SATURATION OF UNCERTAINTY RELATIONS FOR TWISTED ACCELERATION-ENLARGED NEWTON–HOOKE SPACE-TIMES

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Using Fock representation we construct states saturating uncertainty relations for twist-deformed acceleration-enlarged Newton–Hooke space-times.

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1. Introduction

In the last time, there appeared a lot of papers dealing with classical and quantum mechanics (see *e.g.* [1,2,3]) as well as with field theoretical models (see *e.g.* [4]), in which the quantum space-time plays a crucial role. The idea to use non-commutative coordinates is quite old — it goes back to Heisenberg and was firstly formalized by Snyder in [5]. Recently, however, there were found new formal arguments based mainly on Quantum Gravity [6] and String Theory models [7], indicating that space-time at Planck scale should be noncommutative, *i.e.* it should have a quantum nature. Besides, the main reason for such considerations follows from the suggestion that relativistic space-time symmetries should be modified (deformed) at Planck scale, while the classical Poincare invariance still remains valid at larger distances [8,9].

Currently, it is well known, that in accordance with the Hopf-algebraic classification of all deformations of relativistic and non-relativistic symmetries, one can distinguish three types of quantum spaces [10,11] (for details see also [12]):

(1) Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [13,14,15]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \,, \tag{1}$$

(2) Lie-algebraic modification of classical space-time [15, 16, 17, 18]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho} x_\rho \,, \tag{2}$$

(3) Quadratic deformation of Minkowski and Galilei spaces [15, 18, 19, 20]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau \,, \tag{3}$$

with coefficients $\theta_{\mu\nu}$, $\theta_{\mu\nu}^{\rho}$ and $\theta_{\mu\nu}^{\rho\tau}$ being constants.

Besides, it has been demonstrated in [21], that in the case of so-called acceleration-enlarged Newton–Hooke Hopf algebras $\mathcal{U}_0(\widehat{\mathrm{NH}}_\pm)$ the twist deformation provides the new space-time non-commutativity of the form[1,2,3]

(4)

$$[t, x_i] = 0 \,, \qquad [x_i, x_j] = if_\pm\left(\frac{t}{\tau}\right)\theta_{ij}(x) \,, \tag{4}$$

with time-dependent functions

$$\begin{aligned} f_+\left(\frac{t}{\tau}\right) &= f\left(\sinh\left(\frac{t}{\tau}\right), \cosh\left(\frac{t}{\tau}\right)\right) \,, \\ f_-\left(\frac{t}{\tau}\right) &= f\left(\sin\left(\frac{t}{\tau}\right), \cos\left(\frac{t}{\tau}\right)\right) \,, \end{aligned}$$

$\theta_{ij}(x) \sim \theta_{ij} = \mathrm{const.}$ or $\theta_{ij}(x) \sim \theta_{ij}^{k} x_k$ and τ denoting the time scale parameter — the cosmological constant.

It should be also noted that different relations between all mentioned above quantum spaces (1), (2), (3) and (4) have been summarized in paper [12].

From historical point of view, the studies on so-called coherent states were started by Schrödinger, who minimalized uncertainty relations for position and momenta operator in the case of harmonic oscillator model [25]. The result of these investigations has been applied in the 60s by Glauber [26] to provide a complete quantum-theoretical description of coherence for electromagnetic free field. It was a pioneer work in quantum optic theory describing phenomena associated with such processes as laser light emission or laser interferometry [27]. Recently, in articles [28, 29], the above-mentioned results

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[1] $x_0 = ct$.

[2] The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see [13, 14, 23, 24]), for quantum acceleration-enlarged Newton–Hooke Hopf algebras.

[3] The twisted (usual) Newton–Hooke quantum space-times have been provided in [22].

have been extended to the case of canonically deformed space-time (1). Particularly, it has been constructed the proper Fock space of quantum states and the deformed coherent wave functions.

In this article, following [28] and [29], we find coherent states for twisted acceleration-enlarged Newton–Hooke space-times (31), *i.e.* we provide states which saturate the deformed uncertainty relations (39)–(41). In the first section we recall basic facts associated with saturation of Heisenberg relations for commutative space-time. Section 2 concerns the saturation of twist-deformed uncertainty relations (39)–(41) — it contains the construction of Fock space and twisted coherent states. The final remarks are discussed in the last section.

2. Saturation of uncertainty relations and coherent states in commutative space-time

2.1. General prescription

Let us start with general algorithm for saturation of uncertainty principles described in [30, 31]. Hence, it is well-known that for arbitrary two observables $\hat{a}$, $\hat{b}$ such that

$$\left[\hat{a}, \hat{b}\right] = i\hat{c}\,, \tag{5}$$

one can derive the following (so-called generalized Heisenberg principle) inequality

$$(\Delta\hat{a})_\psi \cdot \left(\Delta\hat{b}\right)_\psi \geq \tfrac{1}{2}|\langle\hat{c}\rangle_\psi|\,, \tag{6}$$

where $|\psi\rangle$ denotes quantum state normalized to unity and

$$(\Delta\hat{o})_\psi = \sqrt{\langle\psi|(\hat{o} - \langle\hat{o}\rangle_\psi\mathbb{I})^2|\psi\rangle}\,; \qquad \hat{o} = \hat{a}, \hat{b}\,. \tag{7}$$

The Heisenberg relation (6) is saturated when the following condition is satisfied

$$(\hat{a} - \langle\hat{a}\rangle_\psi\mathbb{I})\,|\psi\rangle = -i\xi\left(\hat{b} - \left\langle\hat{b}\right\rangle_\psi\mathbb{I}\right)|\psi\rangle\,. \tag{8}$$

Further, by acting with $\hat{a} - \langle\hat{A}\rangle_\psi\mathbb{I}$ on both sides of equation (8), using formula (5) and again (8), one can rewrite the above condition as follows

$$(\hat{a} - \langle\hat{a}\rangle_\psi\mathbb{I})^2|\psi\rangle = -\xi^2\left(\hat{b} - \left\langle\hat{b}\right\rangle_\psi\mathbb{I}\right)^2|\psi\rangle + \xi\hat{c}|\psi\rangle\,, \tag{9}$$

or, equivalently, on multiplying by $|\psi\rangle$ from the left as

$$(\Delta\hat{a})^2_\psi + \xi^2\left(\Delta\hat{b}\right)^2_\psi = \xi\langle\hat{c}\rangle_\psi\,. \tag{10}$$

It is easy to check that the relation (10) together with the saturated form of Heisenberg principle (6) gives

$$(\Delta\hat{a})^2_\psi = \frac{\xi}{2}\langle\hat{c}\rangle_\psi\,, \qquad \left(\Delta\hat{b}\right)^2_\psi = \frac{1}{2\xi}\langle\hat{c}\rangle_\psi\,, \tag{11}$$

which explains the meaning of ξ.

2.2. The standard Heisenberg relation case

Let us now apply the above scheme to the standard Heisenberg relation

$$[\hat{x},\hat{p}] = i\hbar\,, \tag{12}$$

which yields inequality

$$\Delta\hat{x}\cdot\Delta\hat{p} \geq \frac{\hbar}{2}\,. \tag{13}$$

In accordance with the formula (8) one can observe that the uncertainty relation (13) is saturated iff

$$(\hat{x}-\alpha\mathbb{I})|\psi\rangle = -i\xi(\hat{p}-\beta\mathbb{I})|\psi\rangle\,, \tag{14}$$

where $\alpha = \langle\hat{x}\rangle_\psi$ and $\beta = \langle\hat{p}\rangle_\psi$. Next, we define in a standard way the creation/annihilation operators[4]

$$a \equiv \frac{1}{\sqrt{2\hbar}}(\hat{x}+i\hat{p})\,, \qquad a^\dagger \equiv \frac{1}{\sqrt{2\hbar}}(\hat{x}-i\hat{p})\,, \tag{15}$$

satisfying

$$\left[a,a^\dagger\right] = 1\,, \tag{16}$$

and then, the Hilbert space of states is spanned by the vectors

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle\,. \tag{17}$$

Firstly, in order to find the general solution of equation (14) one should notice that ξ is bigger than zero[5]. Further, we consider $\xi = 1$ and observe that in such a case the equation (14) can be rewritten as follows

$$a|\psi\rangle = z|\psi\rangle \qquad \text{with} \qquad z = \frac{\alpha+i\beta}{\sqrt{2\hbar}}\,. \tag{18}$$

[4] We use $\omega = m = 1$ units.

[5] In fact, parameter ξ is different than zero because operator $\hat{x}-\alpha\mathbb{I}$ cannot have normalized eigenvectors (operators commuting to c-number have no normalized eigenvectors in their common invariant domain). Consequently, for $\xi \neq 0$ equation (11) gives ξ bigger than zero.

The solutions of (18), *i.e.* the eigenstates of annihilation operator a are called coherent states and, particularly, the vacuum vector is coherent state corresponding to the eigenvalue z equal zero. In order to find remaining solutions of (18) one defines, for any complex value of z, the unitary operators

$$U(z) \equiv e^{za^\dagger - \bar{z}a} = e^{-\frac{1}{2}|z|^2} e^{za^\dagger} e^{-\bar{z}a} . \tag{19}$$

Next, one can easily check that

$$U^\dagger(z) a U(z) = a + z \cdot \mathbb{I} , \tag{20}$$

what means that any coherent state for $\xi = 1$ is given by

$$|z\rangle \equiv U(z)|0\rangle = e^{-\frac{1}{2}|z|^2} e^{za^\dagger}|0\rangle = e^{-\frac{1}{2}|z|^2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle . \tag{21}$$

Let us now turn to the case $\xi \neq 1$ for which formula (14) can be written as

$$a_\xi|\psi\rangle = z|\psi\rangle , \tag{22}$$

with

$$a_\xi = \frac{1}{\sqrt{2\hbar}}\left(\frac{\hat{x}}{\sqrt{\xi}} + i\sqrt{\xi}\hat{p}\right) , \qquad a_\xi^\dagger = \frac{1}{\sqrt{2\hbar}}\left(\frac{\hat{x}}{\sqrt{\xi}} - i\sqrt{\xi}\hat{p}\right) , \tag{23}$$

and

$$z = \frac{1}{\sqrt{2\hbar}}\left(\frac{\alpha}{\sqrt{\xi}} + i\beta\sqrt{\xi}\right) . \tag{24}$$

It is also easy to verify that

$$\left[a_\xi, a_\xi^\dagger\right] = 1 , \tag{25}$$

and that for $\xi = 1$ we have

$$a_{\xi=1} = a . \tag{26}$$

Solutions of equation (22) can be find with use of ξ-creation/annihilation $a_\xi^\dagger/a_\xi$ operators and ξ-vacuum state $|0\rangle_\xi$. However, all representations of Fock algebra are unitarily equivalent and, indeed, one can check that

$$V(\xi) a V^\dagger(\xi) = a_\xi , \qquad V(\xi) a^\dagger V^\dagger(\xi) = a_\xi^\dagger , \tag{27}$$

for the unitary operator $V(\xi)$ defined by

$$V(\xi) = e^{-\frac{1}{4}\ln\xi\left(a^2 - (a^\dagger)^2\right)} . \tag{28}$$

Consequently, the solution of equation (14) can be written as

$$|z,\xi\rangle = V(\xi)U(z)|0\rangle = e^{-\frac{1}{2}|z|^2} e^{-\frac{1}{4}\ln\xi\left(a^2-\left(a^\dagger\right)^2\right)} e^{za^\dagger}|0\rangle \tag{29}$$

with complex parameter z related to the mean values of $\hat{x}$ and $\hat{p}$ operators, and ξ describing their dispersions (see formulas (11) and (18), respectively)

$$(\Delta\hat{x})^2 = \frac{\xi\hbar}{2}\,, \qquad (\Delta\hat{p})^2 = \frac{\hbar}{2\xi}\,. \tag{30}$$

3. Coherent states for twist-deformed acceleration-enlarged Newton–Hooke space-times

In this section we turn to the twisted acceleration-enlarged Newton–Hooke space-times equipped with classical time and quantum spatial directions, *i.e.* we consider spaces of the form

$$[t,\bar{x}_i] = 0\,, \qquad [\bar{x}_1,\bar{x}_2] = if(t)\,; \qquad i=1,2\,, \tag{31}$$

with positive defined function $f(t)$ given by[6]

$$f(t) \;=\; f_{\kappa_1}(t) = f_{\pm,\kappa_1}\left(\frac{t}{\tau}\right) = \kappa_1\, C_\pm^2\left(\frac{t}{\tau}\right)\,, \tag{32}$$

$$f(t) \;=\; f_{\kappa_2}(t) = f_{\pm,\kappa_2}\left(\frac{t}{\tau}\right) = \kappa_2\tau^2\, S_\pm^2\left(\frac{t}{\tau}\right)\,, \tag{33}$$

$$f(t) \;=\; f_{\kappa_3}(t) = f_{\pm,\kappa_3}\left(\frac{t}{\tau}\right) = 4\kappa_3\tau^4\left(C_\pm\left(\frac{t}{\tau}\right)-1\right)^2\,, \tag{34}$$

$$C_{+/-}\left(\frac{t}{\tau}\right) = \cosh/\cos\left(\frac{t}{\tau}\right) \qquad \text{and} \qquad S_{+/-}\left(\frac{t}{\tau}\right) = \sinh/\sin\left(\frac{t}{\tau}\right)\,.$$

As it was already mentioned in Introduction, in $\tau\to\infty$ limit, the above quantum spaces reproduce the canonical (1), quadratic (3) and quartic type of space-time non-commutativity, with[7]

$$f_{\kappa_1}(t) \;=\; \kappa_1\,, \tag{35}$$

$$f_{\kappa_2}(t) \;=\; \kappa_3\, t^2\,, \tag{36}$$

$$f_{\kappa_3}(t) \;=\; \kappa_4\, t^4\,. \tag{37}$$

[6] $\kappa_a > 0$.

[7] Space-times (35), (36) correspond to the twisted Galilei Hopf algebras provided in [15], while the quantum space (37) is associated with acceleration-enlarged Galilei Hopf structure [21].

Of course, for all parameters κ_a running to zero the above deformations disappear.

The above spaces can be extended to the whole algebra of position and momentum operators as follows

$$[\bar{x}_1, \bar{x}_2] = i f_{\kappa_a}(t)\,, \quad [\bar{p}_i, \bar{p}_j] = 0\,, \quad [\bar{x}_i, \bar{p}_j] = i\hbar\delta_{ij}\,, \qquad i,j = 1,2\,, \tag{38}$$

and then, the corresponding uncertainty relations take the form

$$\Delta\bar{x}_1\Delta\bar{x}_2 \ \geq\ \frac{f_{\kappa_a}(t)}{2}\,, \tag{39}$$

$$\Delta\bar{x}_1\Delta\bar{p}_1 \ \geq\ \frac{\hbar}{2}\,, \tag{40}$$

$$\Delta\bar{x}_2\Delta\bar{p}_2 \ \geq\ \frac{\hbar}{2}\,. \tag{41}$$

In the next two subsections we construct the quantum-mechanical states saturating the deformed Heisenberg principles (39)–(41). Partially, we use algorithm described in pervious section and the results of articles [28, 29].

3.1. Oscillator representations

In order to find the coherent states associated with twisted commutation relations (38), we provide their oscillator (irreducible) representations. First of all, we observe that position and momentum operators $\bar{x}_i$ and $\bar{p}_i$ can be written in terms of canonical ones ($\hat{x}_i$, $\hat{p}_i$) as follows

$$\bar{x}_i \equiv \hat{x}_i - \frac{f_{\kappa_a}(t)}{2\hbar}\epsilon_{ij}\hat{p}_j\,, \qquad \bar{p}_i \equiv \hat{p}_i\,, \tag{42}$$

with $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$. Then, it seems sensible to introduce the following definition of creation/anihilation operators

$$a_i(t) \ \equiv\ \frac{1}{\sqrt{2\hbar}}\left[\bar{x}_i + \left(i\delta_{ij} + \frac{f_{\kappa_a}(t)}{2\hbar}\epsilon_{ij}\right)\bar{p}_j\right]\,, \tag{43}$$

$$a_i^\dagger(t) \ \equiv\ \frac{1}{\sqrt{2\hbar}}\left[\bar{x}_i + \left(-i\delta_{ij} + \frac{f_{\kappa_a}(t)}{2\hbar}\epsilon_{ij}\right)\bar{p}_j\right]\,, \tag{44}$$

which satisfy

$$\left[a_i(t), a_j^\dagger(t)\right] = \delta_{ij}\,. \tag{45}$$

In such a way we arrive at Fock space spanned by the orthonormal vectors of the form

$$|n_1, n_2, t\rangle = \frac{1}{\sqrt{n_1!}}\frac{1}{\sqrt{n_2!}}\left(a_1^\dagger(t)\right)^{n_1}\left(a_2^\dagger(t)\right)^{n_2}|0\rangle\,. \tag{46}$$

For later convenience we also provide the modified operators

$$a_\pm(t) \equiv \frac{1}{\sqrt{2}}\left(a_1(t) \mp i a_2(t)\right)\,, \qquad a_\pm^\dagger(t) \equiv \frac{1}{\sqrt{2}}\left(a_1^\dagger(t) \pm i a_2^\dagger(t)\right)\,, \tag{47}$$

leading to the following (new) basis

$$|n_+, n_-, t\rangle = \frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}\left(a_+^\dagger(t)\right)^{n_+}\left(a_-^\dagger(t)\right)^{n_-}|0\rangle\,. \tag{48}$$

3.2. Saturating of uncertainty relations

Let us construct all states which saturate the uncertainty relations (39). To this end, in accordance with algorithm proposed in [28] and [29], we define the following set of independent creation/anihilation operators

$$\begin{aligned}
b(t) &\equiv \sqrt{\frac{\hbar}{2f_{\kappa_a}(t)}}\left[\left(1+\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_- + \left(1-\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_+^\dagger\right]\,,\\
b^\dagger(t) &\equiv \sqrt{\frac{\hbar}{2f_{\kappa_a}(t)}}\left[\left(1+\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_-^\dagger + \left(1-\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_+\right]\,,\\
c(t) &\equiv \sqrt{\frac{\hbar}{2f_{\kappa_a}(t)}}\left[\left(1+\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_+ + \left(1-\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_-^\dagger\right]\,,\\
c^\dagger(t) &\equiv \sqrt{\frac{\hbar}{2f_{\kappa_a}(t)}}\left[\left(1+\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_+^\dagger + \left(1-\frac{f_{\kappa_a}(t)}{2\hbar}\right)a_-\right]\,.
\end{aligned} \tag{49}$$

Next, by straightforward calculations we get

$$b(t) = \frac{1}{\sqrt{2f_{\kappa_a}(t)}}\left(\bar{x}_1 + i\bar{x}_2\right), \qquad b^\dagger(t) = \frac{1}{\sqrt{2f_{\kappa_a}(t)}}\left(\bar{x}_1 - i\bar{x}_2\right), \tag{50}$$

what means that both b-operators are spanned in a standard way by non-commutative positions $\bar{x}_1$ and $\bar{x}_2$. Consequently, due to the commutation relations (31)[8] one can applied the standard scheme proposed in Section 2. Then, in accordance with formula (29) we have

$$|z, \xi, t\rangle = e^{-\frac{1}{2}|z|^2} e^{+\frac{1}{4}\ln\xi\left(\left(b^\dagger(t)\right)^2 - b^2(t)\right)} e^{z b^\dagger(t)}|0, t\rangle_b\,, \tag{51}$$

where symbol $|0, t\rangle_b$ denotes the vacuum state for annihilator $b(t)$, *i.e.*

$$b(t)|0, t\rangle_b = 0\,. \tag{52}$$

[8] They are the same as canonical commutation relations (12) with operator $\hat{p}$ replaced by $\hat{x}_2$ and $\hbar$ replaced by function $f_{\kappa_a}(t)$.

It should be noted that the choice of state (52) is not unique, *i.e.* it may contain an arbitrary number of $c^\dagger(t)$ excitation. Besides, one should observe that in accordance with formal arguments proposed in [28,29], the representation given by $b(t)$, $b^\dagger(t)$, $c(t)$ and $c^\dagger(t)$ operators is unitary equivalent to that defined by $a_\pm(t)$ and $a_\pm^\dagger(t)$. The corresponding transformation is given by[9]

$$\begin{aligned} b(t) &= T(t)\,[a_-(t)]\,T^\dagger(t)\,, &(53)\\ b^\dagger(t) &= T(t)\left[a_-^\dagger(t)\right]T^\dagger(t)\,, &(54)\\ c(t) &= T(t)\,[a_+(t)]\,T^\dagger(t)\,, &(55)\\ c^\dagger(t) &= T(t)\left[a_+^\dagger(t)\right]T^\dagger(t)\,, &(56)\end{aligned}$$

with

$$T(t) = e^{\frac{1}{2}\ln\left(\frac{2\hbar}{f_{\kappa_a}(t)}\right)\left(a_+(t)a_-(t)-a_+^\dagger(t)a_-^\dagger(t)\right)}\,. \tag{57}$$

Consequently, it means that the states saturating (31) are linear combinations with respect to n_+ of the vectors[10]

$$|z,\xi,n_+,t\rangle = e^{-\frac{1}{2}|z|^2}T(t)e^{-\frac{1}{4}\ln\xi\left(a_-^2(t)-\left(a_-^\dagger(t)\right)^2\right)}e^{za_-^\dagger(t)}|n_+,0,t\rangle\,. \tag{58}$$

Let us now turn to the states saturating

$$\Delta\bar{x}_1\Delta\bar{p}_1 \geq \frac{\hbar}{2}\,. \tag{59}$$

Firstly, as in the pervious case, we define the new creation/anihilation operators as follows

$$\begin{aligned} d(t) &= a_1(t) + \frac{if_{\kappa_a}(t)}{4\hbar}\left(a_2(t)-a_2^\dagger(t)\right)\,, &(60)\\ d^\dagger(t) &= a_1^\dagger(t) + \frac{if_{\kappa_a}(t)}{4\hbar}\left(a_2(t)-a_2^\dagger(t)\right)\,, &(61)\\ e(t) &= a_2(t) + \frac{if_{\kappa_a}(t)}{4\hbar}\left(a_1(t)-a_1^\dagger(t)\right)\,, &(62)\\ e^\dagger(t) &= a_2^\dagger(t) + \frac{if_{\kappa_a}(t)}{4\hbar}\left(a_1(t)-a_1^\dagger(t)\right)\,, &(63)\end{aligned}$$

which in d-sector take the form

$$d(t) = \frac{1}{\sqrt{2\hbar}}(\bar{x}_1 + i\bar{p}_1)\,, \qquad d^\dagger(t) = \frac{1}{\sqrt{2\hbar}}(\bar{x}_1 - i\bar{p}_1)\,. \tag{64}$$

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[9] It can be find by analogy to the algorithm proposed in [28] and [29] for the case of canonical deformation (1).

[10] z and ξ are fixed.

Further, one can find unitary transformation connecting $d(t)$ and $e(t)$ operators with old ones. It looks as follows

$$d(t) = S(t)\,[a_1(t)]\,S^{\dagger}(t)\,, \qquad e(t) = S(t)\,[a_2(t)]\,S^{\dagger}(t)\,, \tag{65}$$

where

$$S(t) = e^{\frac{if_{\kappa_a}(t)}{4\hbar}\left(a_1(t)-a_1^{\dagger}(t)\right)\left(a_2(t)-a_2^{\dagger}(t)\right)}\,. \tag{66}$$

Consequently, the states saturating (59) can be written as linear combinations, with respect to n_2 but with parameters z and ξ fixed, of the following vectors

$$|z,\xi,n_2,t\rangle = e^{-\frac{1}{2}|z|^2}S(t)e^{-\frac{1}{4}\ln\xi\left(a_1(t)^2-\left(a_1^{\dagger}(t)\right)^2\right)}e^{za_1^{\dagger}(t)}|0,n_2,t\rangle\,, \tag{67}$$

It is easy to see, that the states saturating (41) are obtained by exchanging index "1" to "2" and functions $f_{\kappa_a}(t)$ to $-f_{\kappa_a}(t)$ in formula (67), *i.e.*

$$|z,\xi,n_1,t\rangle = e^{-\frac{1}{2}|z|^2}S^{\dagger}(t)e^{-\frac{1}{4}\ln\xi\left(a_2(t)^2-\left(a_2^{\dagger}(t)\right)^2\right)}e^{za_2^{\dagger}(t)}|0,n_1,t\rangle\,. \tag{68}$$

4. Final remarks

In this article we construct states saturating uncertainty relations for twisted acceleration-enlarged Newton–Hooke space-times (4). Particularly, for very special choice of considered quantum spaces we get the results obtained in [28, 29] for canonical deformation (1).

It should be noted that presented investigation has been performed for the case of quite simple deformation with two spatial directions commuting to function of classical time. However, the mentioned studies can be extended in non-trivial way to much more complicated space-time models, such as Lie-algebraic or quadratic type of non-commutativity with two spatial directions commuting to space. Besides, one should better understand the obtained results in context of (for example) wave-gravitational interferometry processes, for which saturating states play a prominent role [27]. The works in these directions already started and are in progress.

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