\left(\xi_j^2-\eta_j^2\right)t + b_j\,.$$

Then the elements of D are

$$D_{12} = \frac{2}{k_2-\bar{k}_1}\exp\left\{\tfrac{1}{2}\left[(a_2+a_1)+i(b_2-b_1)\right]\right\}\cosh\left\{\tfrac{1}{2}\left[(z_2+z_1)+i(\varphi_2-\varphi_1)\right]\right\}\,,$$

$$D_{21} = -\bar{D}_{12}\,, \qquad D_{11} = \frac{\cosh z_1}{i\eta_1}\,, \qquad D_{22} = \frac{\cosh z_2}{i\eta_2}\,.$$

Through a direct calculation, we can obtain

$$\Gamma_{12}^{(1)} = \frac{V_1+iV_2}{W_2}\,, \quad (45)$$

where

$$\begin{aligned} V_1 &= 2\eta_1\eta_2(\xi_2-\xi_1)e^{a_1+a_2}\left(\tanh z_1 \text{sech}\, z_2 e^{i\varphi_2} - \tanh z_2 \text{sech}\, z_1 e^{i\varphi_1}\right)\,, \\ V_2 &= \left[(\xi_2-\xi_1)^2+(\eta_1+\eta_2)^2-2\eta_2(\eta_1+\eta_2)e^{a_2}\right]\eta_1 e^{a_1+i\varphi_1}\text{sech}\, z_1 \\ &\quad + \left[(\xi_2-\xi_1)^2+(\eta_1+\eta_2)^2-2\eta_1(\eta_1+\eta_2)e^{a_1}\right]\eta_2 e^{a_2+i\varphi_2}\text{sech}\, z_2\,, \\ W_2 &= -(\xi_2-\xi_1)^2-(\eta_1+\eta_2)^2+2\eta_1\eta_2 e^{a_1+a_2} \\ &\quad \times\left[\text{sech}\, z_1 \text{sech}\, z_2 \cosh(z_1+z_2) + \text{sech}\, z_1 \text{sech}\, z_2 \cos(\varphi_2-\varphi_1)\right]\,. \end{aligned}$$

With $\varGamma_{12}^{(1)}$ in hand, we can obtain the two-soliton solution of GFNLS equation from (41) and (42).

In order to obtain the breather solution, we firstly take $\xi_j = 0$ and $a_j = 0$, then

$$\varGamma_{12}^{(1)} = i\frac{\widetilde{V}_2}{\widetilde{W}_2}\,,$$

where

$$\begin{aligned}
\widetilde{V}_2 &= \left(\eta_1^2 - \eta_2^2\right)\left[\frac{\cosh(2\eta_2 x)}{2\eta_2}e^{i\left(4\eta_1^2 t + b_1\right)} - \frac{\cosh(2\eta_1 x)}{2\eta_1}e^{i\left(4\eta_2^2 t + b_2\right)}\right],\\
\widetilde{W}_2 &= \cosh\left[2(\eta_1+\eta_2)x\right] + \cos\left[4\left(\eta_2^2 - \eta_1^2\right)t + b_2 - b_1\right]\\
&\quad - \frac{(\eta_1+\eta_2)^2}{2\eta_1\eta_2}\cosh(2\eta_1 x)\cosh(2\eta_2 x)\,.
\end{aligned}$$

Secondly, setting $\eta_1 = 3/2$, $\eta_2 = 1/2$ and $b_1 = 0$, $b_2 = \pi$, one has the breather solution of GFNLS equation (1), where

$$\varGamma_{12}^{(1)} = 2e^{i\left(t-\frac{\pi}{2}\right)}\frac{\cosh 3x + 3e^{8it}\cosh x}{\cosh 4x + 4\cosh 2x + 3\cos 8t}\,.$$

It is remarked that the assumption $\xi_j = 0$, $j = 1, 2$ means that the relative velocity of the solitons is zero, and we put $a_j = 0$, $j = 1, 2$ because the maxima of both solitons coincide. It is noted that the initial condition $\varGamma_{12}^{(1)}(x, 0) = -i\operatorname{sech} x$ guarantees the evolution of GFNLS breather. Figure 1 demonstrates the temporal evolution of GFNLS breather.

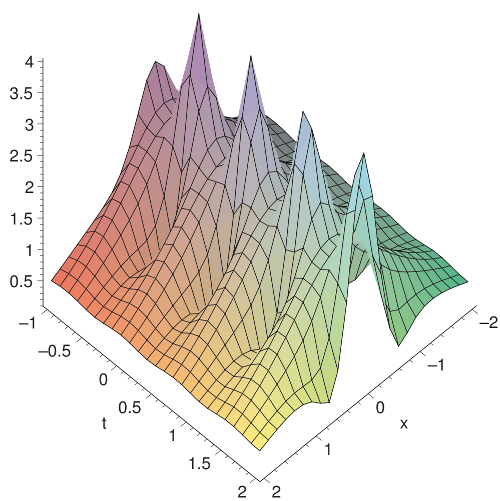

Fig. 1. Modular of the breather solution takes the form $|u| = 2\left|\varGamma_{12}^{(1)}\right|$.

5. Summary and discussion

In this paper, we have studied a generalized focusing NLS equation by using the dressing method which is based on the local RH problem. We have briefly discussed the direct scattering problem and established the expression of potential about the solution of RH problem. In order to obtain the soliton solutions, the RH problem with zeros is considered, and the method of regularization is introduced to reconstruct the expression of potential about the dressing factor. For the rational matrix, we have decomposed the dressing matrix into sample fractions where all the involved vectors have explicit (x, t) dependence.

The "potential" matrix in spectral problem (8) has non-zero diagonal elements, which led to the normalization (9) as introduced in our paper. It is noted that this normalization can be reduced to the canonical one. While, if the canonical normalization is considered, the associated discussion will be complicated and difficult. In the latter case, for example, it will be difficult to construct the potential about the solution of RH problem.

We have used dressing method via RH problem, but a few results of RH problem are involved. It is known that the RH problem in the inverse scattering transform (IST) is involved to reconstruct the potential from knowledge of the scattering data, this can be done by taking the minus projection of the RH problem and finding the asymptotic behaviour of the former result in the inverse scattering problem, (for a review see [4], and references therein). In this paper, instead of discussing the inverse scattering problem, we have used the procedure of regularization and decomposition to reconstruct the potential from the dressing factor which involves the (x, t) dependent vectors. Following this procedure, it has been shown that it is more convenient to study the higher order soliton solutions of the integrable nonlinear PDEs [20, 21].

Since the GFNLS (1) is invariant under the transformation $u \to ue^{i\alpha}$, where α is a real constant, then the conservation law (7) may be regarded as conservation of particle numbers. So, for the GFNLS equation, the linear spectral problem (8) contains not only the potential u or Q but also one of the conserved densities $i\beta Q^2$ which determines the asymptotic condition (9) of the Jost solutions. The evolution equation of the latter part of the "potential" matrix is the conservation law (7).

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