

# REINTERPRETATION OF GLUON CONDENSATE IN DYNAMICS OF HADRONIC CONSTITUENTS

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We describe an approximate quantum mechanical picture of hadrons in Minkowski space in the context of a renormalization group procedure for effective particles (RGPEP) in a light-front Hamiltonian formulation of QCD. The picture suggests that harmonic oscillator potentials for constituent quarks in lightest mesons and baryons may result from the gluon condensation inside hadrons, rather than from an omnipresent gluon condensate in vacuum. The resulting boost-invariant constituent dynamics at the renormalization group momentum scales comparable with  $\Lambda_{\text{QCD}}$ , is identified using gauge symmetry and a crude mean-field approximation for gluons. Besides constituent quark models, the resulting picture also resembles models based on AdS/QCD ideas. However, our hypothetical picture significantly differs from the models by the available option for a systematic analysis in QCD, in which the new picture may be treated as a candidate for a first approximation. This option is outlined by embedding our presentation of the crude and simple hadron picture in the context of RGPEP and a brief outlook on hadron phenomenology. Several appendices describe elements of the formalism required for actual calculations in QCD, including an extension of RGPEP beyond perturbation theory.

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## 1. Introduction

The glaring contrast between the simplicity of the constituent quark model (CQM) classification scheme for hadrons in particle data tables [1] and the many complexities and subtleties of QCD has not been resolved since quarks were proposed to explain hadrons [2] and QCD was proposed as a theory of strong interactions [3, 4, 5]. This contrast can be illustrated by the simplicity of attempts to build relativistic quark models, where one introduces harmonic oscillator potentials [6], in comparison with the complexity of QCD sum rules [7], in which even the state without any hadron, *i.e.*, vacuum, is meant to contain a complex structure. The complex structure is reflected in the concept of condensates.

The vacuum structure in quantum field theory (QFT) eludes theorists [8] despite the fact that they have a complete command of perturbative calculations. For example, in the case of QCD, theorists can use asymptotic freedom [9,10] to calculate quark and gluon scattering amplitudes but they also have to use a parton model [11] to relate their calculations to observables. The point is that the mechanism of binding of partons continues to be a mystery. The mystery is understood in principle as related to the complex features of the theory that lie beyond reach of perturbative approaches. Regarding these complex features, palpable progress in calculating hadronic masses is achieved using lattice formulation of QCD [12,13]. However, so far it is not clear how to use QCD for producing quark and gluon wave functions of hadrons in the Minkowski space and thus provide a picture of hadrons of comparable precision to the picture of atoms achieved in QED.

In this situation, it is interesting to note that there exists a formulation of QCD which uses Dirac's light-front (LF) form of Hamiltonian dynamics [14,15] and does not introduce any complex vacuum structure [16]. How could then the LF formulation of QCD produce the effects that in other approaches are associated with the concept of a complex vacuum state, including effects associated with spontaneous symmetry breaking or the gluon condensate? Ref. [16] points out that an effective Hamiltonian expected to result from a renormalization group analysis of the canonical LF Hamiltonian of QCD with all counterterms that are necessary for stabilizing the analysis, may contain new terms that provide such effects.

Here it is argued that the renormalization group procedure for effective particles (RGPEP, see Sec. 2) provides a new way of thinking about light quarks in LF QCD. In this new way, the harmonic oscillator potentials, previously associated in Ref. [17] with a gluon condensate in vacuum, can be interpreted as coming from the gluon content of a hadron rather than a vacuum. Previous reasoning in Ref. [17] was developed using the instant form [14] of dynamics, *i.e.*, the standard Hamiltonian evolution in time. Now, the possibility of considering harmonic oscillator potentials for light quarks at distances comparable with the characteristic size of hadrons, meant here to be equivalent to the distances on the order of  $1/\Lambda_{\text{QCD}}$  in the RGPEP scheme, is pointed out using a decomposition of hadronic states into the LF Fock space components, *i.e.*, states obtained from empty vacuum using creation operators defined in the LF quantization of fields rather than the standard canonical quantization of the instant form. These Fock components contain various numbers of virtual effective particles whose interactions depend on the renormalization group scale in the RGPEP scheme. The scale is denoted by  $\lambda$ . One can think about  $\lambda$  as having dimension of momentum. Low-energy features of light states in QCD made of light quarks are expected describable using an effective Hamiltonian that corresponds to

$\lambda$  comparable with  $\Lambda_{\text{QCD}}$ . Readers interested in the details of RGPEP that are already known in the case of heavy quark dynamics, can consult Ref. [18], where an approximate QCD calculation of masses of heavy quarkonia is described. Here only the case of light quarks is discussed.

According to Ref. [19], if gluon condensate effects are coming from the gluon content of hadrons rather than a vacuum<sup>1</sup>, one can avoid the problem of an excessively large vacuum energy density in cosmology. However, the new reasoning presented here does not deal with cosmological issues. Instead, this article explains how the mean-field mechanism that was described in Ref. [17] can be reinterpreted in the context of RGPEP in QCD. In the mechanism in Ref. [17], the gluon condensate meant to exist in a vacuum provided harmonic oscillator potentials for constituent quarks in light mesons and baryons. The potentials agreed with known phenomenological CQMs [21, 22, 23] when the expectation value of the gluon field strength squared was equated to the vacuum gluon condensate value used in the QCD sum rules [7]. It is argued below that it was not necessary to assume, as it was done in Ref. [17], that the gluonic expectation value came from the empty vacuum. The point of view developed here is that the gluon expectation value might be coming from the content of a hadron itself, and the relevant content can in principle be identified in LF QCD using RGPEP. Potentially broad implications of such change in interpretation are not discussed here. The hypothesis put forward in this article requires a considerable amount of work to verify and it would be premature to draw broad conclusions before relevant RGPEP calculations are completed.

The calculations described here arrive at similar harmonic potentials to those found in Ref. [17] but in a different way and with a significantly different interpretation. Namely, instead of starting from the non-relativistic (NR) Schrödinger equation and gluon vacuum condensate, one starts from the LF QCD Hamiltonian eigenvalue equation for mesons and baryons. Canonical quark and gluon field operators are formally transformed using RGPEP to effective operators at the running cutoff scales that are comparable with  $\Lambda_{\text{QCD}}$ . Using color gauge invariance principle, together with a mean-field approximation that still needs justification and is not verified yet by a rigorous *ab initio* calculation, one arrives at the eigenvalue equations in which expectation values of effective gluon fields are evaluated in a gluonic component of hadronic states rather than in the vacuum. The gluonic component is postulated to be universal. Then, using new relative momentum variables, one obtains LF wave functions that describe the relative motion of hadronic effective constituents in a boost-invariant way. In summary, the reasoning described here implies a relativistic picture of hadrons in which

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<sup>1</sup> The quark condensate has also been suggested to originate in hadronic content instead of a vacuum [20].

the same harmonic oscillator potentials that agree with CQMs phenomenology may result from the gluon content of hadrons in LF QCD rather than from the vacuum *per se*.

It should be clarified that harmonic potentials among effective constituents in lightest hadrons apply in a situation where color charges are close to each other. If they were separated by a distance considerably larger than the size of a light hadron, their invariant mass would increase considerably due to the potential. Interactions other than the potential term alone would be activated, able to create additional particles and changing the dynamics, perhaps including a string of effective gluons.

The new way of thinking about effective, low-energy dynamics of light quarks, introduced here through the RGPEP, differs from the strategy described in Ref. [16] in several ways besides identification of vacuum-like effects. The new elements that are extensively discussed in the following sections include: the dynamical transformation from bare to effective particles of size  $s = 1/\lambda$ , construction of the corresponding effective fields, possibility of using NR approximation at small  $\lambda$ , identification of effective interactions through gauge symmetry (instead of introducing an entirely *ad hoc* potential), extension beyond perturbation theory, and invariance with respect to 7 kinematical LF symmetries.

It follows from the RGPEP that the low-energy (actually, small invariant mass) LF dynamics of quarks should include a gluon component. We use a mean-field approximation to show that such gluon component may provide expectation values of the type that is associated with vacuum in the instant form of quark dynamics. As a result, the first-approximation potential for a LF mass-squared operator of quarks at small  $\lambda$  is obtained in the form of a quadratic function of distances among the effective quarks, not a linear one. Note, however, that the LF operator where the quadratic potential enters is mass squared. This means that, for a large distance between static constituents, the mass squared is proportional to the distance squared. This in turn means that the mass is proportional to the distance. Thus, the LF oscillator potential actually corresponds to a linear potential in a conventional way of thinking in terms of Hamiltonians in the instant form of dynamics.

This new feature of the LF oscillator potential was entirely absent in the Ref. [17], which introduced the oscillator potential only in the instant form of dynamics. Therefore, although the introduction of the mean-field approximation in the LF mass-squared operator at small  $\lambda$  shares the step of evaluating expectation values of the gluon field with Ref. [17], the state that provides the expectation value is quite different and the interpretation of the potential is quite different. In addition, the introduction of the potential in the LF mass-squared operator requires new relative momentum variables for constituent quarks in mesons and baryons, and the required LF definitions are provided in the text.

A surprising new outcome of the present article is that the new variables used to define the effective LF oscillator potential correspond to the variables that Brodsky and Teramond used to show how AdS/QCD ideas may be incorporated in the LF dynamics [24, 25, 26], including the soft-wall model oscillator potential [27]. The present reasoning suggests that the coefficient of the SW-model oscillator potential in Brodsky–Teramond holography may be identified with the condensate of gluons inside hadrons.

Most importantly, however, the LF Hamiltonian formulation of QCD must include an *ab initio* renormalization group procedure. This is a necessary condition for the whole theory to become well-defined and for the issue of effective gluon expectation values in hadronic states to become well-posed. The whole RGPEP scheme is further required for carrying out more detailed calculations than the crude mean-field ones described in this article. The more detailed calculations are required to verify the extent to which the heuristic picture of quarks oscillating in the mean-field of hadronic gluons may be a good approximation to a full solution for light mesons and baryons in LF QCD. The complexity of the required scheme and the need for placing the discussion of gluon condensation within the scheme of RGPEP, are the actual reasons for a considerable length of this article.

The article is organized in the following way. Section 2 describes the concept of effective particle in the context of RGPEP and explains why the vertex form factors (denoted by  $f_\lambda$ ) eliminate changes of the number of massive virtual effective particles. This feature is required for thinking that a hadron can be represented by a convergent expansion in the basis of the Fock space. The basis that counts is not built in terms of canonical quark and gluon operators but in terms of the effective ones, corresponding to relatively small scale parameter  $\lambda$ . The convergence is not proven, but it is deemed not excluded given the exponential falloff of the RGPEP form factors  $f_\lambda$  as functions of constituent invariant masses. Section 3 outlines the representation of states of light hadrons in the effective particle basis in the Fock space. Section 4 discusses a perturbative expansion for the unitary transformation  $U_\lambda$  that connects current, or canonical quarks and gluons with their CQM counterparts, and for the transformations  $W_{\lambda_1\lambda_2}$  that connect effective particles that correspond to different values of the scale  $\lambda$ . The eigenvalue problem for light hadrons is considered in Sec. 5. This section describes our derivation of LF invariant mass operators for quarks including their minimal coupling to the gluons condensed in a hadron. Next section shows how the RGPEP can be applied in phenomenology of hadrons, with emphasis on form factors, structure functions, and connection with AdS/QCD ideas. Section 7 concludes the paper. Finally, four appendices are provided in order to support the claim that RGPEP can be used to verify our reinterpretation of the gluon condensate in QCD. Appendix A outlines how CQMs can be viewed as limited representatives of the same universality class

that QCD belongs to. Appendix B describes details concerning perturbative calculations of RGPEP transformations. Presentation of RGPEP beyond perturbation theory is given in Appendix C. The last Appendix D offers a visualization of the RGPEP scale dependence of hadronic structure.

## 2. Effective particles and vertex form factors

Reasoning described in the next sections requires elements of RGPEP [28]<sup>2</sup>. This section focuses on the concept of effective particles and the role of vertex form factors that RGPEP introduces in the interactions of effective particles, starting from QCD.

In QCD, RGPEP begins with the canonical LF Hamiltonian in which the dynamically independent quark and gluon quantum fields are expanded into their Fourier components. These components are the operators that create or annihilate single bare quarks or gluons of definite momentum. For brevity, creation and annihilation operators will be commonly called particle operators, in order to distinguish them where necessary from quark and gluon field operators.

The canonical LF Hamiltonian contains singular interaction terms. It must be regulated to avoid infinities. This is done by introducing regularization factors in interaction terms. These factors are constructed to limit relative motion of particles that participate in interactions. For example, let the total momentum of all particles in an interaction term have components  $P^+$  and  $P^\perp$ . The notation means:  $P^+ = P^0 + P^z$ ,  $z$ -axis is the direction distinguished in the definition<sup>3</sup> of the LF,  $\perp$  denotes transverse components, *i.e.*, the components  $x$  and  $y$  that are transverse to the  $z$ -axis. Let a selected particle have momentum components  $p^+ = xP^+$  and  $p^\perp = xP^\perp + k^\perp$ . Then, the regulating factor for this particle that is sufficient for taming logarithmic divergences in QCD can be of the form  $x^\delta \exp(-k^{\perp 2}/\Delta^2)$  [29], where  $\delta \rightarrow 0$  and  $\Delta \rightarrow \infty$  are the small- $x$  and ultraviolet regularization parameters, respectively. Initially, one also introduces an absolute infinitesimal lower bound  $\epsilon^+ < p^+$  for all particle operators in order to eliminate the LF zero modes. This step amounts, however, to saying that all particles must have positive momentum component  $p^+$ . This requirement eliminates complex vacuum. Subsequently, one demands that for every particle in every interaction term  $x > \epsilon$ , with an infinitesimal number  $\epsilon$ . The number  $\epsilon$  is a

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<sup>2</sup> Appendix A describes RGPEP in the context of asking how QCD can be transformed into an effective theory that resembles CQMs as approximate representatives of the same universality class.

<sup>3</sup> LF is a hyper-plane in space-time that is swept by a wave front of a plane wave of light. In standard notation, the frame of reference of the inertial observer who develops a quantum theory is set up so that the plane wave moves against the  $z$ -axis in the chosen frame.

fraction of momentum. It is not related to  $\epsilon^+$ . Then, when one works in the limit  $\epsilon x^{-\delta} \rightarrow 0$  point-wise in  $x$  when  $\epsilon$  is sent to 0, the factor  $x^\delta$  takes over the regulatory function from  $\epsilon$ . As a result, the variables  $x$  range from 0 to 1 and there are regulating factors  $x^\delta$  and  $(1-x)^\delta$  at the end points in all interaction vertices. The variable  $x = p^+/P_h^+$  for a particle of momentum  $p$  in a hadron of momentum  $P_h$  coincides with the parton model longitudinal momentum fraction in a hadron in the infinite momentum frame (IMF). Variable  $k^\perp$  is equal to the transverse momentum of a parton in the IMF when the parent hadron of the parton moves precisely along  $z$ -axis. Note, however, that LF QCD is formulated in terms of the same variables  $x$  and  $k^\perp$  no matter how hadrons move, *i.e.*, the LF Hamiltonian theory is explicitly invariant with respect to the 7 Poincaré transformations that preserve the LF. This means that the theory of a hadron structure in a rest frame of the hadron has the same form as in the IMF.

When the regularization is being removed,  $\delta \rightarrow 0$  and  $\Delta \rightarrow \infty$ , RGPEP establishes the required ultraviolet counterterms [28]. The small- $x$  regularization drops out from dynamics of colorless states because they cannot produce long-distance interactions along the LF.

Let the regularized LF Hamiltonian of QCD with counterterms be denoted by  $H$  and let  $b$  denote bare particle operators. RGPEP introduces effective particles of scale  $\lambda$  through a unitary transformation

$$b_\lambda = U_\lambda b U_\lambda^\dagger. \quad (1)$$

The corresponding Hamiltonian operator,

$$H_\lambda(b_\lambda) = H(b), \quad (2)$$

is a combination of products of operators  $b_\lambda$  with coefficients  $c_\lambda$  that are different from coefficients  $c$  of corresponding products of operators  $b$  in the canonical Hamiltonian with counterterms,  $H(b)$ . Since  $\lambda$  is related to an upper limit on momentum transfers in interactions, the operators  $b$  correspond to  $\lambda = \infty$ . For the infinite  $\lambda$ , we have  $b_\infty = b$  and  $H(b) = H_\infty(b_\infty)$ <sup>4</sup>.

RGPEP provides differential equations that produce expressions for the coefficients  $c_\lambda$  in  $H_\lambda$ . The operator  $U_\lambda$  is calculable in RGPEP order-by-order in an effective coupling constant [28] in the form of normal-ordered products of particle operators  $b_\lambda$ . Appendix B illustrates how it is done in lowest orders. Appendix C shows how non-perturbative calculations can be attempted. Appendices A, B, and C, provide a formal background for the entire discussion that follows.

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<sup>4</sup> Where it is unlikely to lead to a confusion,  $H_\lambda(b_\lambda)$  is abbreviated to  $H_\lambda$ . Later, also operators such as  $H_{\lambda_1}(b_{\lambda_2})$  occur, meaning an operator that is a combination of products of particle operators  $b_{\lambda_2}$  with coefficients  $c_{\lambda_1}$  instead of  $c_{\lambda_2}$ .

The key feature of  $H_\lambda(b_\lambda)$  that emerges from RGPEP is that all coefficients  $c_\lambda$  contain a form factor  $f_\lambda$ . The form factor prevents changes of the total invariant mass of effective particles undergoing interaction by more than  $\lambda$ . The RGPEP parameter  $\lambda$  plays thus the role of a momentum-space width of vertex form factors in all interaction terms. In fact, RGPEP is designed to work this way. When  $\lambda$  is small, the form factors suppress all interactions among massive particles besides the terms one can call potentials, *i.e.*, the interaction terms that do not change the number of massive particles (see below). In other words, the RGPEP transformation  $U_\lambda$  is designed to identify the dynamical relationship between nearly massless current quarks and their field-theoretic interactions at formally infinite  $\lambda$  with massive constituent quarks and their interactions through potentials at  $\lambda$  so small that potential models may apply as an approximation to solutions of the whole theory for states of smallest masses.

If one so desires, transformation  $U_\lambda$  can be kinematically complemented with the Melosh transformation [30], which relates the spinors typically used for description of constituent quarks in the constituents rest frame (CRF<sup>5</sup>), with the spinors one can conveniently use in the IMF<sup>6</sup>. Melosh transformations for spinors are automatically incorporated in the LF spinors used here.

It follows from its definition that the form factor  $f_\lambda$  prevents the number of effective particles from changing if their masses  $m_\lambda$  exceed  $\lambda$ . For example, suppose that a virtual particle of mass  $m_\lambda$  emits another virtual particle with mass  $m_\lambda$ . The change of invariant mass of particles in the interaction is at least  $m_\lambda$ . The form factor  $f_\lambda$  is designed to quickly tend to 0 for invariant-mass changes greater than  $\lambda$ . So, if  $m_\lambda$  exceeds  $\lambda$ , the form factor eliminates the emission. In particular, if effective gluons are assumed to have effective

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<sup>5</sup> The CRF frame is an inertial reference frame in which the total momentum of constituents treated as free particles of definite masses has only time component different from zero. The concept of CRF in LF dynamics is different from a similar concept in the instant form of dynamics, because conservation of  $P^+$  in interactions makes the CRF differ from the center-of-mass system (CMS) of a hadron in the LF dynamics, while the CRF and CMS are the same in the instant form of dynamics, where  $P^z$  is conserved by interactions.  $P^\perp$  is preserved in interactions in both forms of dynamics equally.

<sup>6</sup> Ref. [4] is of interest here in a context of the dynamical connection between canonical (or current) quarks and effective (or constituent) quarks that Melosh sought, because the last sentence in Ref. [4] suggests one might perhaps use some collective coordinates as a satisfactory method for truncating QCD, instead of “the brute-force lattice gauge theory approximation!” While Ref. [4] is quite new to the author at the time of writing this article, it should be noted that RGPEP can be viewed as an attempt of the type advocated by Gell-Mann. Namely, the effective quarks and gluons, as constituents of size  $s = \lambda^{-1} \sim \Lambda_{\text{QCD}}^{-1}$ , describe collective modes in dynamics of many small quarks, anti-quarks, and gluons, while the latter may still be individually active in virtual processes characterized by large invariant-mass changes, allowed in a single interaction only when  $\lambda \gg \Lambda_{\text{QCD}}$ .

masses  $m_\lambda$ , the Hamiltonian  $H_\lambda$  will not include interactions that can change the number of effective gluons when  $m_\lambda > \lambda$ . A sizeable mass of effective gluons in  $H_\lambda$  is what one may expect to happen at small values of  $\lambda$  in QCD because such effective gluon mass is a candidate for explaining the absence of small spacing in the spectrum of hadronic masses. There should be small spacing in the presence of massless gluons, as it happens in atoms described by QED with massless photons. In contrast to atoms, hadrons do not exhibit such small spacing.

Note that a smooth form factor  $f_\lambda$  must allow for some small, transitional range of values of  $\lambda < m_\lambda$ . In the transitional range, interactions that change the number of effective gluons gradually disappear when  $\lambda$  is lowered below  $m_\lambda$ . In the reversed RGPEP evolution in  $\lambda$  from small to large values, gluons will gradually appear in the dynamics as  $\lambda$  increases and  $m_\lambda$  decreases so that at some point  $\lambda$  becomes greater than  $m_\lambda$ .

### 3. RGPEP representation of states of light hadrons

If QCD is to explain precisely the success of classification of light hadrons in terms of constituent quarks with masses on the order of 1/3 of a nucleon mass, a clearly defined procedure must explain how the masses  $m_u$  and  $m_d$  of the lightest quarks increase from their standard model (SM) values of order 5 MeV in a local gauge theory to their common constituent value of order  $\Lambda_{\text{QCD}}$  in a corresponding effective theory<sup>7</sup>. It is not known yet if RGPEP leads to such increase of effective masses of light quarks in QCD when the parameter  $\lambda$  is lowered toward  $\Lambda_{\text{QCD}}$ , or even below  $\Lambda_{\text{QCD}}$ . We assume here that RGPEP does lead to such result<sup>8</sup>. This assumption cannot be rigorously verified yet because the domain of  $\lambda \sim \Lambda_{\text{QCD}}$  cannot be reached using low orders of perturbation theory and non-perturbative solutions to RGPEP equations are still not known in QCD.

When  $\lambda$  is large in comparison to  $\Lambda_{\text{QCD}}$ , so that the effective coupling  $g_\lambda$  is small and perturbation theory applies, the eigenvalue problem involves many Fock sectors with many quarks of small Lagrangian masses. In these circumstances, it is hard to analyze the eigenvalue equations precisely. When  $\lambda$  is lowered, the number of necessary Fock sectors is expected to decrease because of the vertex form factors  $f_\lambda$  but the coupling constant  $g_\lambda$  increases and it is hard to evaluate  $H_\lambda$  precisely. A way out of this situation is to calculate  $H_\lambda$  in terms of successive approximations<sup>9</sup>.

<sup>7</sup> How RGPEP is able to relate a local canonical theory to a non-local effective theory is explained in [31].

<sup>8</sup> Quark masses increase as momentum scale is lowered in perturbation theory developed in terms of the Feynman diagrams [32]. Similar results are obtained from Dyson–Schwinger equations [33]. LF constituent models can incorporate running masses [34].

<sup>9</sup> This includes successive approximations for solutions of the non-perturbative RGPEP equations in Appendix C.

A workable RGPEP method for building successive approximations has been outlined including results for hadron masses and wave functions in the case of heavy quarkonia [18], where the number of heavy quarks is nearly constant, equal 2. In that case, one can take advantage of the NR approximation for relative motion of quarks, based on smallness of the effective coupling constant  $g_\lambda$  when  $\lambda$  is comparable with heavy quark masses that are much larger than  $\Lambda_{\text{QCD}}$ .

However, no counterpart for such approximation scheme is available yet for light quarks in LF QCD using RGPEP. The reason, stated already above, is that, in distinction from the case of heavy quarks, the light quark masses are so much smaller than  $\Lambda_{\text{QCD}}$  that many light quarks can *a priori* be created or annihilated in the interactions that limit virtual energy changes by about  $\Lambda_{\text{QCD}}$ . Therefore, one needs to approximate reasonably well a great deal of relativistic dynamics of virtual particles that are constantly created and annihilated by strong interactions. It is suggested below that a reinterpretation of the gluon condensate as a part of a hadron provides a guideline for moving in that direction.

It is a conceptual jump that requires a connection between constituent quarks and structure functions (see Sec. 6.2), but one can observe here that parton distribution functions at small  $Q^2$  are typically very stable as functions of  $Q^2$  and quarks always carry only about half of the nucleon momentum [35, 36, 37, 38, 39]. The gluon component of hadrons, which plays the role of a gluon condensate in the picture described below, may only be responsible for a part of the other half of hadron momentum. Namely, the constituent quarks also contain gluons. The issue of how heavy is the part of a hadron that is made of effective quarks at  $\lambda \lesssim \Lambda_{\text{QCD}}$  and how heavy is the corresponding gluon condensate component, is discussed in Sec. 6 and Appendix D. Firm conclusions require RGPEP calculations that have not been done yet.

In the picture described below, the masses of effective quarks and gluons at some scale  $\lambda \lesssim \Lambda_{\text{QCD}}$  are assumed greater than  $\lambda$  itself. All interactions that change the number of the effective particles are absent and the only interactions left in  $H_\lambda$  with  $\lambda$  near and below  $\Lambda_{\text{QCD}}$  are potentials. These potentials are expected to match potentials used in the CQMs. Our reinterpretation of the gluon condensate is associated with a definite result for the potentials that act among the constituent quarks. In summary, the LF Hamiltonian of QCD with some  $\lambda = \lambda_c \lesssim \Lambda_{\text{QCD}}$  is assumed to describe quarks of mass  $m_c \gtrsim \lambda_c$ . These quarks interact only through potentials, *i.e.*, interactions that do not change the number of constituents. The subscript ‘c’ refers to the word constituent.

An eigenstate  $|\psi\rangle$  of the effective Hamiltonian  $H_{\lambda_c}(b_{\lambda_c})$  that represents a hadron of momentum  $P_h$ , is a superposition of effective-particle basis states ( $n = 2$  for mesons and  $n = 3$  for baryons)

$$|1 \dots n\rangle_{\lambda_c} = \prod_{i=1}^n b_{i\lambda_c}^\dagger |0\rangle, \quad (3)$$

$$|\psi\rangle = \sum_{1\dots n} \psi_{\lambda_c}(1, \dots, n) |1 \dots n\rangle_{\lambda_c}. \quad (4)$$

The arguments 1 to  $n$  of the wave function provide a shorthand notation for momenta, spins, flavors, and colors of the corresponding particles. For example, 2 as an argument of a wave function stands for the three-momentum, spin projection on  $z$ -axis, flavor, and color of the particle number 2. Summation over numbers 1 to  $n$  is a shorthand notation for summation over the quantum numbers of the corresponding particles including integration over their momenta. In this abbreviated notation, the eigenvalue equation for a mass and a wave function of a hadron with momentum  $P_h^+$  and  $P_h^\perp$  built from  $n$  constituents of scale  $\lambda_c$ , takes the form

$$\lambda_c \langle 1 \dots n | \left( P_h^+ H_{\lambda_c} - P_h^\perp{}^2 \right) |\psi\rangle = M^2 \lambda_c \langle 1 \dots n | \psi\rangle. \quad (5)$$

This eigenvalue equation with  $n = 2$  or  $n = 3$  is meant to correspond to the CQM picture of hadrons as built from 2 or 3 constituent quarks.

Note that the vacuum state  $|0\rangle$  is not changed when  $\lambda$  changes. This is a unique feature of LF dynamics. In the standard, instant form of dynamics, a change in particle operators must be accompanied with a change of the corresponding vacuum state.

One can write the same eigenvalue equation for the same states using other values of  $\lambda$  than  $\lambda_c$ . The eigenstate  $|\psi\rangle$  is the same for all values of  $\lambda$  but the basis states and corresponding wave functions depend on  $\lambda$ . For  $\lambda \gg \Lambda_{\text{QCD}}$ , the state  $|\psi\rangle$  may contain a giant number of multi-particle Fock components built by acting with creation operators  $b_\lambda^\dagger$  on the vacuum, while when  $\lambda = \lambda_c$ , the entire state contains only 2 or 3 constituent quarks.

Using RGPEP, one can express constituent quarks at scale  $\lambda_c$  in terms of effective quarks and gluons corresponding to  $\lambda > \lambda_c$ ,

$$b_{\lambda_c} = W b_\lambda W^\dagger, \quad (6)$$

$$W = U_{\lambda_c} U_\lambda^\dagger. \quad (7)$$

Momenta, spins, and izospins of the effective particles are the same, irrespective of the change in parameter  $\lambda$ . Since  $W|0\rangle = W^\dagger|0\rangle = |0\rangle$ , one has

$$|1 \dots n\rangle_{\lambda_c} = W \prod_{i=1}^n b_{i\lambda}^\dagger |0\rangle, \quad (8)$$

and the same eigenstate can be written as

$$|\psi\rangle = \sum_{1\dots n} \psi_{\lambda_c}(1, \dots, n) W |1 \dots n\rangle_{\lambda}. \quad (9)$$

The result of action of  $W$  on the basis state with  $n$  constituents corresponding to scale  $\lambda$ , with  $\lambda > \lambda_c$ , is a coherent slew of Fock sectors with various particle numbers, with the number of quarks greater or equal to the minimal constituent number  $n$  for a hadron. The momentum space wave functions of the resulting Fock components are determined by the bound-state eigenvalue condition of  $H_{\lambda}$ .

Although a precise structure of  $H_{\lambda}$  is beyond insight of perturbation theory for  $\lambda \sim \Lambda_{\text{QCD}}$  and  $W$  at such scales also contains non-perturbative dynamics, some generic features of the Fock wave functions in Eq. (9) can be assessed on the basis of generic features of  $W$  implied by properties of operators  $U_{\lambda_c}$  and  $U_{\lambda}$  that are visible already in perturbation theory [40]. Operator  $W$  conserves momentum.  $W$  can replace a single effective particle with a bunch of other particles which carry together the same momentum as the replaced particle.  $W$  can also annihilate a whole set of particles and create a new set with the same momentum.

Hence, the colorless meson ( $M$ ) and baryon ( $B$ ) states  $|\psi\rangle$  that correspond to the CQM picture (color factors are written explicitly),

$$|\psi\rangle_M = \sum_{12} \psi_{\lambda_c}(1, 2) \frac{\delta^{ab}}{\sqrt{3}} b_{1\lambda_c}^{a\dagger} d_{2\lambda_c}^{b\dagger} |0\rangle, \quad (10)$$

$$|\psi\rangle_B = \sum_{123} \psi_{\lambda_c}(1, 2, 3) \frac{\epsilon^{abc}}{\sqrt{6}} b_{1\lambda_c}^{a\dagger} b_{2\lambda_c}^{b\dagger} b_{3\lambda_c}^{c\dagger} |0\rangle, \quad (11)$$

are equal to, respectively,

$$|\psi\rangle_M = \sum_{12} \psi_{\lambda_c}(1, 2) \frac{\delta^{ab}}{\sqrt{3}} W b_{1\lambda}^{a\dagger} d_{2\lambda}^{b\dagger} |0\rangle, \quad (12)$$

$$|\psi\rangle_B = \sum_{123} \psi_{\lambda_c}(1, 2, 3) \frac{\epsilon^{abc}}{\sqrt{6}} W b_{1\lambda}^{a\dagger} b_{2\lambda}^{b\dagger} b_{3\lambda}^{c\dagger} |0\rangle. \quad (13)$$

The representation of meson and baryon states in Eqs. (10) and (11) in terms of only 2 or 3 effective constituents at scale  $\lambda_c$ , is thus transformed into the representation of the same states in Eqs. (12) and (13) that are built from effective particles of scale  $\lambda$ . The resulting Fock space wave functions for effective particles of size  $\lambda^{-1}$  involve functions  $\psi_{\lambda_c}(1, 2)$  and  $\psi_{\lambda_c}(1, 2, 3)$  through convolutions that emerge from the sums in Eqs. (12) and (13) and the structure of  $W$ . This is how the complex Fock-space structure of QCD hadrons is supposed to contain information about a simple CQM picture.

#### 4. Structure of $W$

In order to provide an illustration of how  $W$  may act on quark and gluon states, this section explains the structure of  $W$  obtained just in first-order perturbation theory in the effective coupling constant  $g_\lambda$ . The perturbative expression is obtained starting from Eqs. (1) and (A.1), which produce together

$$\frac{d}{d\lambda}\mathcal{H}_\lambda = [\mathcal{H}_\lambda, \mathcal{T}], \quad (14)$$

where  $\mathcal{T} = \mathcal{U}_\lambda^\dagger \frac{d}{d\lambda} \mathcal{U}_\lambda$  and  $\mathcal{U}_\lambda = U_\lambda(b_\infty)$ . This means that the functional  $\mathcal{F}_\lambda$  in Eq. (A.1) is set to have the form

$$F_\lambda[\mathcal{H}_\lambda] = [\mathcal{H}_\lambda, \mathcal{T}], \quad (15)$$

and the key to RGPEP is the dependence of  $\mathcal{T}$  on  $\mathcal{H}_\lambda$  [28]. This dependence is designed keeping in mind that the resulting Hamiltonian  $H_\lambda(b_\lambda)$  should contain the form factor  $f_\lambda$  in each and every interaction term. Suitable notation for this condition is provided by writing  $H_\lambda = f_\lambda G_\lambda$ , where  $H_\lambda$  contains the form factor  $f_\lambda$  in interaction vertices while  $G_\lambda$  does not. If an operator  $G_\lambda$  contains a product of creation and annihilation operators  $b_\lambda$  with a coefficient  $c_\lambda$ , the operator  $H_\lambda = f_\lambda G_\lambda$  contains exactly the same product with coefficient  $f_\lambda c_\lambda$ , where  $f_\lambda$  depends on the difference between the invariant mass squared of particles annihilated by annihilation operators in the product and the invariant mass squared of particles created by creation operators in the product. Both invariant masses are calculated using the particle kinematical momentum variables and eigenvalues of part  $H_{0\lambda} = G_{0\lambda}$  of the Hamiltonian  $H_\lambda$ .

Thus, using  $\mathcal{H}_\lambda = H_\lambda(b)$  and omitting  $\lambda$ , one writes

$$\mathcal{H} = f\mathcal{G} = \mathcal{G}_0 + f\mathcal{G}_I. \quad (16)$$

Differentiating  $\mathcal{H}$  with respect to  $\lambda$  one arrives at the equation that defines  $\mathcal{T}$  in such a way that it must vanish when interactions vanish (this guarantees that  $\mathcal{T}$  is expandable in powers of the coupling constant) and that its matrix elements between eigenstates of  $\mathcal{G}_{0\lambda}$  vanish when the differences between the corresponding eigenvalues vanish [28]

$$[\mathcal{T}, \mathcal{G}_0] = [(1-f)\mathcal{G}_I]'. \quad (17)$$

Prime denotes differentiation with respect to  $\lambda$ . The solution is denoted by

$$\mathcal{T} = \{[(1-f)\mathcal{G}_I]'\}_{\mathcal{G}_0}, \quad (18)$$

where the curly braces with subscript  $\mathcal{G}_0$  indicate the energy difference (difference of eigenvalues of  $\mathcal{G}_0$ ) in denominator that results from the commutator of  $\mathcal{T}$  with  $\mathcal{G}_0$  on the left-hand side in Eq. (17). Then  $\mathcal{U}$  satisfies

$$\mathcal{U}' = \mathcal{U} \{[(1-f)\mathcal{G}_I]'\}_{\mathcal{G}_0} \quad (19)$$

with condition  $\mathcal{U}_\infty = 1$ . Eq. (19) can be expanded in powers of the coupling constant.

#### 4.1. Perturbative expansion for $U_\lambda$

Let us assume here that  $\mathcal{G}_0$  does not depend on the coupling constant to all orders of perturbation theory. This means that perturbative self-interaction terms are included in  $\mathcal{G}_I$  and  $\mathcal{G}_0$  is actually independent of  $\lambda$ <sup>10</sup>. While this condition may seem quite restrictive, note that it allows for appearance of mass terms that are proportional to positive powers of  $\Lambda_{\text{QCD}}$ , since  $\Lambda_{\text{QCD}}$  vanishes to all orders of perturbation theory. Writing expansions in the bare coupling constant,

$$\mathcal{U}_\lambda = 1 + g u_{\lambda 1} + g^2 u_{\lambda 2} + \dots, \quad (20)$$

$$\mathcal{G}_{I\lambda} = g \mathcal{G}_{I\lambda 1} + g^2 \mathcal{G}_{I\lambda 2} + \dots, \quad (21)$$

one obtains equations (for convenience of notation, the prime is now put outside the curly braces, which is justified because eigenvalues of  $\mathcal{G}_0$  are independent of  $\lambda$ )

$$u'_{\lambda 1} = \{(1-f)\mathcal{G}_{I\lambda 1}\}'_{\mathcal{G}_0}, \quad (22)$$

$$u'_{\lambda 2} = u_{\lambda 1} \{(1-f)\mathcal{G}_{I\lambda 1}\}'_{\mathcal{G}_0} + \{(1-f)\mathcal{G}_{I\lambda 2}\}'_{\mathcal{G}_0}, \quad (23)$$

*etc.*, with solutions of the form [40]

$$u_{\lambda 1} = \{(1-f)\mathcal{G}_{I\lambda 1}\}_{\mathcal{G}_0}, \quad (24)$$

$$u_{\lambda 2} = \frac{1}{2} u_{\lambda 1}^2 + \frac{1}{2} \int_{\infty}^{\lambda} ds [u_{s1}, u'_{s1}] + \{(1-f)\mathcal{G}_{I\lambda 2}\}_{\mathcal{G}_0}, \quad (25)$$

*etc.* This expansion can be rewritten in terms of the effective coupling  $g_\lambda$  for the purpose of eliminating ultraviolet divergences involved in the definition of the bare coupling constant and the corresponding counterterm. However, for  $g = g_\lambda + O(g_\lambda^3)$ , one can simply replace  $g$  by  $g_\lambda$  in the terms explicitly

<sup>10</sup> This way of proceeding is not necessary, but it considerably simplifies the discussion that follows because there is no need to expand the form factor  $f_\lambda$  and the energy denominators in powers of the coupling constant.

listed in Eqs. (24) and (25). In addition,  $U_\lambda$  is unitary by construction. Therefore,  $\mathcal{U}_\lambda = U_\lambda(b_\infty) = U_\lambda(b_\lambda)$  and one can freely replace  $b_\infty$  by  $b_\lambda$  in these equations, obtaining thus first two terms in expansion of  $U_\lambda$  in powers of  $g_\lambda$ .

4.2. *Perturbative expansion for W*

Now consider  $\lambda_c$  and  $\lambda \geq \lambda_c$ . Eq. (7) implies  $W(b_\infty) = U_{\lambda_c}(b_\infty)U_\lambda^\dagger(b_\infty)$ , while in Eqs. (12) and (13) the operator  $W$  acts on states created from the vacuum by operators  $b_\lambda$ . It is therefore convenient to use an equivalent expression

$$W(b_\infty) = U_\lambda^\dagger(b_\lambda) U_{\lambda_c}(b_\lambda) \tag{26}$$

$$= \left[ 1 + g_\lambda u_{\lambda_1}^\dagger + g_\lambda^2 u_{\lambda_2}^\dagger + \dots \right] \left[ 1 + g_\lambda u_{\lambda_{c1}} + g_\lambda^2 u_{\lambda_{c2}} + \dots \right], \tag{27}$$

where  $b_\infty$  in every  $u$  on the right-hand side is replaced by  $b_\lambda$ . Thus,

$$W = 1 + g_\lambda W_1 + g_\lambda^2 W_2 + \dots, \tag{28}$$

$$W_1 = \{(f_\lambda - f_{\lambda_c})\mathcal{G}_{I\infty 1}\}_{\mathcal{G}_0}, \tag{29}$$

$$W_2 = \frac{1}{2} (u_{\lambda 1} - u_{\lambda_{c1}})^2 + \frac{1}{2} [u_{\lambda_{c1}}, u_{\lambda 1}] + \frac{1}{2} \int_{\lambda_c}^{\lambda} ds [u_{s1}, u'_{s1}]$$

$$+ \{(1 - f_{\lambda_c})\mathcal{G}_{I\lambda_{c2}}\}_{\mathcal{G}_0} - \{(1 - f_\lambda)\mathcal{G}_{I\lambda 2}\}_{\mathcal{G}_0}, \tag{30}$$

where  $b_\infty$  is replaced by  $b_\lambda$  everywhere on the right-hand sides.

The perturbative expansion can be similarly carried out to higher orders. The term of focus here is  $W_1$ , as an illustration of how  $W$  acts on quark and gluon states. This illustration is valid when the coupling constant  $g_\lambda$  is very small, which means that  $\lambda \gg \Lambda_{\text{QCD}}$  in the RGPEP scheme.

To some extent,  $W_1$  is also useful as an indicator of how  $W$  acts on states when  $\lambda$  approaches  $\lambda_c \sim \Lambda_{\text{QCD}}$ . In this case,  $g_\lambda$  is relatively large. Although the size of contribution of terms containing  $W_k$  with  $k > 1$  in comparison to the size of contribution of the term with  $W_1$  for such small  $\lambda$  is not known, it is certain that all these terms together contribute 0 when  $\lambda = \lambda_c$ . They vanish no matter how large is  $g_\lambda$  because integrals in them effectively range from  $\lambda_c$  to  $\lambda$  and thus vanish when  $\lambda$  tends to  $\lambda_c$ . On the other hand, the higher power of  $g_\lambda$  the larger number of integrals involved. All these integrals must effectively range from  $\lambda_c$  to  $\lambda$ , so that they vanish when  $\Delta\lambda = \lambda - \lambda_c$  tends to 0 as the appropriate power of  $\Delta\lambda$ . The first term in the expansion in  $\Delta\lambda$  must be linear in  $\Delta\lambda$  and  $W_1$  is such.

Regarding the size of  $g_\lambda$ , the range of  $\lambda$  right above  $\lambda_c$  is a region where  $g_\lambda$  in the RGPEP scheme in QCD may be limited in size, instead of having large values that one may expect on the basis of a straightforward

extrapolation of low-order expressions obtained in the region of large  $\lambda$  [41]. Such limitation of  $g_\lambda$  from above is certainly observed in simple mathematical models that include asymptotic freedom and bound states in a renormalization group procedure for Hamiltonians [42]. Similar limitation of  $g_\lambda$  is also suggested in phenomenology of non-perturbative running couplings based on AdS/QCD ideas and LF holography [43]. New theoretical information about the size of  $g_\lambda$  at small  $\lambda$  is expected to follow from a non-perturbative RGPEP equations described in Appendix C.

#### 4.3. $W_1$ for quarks and gluons

According to Eq. (29), the first-order term in  $W = 1 + g_\lambda W_1 + O(g_\lambda^2)$ , is given by

$$W_1 = \{(f_\lambda - f_{\lambda_c})\mathcal{H}_{I\infty 1}\}_{\mathcal{H}_0}, \quad (31)$$

where  $\mathcal{H}_0$  is the term independent of the bare coupling constant  $g$  and  $\mathcal{H}_{I\infty 1}$  is a regulated interaction term proportional to the first power of  $g$  in the canonical LF QCD Hamiltonian,  $\mathcal{H}_\infty = \mathcal{H}_{0\infty} + g\mathcal{H}_{I\infty 1} + O(g^2)$ , derived using gauge  $A^+ = 0$  from the Lagrangian for QCD,  $\mathcal{L} = \bar{\psi}(i\mathcal{D} - m)\psi - \frac{1}{2}\text{Tr}F^{\mu\nu}F_{\mu\nu}$ . Thus,

$$\mathcal{H}_\infty = \int dx^- d^2x^\perp [h_{0\infty} + gh_{I\infty 1} + O(g^2)], \quad (32)$$

where the Hamiltonian density terms independent of  $g$  are

$$h_{0\infty} = \frac{1}{2}\bar{\psi}\gamma^+ \frac{-\partial^\perp{}^2 + m^2}{i\partial^+} \psi - \text{Tr} A^\perp (\partial^\perp)^2 A^\perp, \quad (33)$$

and the interaction terms order  $g$  are

$$h_{I\infty 1} = \bar{\psi} \mathcal{A} \psi + 2i\text{Tr} \partial_\alpha A_\beta [A^\alpha, A^\beta], \quad (34)$$

with SU(3) color notation  $A = A^a t^a$ ,  $[t^a, t^b] = if^{abc} t^c$ ,  $\text{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ . The calculation of  $W_1$  starts with the above expressions and proceeds as described in Appendix B.

Appendix B also argues that  $W$  for  $\lambda \gtrsim \lambda_c \sim \Lambda_{\text{QCD}}$  contains all the terms in  $\mathcal{H}_\lambda$  that change the number of particles. This means that  $W$  can create a coherent slew of the effective Fock components at scale  $\lambda$  in Eqs. (12) and (13) that are implied by the 2 or 3 constituent quark wave functions at  $\lambda_c$  in Eqs. (10) and (11). In turn, this means that the result of action of  $W$  is fully encoded through the Hamiltonian in the wave functions  $\psi_{\lambda_c}(1, 2)$  and  $\psi_{\lambda_c}(1, 2, 3)$  that correspond to the CQM. These wave functions are solutions to the eigenvalue problem for  $H_{\lambda_c}$ , which does not change the number of constituents. But the larger  $\lambda$  the closer  $H_\lambda$  to the canonical LF Hamiltonian of QCD and the more complex the states generated by  $W$  in Eqs. (12) and (13).

### 5. Eigenvalue problem for light hadrons

According to previous sections, the same eigenvalue problem for light hadrons can be written in different ways when one uses different Hamiltonian expressions that correspond to different values of  $\lambda$ . Three values are distinguished in further discussion:  $\lambda = \infty$ ,  $\lambda \gtrsim \lambda_c$ , and  $\lambda = \lambda_c$ . The corresponding eigenvalue problems are equivalent if RGPEP equations are solved exactly. However, since exact solutions are not available, one is forced to guess plausible candidates for the Hamiltonian expressions at different values of  $\lambda$ .

At large  $\lambda$ , insight comes from the perturbative expansion of RGPEP, using regulated canonical LF QCD Hamiltonian with counterterms to start with at  $\lambda = \infty$  and taking advantage of asymptotic freedom. At small  $\lambda$ , one has to guess a first approximation. Phenomenology suggests that some form of a Hamiltonian for constituent quarks bound in a potential well is a good candidate for a first approximation to the QCD Hamiltonian with  $\lambda = \lambda_c$  for hadron states of smallest masses.

These two extreme regions of  $\lambda$  need to be connected to each other within the RGPEP framework in terms of some interpolation<sup>11</sup>. The size of calculable corrections to any first approximation constructed this way will eventually tell us how far from a true RGPEP solution such first approximation can be. The candidate for the first approximation that is developed in this article suggests that the concept of a gluon condensate may be interpreted in a new way.

#### 5.1. Three ways of writing the eigenvalue problem

The first of the three ways, which uses the regulated canonical Hamiltonian for LF QCD with all due counterterms,  $H_\infty$ , amounts to formal writing of the eigenvalue problem in terms of Fock states created by products of operators  $b_\infty^\dagger$ ,  $d_\infty^\dagger$ , and  $a_\infty^\dagger$  from the vacuum state  $|0\rangle$  in the limit of regularization being removed

$$H_\infty|\psi\rangle = P_h^-|\psi\rangle. \tag{35}$$

In this equation, a reasonably accurate description of the eigenstate  $|\psi\rangle$  presumably requires a very large number of wave functions for many significant bare-particle Fock components. The number of such components is expected to grow when the regularization is lifted. Although the full structure of  $|\psi\rangle$  is not known precisely, one can assume some model for one part of it, such as a Fock sector with a smallest possible number of bare particles

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<sup>11</sup> A similar reasoning applies also in the case of Yukawa theory, which is not asymptotically free, when one considers a limited range of scales and effective coupling constant is sufficiently small from theoretical point of view to use perturbative RGPEP [44].

in a meson or a baryon, and try to determine another part, such as a component with one additional gluon, using perturbation theory. Such approach is useful in description of exclusive processes that involve large momentum transfers [45]<sup>12</sup>.

The second of the three ways is designed for the opposite end of the scale for  $\lambda$  in RGPEP, *i.e.*, when  $\lambda = \lambda_c$ . The same Hamiltonian is expected to be expressible there in terms of operators for constituent quarks. In this case, one has the eigenvalue equation of the form

$$H_{\lambda_c}|\psi\rangle = P_h^-|\psi\rangle, \quad (36)$$

in which the state  $|\psi\rangle$  of a single light hadron is represented by Eq. (10) for mesons and Eq. (11) for baryons. This means that the light hadron eigenstates are described by only one wave function,  $\psi_c(1 \dots n)$ , with  $n = 2$  or  $n = 3$ . These wave functions appear in the Fock states built using creation operators  $b_{\lambda_c}^\dagger$  and  $d_{\lambda_c}^\dagger$ . There are no components with effective gluons or additional quark–anti-quark pairs. In other words, the complexity of structure of light hadrons in QCD is encapsulated in the structure of constituent quarks. This means that the operators  $b_{\lambda_c}^\dagger$  and  $d_{\lambda_c}^\dagger$  in Eqs. (10) and (11) are complex combinations of products of operators  $b_\infty^\dagger$ ,  $d_\infty^\dagger$ ,  $a_\infty^\dagger$  and their conjugates.

Using Eqs. (5), (10) and (11), the eigenvalue problem of Eq. (36) for the  $n = 2$  or  $n = 3$  constituents is obtained in the form

$$\left( \sum_{i=1}^n p_i^- + V_{cn}/P_h^+ \right) \lambda_c \langle 1 \dots n | \psi \rangle = P_h^- \lambda_c \langle 1 \dots n | \psi \rangle. \quad (37)$$

The unknown element in this form is the interaction potential term  $V_{cn}$ . In order to derive the structure of  $V_{cn}$  in mesons and baryons, we shall employ the third way of writing the same eigenvalue problem using RGPEP. The third way suggests the reinterpretation of gluon condensate that is the subject of this article. When all gluons considered in the third way are incorporated in the constituent quarks at  $\lambda_c$ , one is left only with the potential  $V_{cn}$ , see Appendix D.

<sup>12</sup> A different way from RGPEP to attack the eigenvalue problem of a LF Hamiltonian for QCD head on is to use discretized light-cone quantization (DLCQ) in which one introduces a finite minimal unit of  $p^+$  momentum,  $\epsilon^+$ , which is the inverse of the size of a periodicity box for fields as functions of the position variable  $x^-$  on the LF. Since the total  $P^+$  of an eigenstate of  $H_\infty$  is conserved, the unit  $\epsilon^+$  naturally limits the number of bare particles in an eigenstate from above by the ratio  $P^+/\epsilon^+$  [46, 47, 48]. Additional cutoffs need to be imposed in DLCQ in order to limit transverse momenta of constituents that may appear in an eigenvalue problem with some simultaneously fixed eigenvalues of the total momentum components  $P^+$  and  $P^\perp$ .

The third way of writing the same eigenvalue problem uses  $H_\lambda$  with  $\lambda \gtrsim \lambda_c$ . The size of  $\lambda$  is assumed to be such that gluons are already somewhat active as participants in the dynamics while the effective quarks are still heavy enough for a NR analysis to apply to their relative motion in a light hadron. The NR approximation is thinkable for a slowly moving hadron according to previous sections, since interactions in the effective Hamiltonian at scale  $\lambda \gtrsim \lambda_c$  contain form factors. The effective particles of sizable masses cannot change their momenta through interactions by large amounts. Thus, an eigenvector corresponding to a slowly moving light hadron can be expected to be described by the wave functions that have significant values only for small momenta of the constituents.

The third picture at some scale  $\lambda \gtrsim \lambda_c$  is considered a candidate for a first approximation to hadrons in RGPEP. The corresponding Hamiltonian will be proposed below using ideas motivated by gauge symmetry in its NR form. Difference between this approximate NR form and the interactions that can be systematically calculated using RGPEP in LF QCD, is to be treated as a perturbation<sup>13</sup>. The approximate picture at  $\lambda \gtrsim \lambda_c$  is described in more detail in the next sections. Here we only suggest that RGPEP provides a scheme that may be used in future calculations of  $V_{cn}$  in Eq. (37) at  $\lambda_c$  by evolving the third-way picture at  $\lambda \gtrsim \lambda_c$  down to  $\lambda_c$  and taking advantage of the RGPEP reinterpretation of the gluon condensate that is described in the next sections.

### 5.2. Light hadron states at $\lambda \gtrsim \lambda_c$

When  $\lambda \gtrsim \lambda_c$ , the meson and baryon states are described by Eqs. (12) and (13), respectively. In these equations, the operator  $W$  acts on the Fock components with 2 or 3 quarks of scale  $\lambda$  and creates additional components. The meson state is simpler than the baryon state and it will be discussed first.

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<sup>13</sup> This strategy is similar to the one described in Ref. [16] with an artificial linear potential. RGPEP introduces new elements: the dynamical transformation from bare to effective particles available at different scales, construction of corresponding effective fields, possibility of using NR approximation at  $\lambda \gtrsim \lambda_c$  for identifying interactions through gauge symmetry as indicated later in the text (instead of introducing an artificial potential), extension beyond perturbation theory (see Appendix C), and invariance with respect to 7 kinematical LF symmetries for the resulting interaction terms and wave functions [31]. As a result, the first approximation potential is proposed to be not a linear but a quadratic function of a relative distance between color charges (see next sections). The quadratic potential is expected to cause creation of additional particles when a distance between two colored particles increases and the net linear increase of energy with distance must ultimately result from the energy of particles created along a line that connects the two widely separated color charges.

The meson state is

$$|\psi_M\rangle = \sum_{12} \psi_{\lambda_c}(1, 2) W b_{1\lambda}^\dagger d_{2\lambda}^\dagger |0\rangle. \quad (38)$$

From Eqs. (B.16), (B.18) and (B.22), it is clear that  $W$  shares properties of the interaction Hamiltonian. For example, if  $H_{\lambda I}$  changes a group of particles to another group, a similar change with additional factors results from action of  $W$ . So, out of 2 effective quarks in a meson (3 in a nucleon) more quarks and gluons are created. In order to use local gauge symmetry to propose the approximate form of the state that results from action of  $W$ , it is useful to represent the state of Eq. (38) in position space. This is done using quantum fields built from effective particle operators at scale  $\lambda$ .

The effective quantum fields are constructed using Eq. (B.1) for quarks and Eq. (B.8) for gluons and replacing operators  $b, d$  and  $a$  with  $b_\lambda, d_\lambda$  and  $a_\lambda$ , respectively. Thus, the operator  $\psi_\lambda(x)$  on the LF is built in the same way from  $b_\lambda$  and  $d_\lambda^\dagger$  as the canonical operator  $\psi(x)$  is built in Eq. (B.1) from the operators  $b$  and  $d^\dagger$  that are equal  $b_\infty$  and  $d_\infty^\dagger$ , respectively

$$\psi_\lambda(x) = \sum_{\sigma c f} \int [k] \left[ \chi_c u_{\lambda f k \sigma} b_{k \sigma c f \lambda} e^{-ikx} + \chi_c v_{\lambda f k \sigma} d_{k \sigma c f \lambda}^\dagger e^{ikx} \right]. \quad (39)$$

The spinors  $u_{\lambda f k \sigma}$  and  $v_{\lambda f k \sigma}$  include corresponding vectors in flavor space. Spinors might depend on  $\lambda$  if the effective quark masses in the field expansion are allowed to depend on  $\lambda$ <sup>14</sup>. Similarly, operator  $A_\lambda^\mu(x)$  on the LF is built from  $a_\lambda$  and  $a_\lambda^\dagger$  as the canonical operator  $A^\mu(x)$  is built in Eq. (B.8) from the operators  $a$  and  $a^\dagger$  that are equal  $a_\infty$  and  $a_\infty^\dagger$ , respectively. Gluon polarization vectors do not depend on  $\lambda$ .

$$A_\lambda^\mu(x) = \sum_{\sigma c} \int [k] \left[ t^c \varepsilon_{k \sigma}^\mu a_{k \sigma c \lambda} e^{-ikx} + t^c \varepsilon_{k \sigma}^{\mu*} a_{k \sigma c \lambda}^\dagger e^{ikx} \right]. \quad (40)$$

As a consequence, the dynamically independent components,  $\psi_\lambda = \Lambda^+ \psi_\lambda$  and  $A_\lambda^\perp$  in  $A^+ = 0$  gauge on the LF  $x^+ = 0$  have the same commutation relations as in a canonical theory.

With particle operators at  $\lambda = \lambda_c$ , an effective constituent quark field operator is constructed in the same way. When this field is used to describe a slowly moving meson, it is useful to write the field as

$$\psi_{\lambda_c} = \begin{bmatrix} U_{\lambda_c}(\vec{x}) \\ V_{\lambda_c}(\vec{x}) \end{bmatrix}, \quad (41)$$

<sup>14</sup> Note that the independent field components,  $\psi_+ = \Lambda^+ \psi$ ,  $\Lambda^+ = (1/2)\gamma^0\gamma^+$ , are independent of the quark masses, and inclusion of effective masses in spinors is merely a way of useful notation for some effects of interactions.

where the upper two-component field  $U_{\lambda_c}$  annihilates quarks and the lower two-component  $V_{\lambda_c}$  creates anti-quarks<sup>15</sup>.

Thus, by inverting the Fourier transforms through integration over the LF hyperplane and using conventions explained in Appendix B, the state in Eq. (10) for a meson of momentum  $P_h$  can be written as

$$\begin{aligned}
 |\psi\rangle_M &= \frac{1}{\sqrt{3}} \int \frac{d^3x_1 d^3x_2}{4m^2} \int [12] 16\pi^3 P_h^+ \delta^3(P_h - k_1 - k_2) \\
 &\quad \times e^{-i(k_1x_1+k_2x_2)} U_{\lambda_c}^\dagger(x_1) \psi_{2\times 2}(\vec{k}_{12}) V_{\lambda_c}(x_2)|0\rangle, \quad (42)
 \end{aligned}$$

where  $\psi_{2\times 2}(\vec{k}_{12})$  denotes the  $2 \times 2$  matrix wave function of relative motion of the quarks. The three-vector  $\vec{k}_{12}$  can be defined as a relative momentum of the quarks in the CRF. Details of the definition of  $\vec{k}_{12}$  are not essential at this point but later discussion will include relevant details.

In order to obtain analogous position representation of the meson state in Eqs. (12) or (38), one needs to apply  $W$  to the expression in Eq. (42). The result is

$$\begin{aligned}
 |\psi\rangle_M &= \frac{1}{\sqrt{3}} \int \frac{d^3x_1 d^3x_2}{4m^2} \int [12] 16\pi^3 P_h^+ \delta^3(P_h - k_1 - k_2) \\
 &\quad \times e^{-i(k_1x_1+k_2x_2)} W U_{\lambda}^\dagger(x_1) \psi_{2\times 2}(\vec{k}_{12}) V_{\lambda}(x_2)|0\rangle. \quad (43)
 \end{aligned}$$

A similar expression is generated in terms of three quark fields and  $W$  for baryons. Integration over momenta in these expressions generates position wave functions as coefficients in the expansion of meson or baryon states into basis states that are created from the LF vacuum by action of a product of two or three quark fields at scale  $\lambda$  and  $W$ .

Continuing with the meson states, the central hypothesis about action of  $W$  is that gauge symmetry forces the result of action of  $W$  on a state of two quarks at scale  $\lambda$  to have the form

$$W \int d^3x_1 d^3x_2 U_{\lambda}^\dagger(x_1) \psi(x_1, x_2) V_{\lambda}(x_2)|0\rangle \quad (44)$$

$$= \int d^3x_1 d^3x_2 d^3x_0 U_{\lambda}^\dagger(x_1) W(x_1, x_2, x_0) V_{\lambda}(x_2)|0\rangle, \quad (45)$$

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<sup>15</sup> The three-vector notation  $\vec{x}$  or  $\vec{k}$  refers here to the LF co-ordinates ( $x^-, x^\perp$ ) in position space. A similar notation is sometimes also used for momentum variables ( $k^+, k^\perp$ ). However, when we proceed later to the Schrödinger eigenvalue problem for effective Hamiltonians for light hadrons, the same notation will be adopted also for three-vectors built from  $\perp$  and  $+$  or  $-$  components in such a way that the standard three-dimensional notation respecting rotational symmetry will be natural.

where the operator  $W(x_1, x_2, x_0)$  is responsible for creating components generated by  $W$ . These components can be of two kinds. One kind is formed by operators that carry the color of initial two quarks. For example, if  $W$  generates a gluon from a quark, the generated gluon and the emerging quark together carry the color of the initial quark. The other kind is formed by colorless operators. For example,  $W$  may cause emission of two gluons by a quark and the gluons may form a color singlet.

These two kinds of contributions are encapsulated in  $W(x_1, x_2, x_0)$  by writing

$$W(x_1, x_2, x_0) = \psi(x_1, x_2, x_0) T(x_1, x_2) G^\dagger(x_0), \quad (46)$$

where  $\psi(x_1, x_2, x_0)$  denotes a new wave function at scale  $\lambda \gtrsim \lambda_c$ ,  $G^\dagger(x_0)$  denotes the colorless component of the state, and

$$T(x_1, x_2) = P \exp \left[ -ig \int_{x_2}^{x_1} dx^\mu A_\mu(x) \right], \quad (47)$$

is the color-transport factor along a straight line between quarks that maintains local gauge symmetry by bringing in required gluon fields<sup>16</sup>. This factor is constructed in analogy with Ref. [17].

The operator that generates the colorless component of a hadron state,  $G^\dagger(x_0)$ , will provide the contribution in dynamics of quarks that is associated with gluon condensation in hadrons, rather than in a vacuum. Namely, instead of the vacuum expectation values of operators considered in Ref. [17], such as  $\langle \Omega | A^i A^j | \Omega \rangle$  with  $i, j = 1, 2$ , the Schrödinger equation for the wave function  $\psi(x_1, x_2, x_0)$  will involve expectation values

$$\langle A^i A^j \rangle_G = \frac{\langle G | A^i A^j | G \rangle}{\langle G | G \rangle}, \quad (48)$$

$$|G\rangle = G^\dagger(x_0) |0\rangle. \quad (49)$$

A similar reasoning is followed regarding baryons. In the case of baryons, one has to deal with three color-transport factors that are constructed in

<sup>16</sup> In the LF gauge  $A^+ = 0$ , one may expect only transverse separation between quarks to count. However, the dependent (constrained) components of fields,  $\psi_-$  and  $A^-$ , contribute to the effective interactions in a non-trivial way and one has to keep in mind that the complete effective theory at small  $\lambda$  should have full rotational symmetry restored. Therefore, a complete RGPEP expression for  $W$  must account for the dynamics along  $x_1^- - x_2^-$  as well as along  $x_1^\perp - x_2^\perp$ . This issue will be addressed later by introducing effective interactions that respect rotational symmetry inside slowly moving hadrons through a new definition of three-dimensional relative momenta of constituents to which the minimal gauge coupling rule can be applied in a rotationally symmetric way.

analogy to Ref. [17]. The operator  $G^\dagger$  in baryons generates the state  $G^\dagger|0\rangle$  that plays the same role that the vacuum state  $|\Omega\rangle$  played in Ref. [17]. It is assumed that the colorless components of mesons and baryons are approximately the same. The issue of universality of expectation values such as in Eq. (48) for mesons will be further discussed below when we come to the construction of the effective Hamiltonian.

In summary, the claim of gauge symmetry regarding the colorless basis states of quarks and gluons at scale  $\lambda$  from which mesons and baryons are made, is that they are of the form (all fields are effective at scale  $\lambda \gtrsim \lambda_c$ )

$$|\vec{x}_1, \vec{x}_2, \vec{x}_G\rangle = \sum_{ab} \frac{\delta^{ab}}{\sqrt{3}} \left(u_{1\lambda}^\dagger T_1\right)^a \left(T_2^\dagger v_{2\lambda}\right)^b G^\dagger|0\rangle, \quad (50)$$

$$|\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_G\rangle = \sum_{abc} \frac{\epsilon^{abc}}{\sqrt{6}} \left(u_1^\dagger T_1\right)^a \left(u_2^\dagger T_2\right)^b \left(u_3^\dagger T_3\right)^c G^\dagger|0\rangle, \quad (51)$$

where

$$T_i = e^{-ig \int_{\underline{x}}^{\underline{x}^i} dx_\mu A^\mu}, \quad (52)$$

and  $\underline{x} = (\vec{x}_1 + \vec{x}_2)/2$  in a meson and  $\underline{x} = (\vec{x}_1 + \vec{x}_2 + \vec{x}_3)/3$  in a baryon. The factor  $T$  is defined here along a straight path in such a way that it reproduces the color-transport factor in mesons in Eq. (47). The path dependence is ignored as an unnecessary complication at the level of mean-field approximation. The operator  $G$  with argument  $\vec{x}_G$  represents the white energy density background called glue. It is assumed to be a scalar boson field with corresponding commutation relations.

There is a trouble on the LF with non-locality of the boson commutation relations in the direction of  $x^-$ , which requires a solution. However, the effective dynamics that will be constructed in next sections will explicitly circumvent this difficulty by using an operator  $G$  that is a function of a three-dimensional position space variables associated with slowly moving hadronic constituents in a slowly moving hadron. Thus, the problem ultimately requiring a solution is not so much the non-locality of a scalar field but how LF QCD can generate a rotationally symmetric effective theory for light hadrons. The construction offered in next sections proposes to treat  $G$  as a field depending on a suitable three-dimensional position variables in which it can be local in a way that respects rotational symmetry.

The quanta of  $G$  are meant to represent excitations of the states of gluons condensed inside hadrons. The quanta of  $G$  are not point-like. Their size is characterized by  $1/\lambda$  and can be considered roughly on the order of  $1/\Lambda_{\text{QCD}}$ , *i.e.*, the quantum extends over the volume of an entire hadron<sup>17</sup>.

<sup>17</sup> The effective glue degree of freedom represents contributions of all white Fock sectors of effective particles that share the hadron momentum to a varying degree as  $\lambda$  varies.

There exists a possibility to associate vector or even higher spin nature with  $G$ , contributing to the hadron spin, but there is no need to do so here.

The basis states are orthonormal in the sense that, for all quarks being different

$$\langle \vec{x}_1, \vec{x}_2, \vec{x}_G | \vec{x}_{1'}, \vec{x}_{2'}, \vec{x}_{G'} \rangle = \delta_{11'} \delta_{22'} \delta_{GG'} , \tag{53}$$

$$\langle \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_G | \vec{x}_{1'}, \vec{x}_{2'}, \vec{x}_{3'}, \vec{x}_{G'} \rangle = \delta_{11'} \delta_{22'} \delta_{33'} \delta_{GG'} \tag{54}$$

and  $\delta_{kk'}$  includes  $\delta^3(\vec{x}_k - \vec{x}_{k'})$ <sup>18</sup>. If quark quantum numbers besides color are not different (this comment concerns only the baryon case), one has to adjust the normalization of basis states by including all permutations that contribute to the scalar products.

In the abbreviated notation used in the remaining part of the article

$$|\vec{x}_1, \vec{x}_2, \vec{x}_G \rangle = |12G \rangle , \tag{55}$$

$$|\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_G \rangle = |123G \rangle . \tag{56}$$

In these abbreviated notation, the meson and baryon states read

$$|\psi \rangle_M = \sum_{12G} \psi(12G) |12G \rangle , \tag{57}$$

$$|\psi \rangle_B = \sum_{123G} \psi(123G) |123G \rangle . \tag{58}$$

### 5.3. Preliminaries concerning Hamiltonian density at $\lambda \gtrsim \lambda_c$

Using the concept of effective quark and gluon fields, one can propose that the LF Hamiltonian at  $\lambda \gtrsim \lambda_c$  be expressible, by analogy to the canonical theory, in terms of an integral over the LF of a density that is a function of the fields. Thus,

$$H_\lambda = \int_{\text{LF}} (h_{0\lambda} + h_{I\lambda}) , \tag{59}$$

where the density  $h_{0\lambda}$  is bilinear in the fields and renders single-particle operators while the density  $h_{I\lambda}$  describes interactions in terms of products of at least three fields (the interactions involve at least three effective particles).

Using power-counting [16], the bilinear density at scale  $\lambda$  can be assumed in the form (all fields at  $x^+ = 0$ )

$$h_{0\lambda} = \frac{1}{2} \bar{\psi}_\lambda \gamma^+ \frac{-\partial^{\perp 2} + m^2}{i\partial^+} \psi_\lambda + \text{Tr} A_\lambda^\perp \left( -\partial^{\perp 2} + m_g^2 \right) A_\lambda^\perp + CT_{I\lambda} , \tag{60}$$

---

<sup>18</sup> With the qualification that locality in the  $x^-$  direction requires proper definition of  $z$ -components in a rotationally invariant effective theory to be discussed in next sections.

where the masses correspond to  $\lambda$ . The symbol  $CT_{I\lambda}$  denotes mass-like terms needed to make the mass parameters  $m$  and  $m_g$  to count as masses in the eigenvalue equations for light mesons and baryons. In other words,  $CT_{I\lambda}$  are by construction canceled by self-interactions in light colorless states<sup>19</sup>.

The interaction density  $h_{I\lambda}$  in Eq. (59) must be non-local in the sense that it involves products of fields at different points<sup>20</sup>. In particular, the minimal coupling between quarks and gluons, which in the canonical gauge theory is of the form

$$H_{MC\infty} = \int d^3x h_{MC\infty}(x), \tag{61}$$

with

$$h_{MC\infty}(x) = g \bar{\psi}(x) \not{A}(x) \psi(x), \tag{62}$$

in the effective theory must take a non-local form that in a lowest-order approximation is of the type [31]

$$H_{MC\lambda} = \int d^3x_1 d^3x_2 d^3x_3 h_{MC\lambda}(x_1, x_2, x_3), \tag{63}$$

with

$$h_{MC\lambda}(x_1, x_2, x_3) = g\lambda f_\lambda(x_2 - x_1, x_3 - x_1) \bar{\psi}_\lambda(x_1) \not{A}_\lambda(x_2) \psi_\lambda(x_3). \tag{64}$$

This non-local interaction term appears with other non-local terms in the Hamiltonian obtained from RGPEP at scale  $\lambda$ . The dependent and independent components of the fields are grouped in Eq. (64) according to a free theory. The dependent components involve the inverse of  $i\partial^+$ , which is a non-local operator. This non-locality is of the same type as in a canonical theory and the RGPEP non-locality appears here on top of the canonical one.

The non-local interaction terms with small  $\lambda$  can be simplified considerably if the domain of action of the Hamiltonian is restricted to slowly moving effective quarks. Namely, consider again the case of Eqs. (63) and (64) and introduce a gradient expansion of the form

<sup>19</sup> There is no need to specify these terms further here. Such terms can be identified explicitly in perturbation theory, *e.g.*, proceeding in a similar way to how it is done in the eigenvalue problem for heavy quarkonia in Ref. [18].

<sup>20</sup> The non-locality discussed here is due to the RGPEP vertex form factors of width  $\lambda$  in momentum space. The RGPEP-induced non-locality must not be confused with the canonical non-locality of LF Hamiltonians in which the dependent parts of fields, such as  $\psi_- = A_- \psi$  and  $A^-$ , involve inverse powers of  $i\partial^+$ , which is a non-local integral operator.

$$\begin{aligned}
h_{\text{MC}\lambda}(x_1, x_2, x_3) &= g_\lambda f_\lambda(x_2 - x_1, x_3 - x_1) \bar{\psi}_\lambda(x_1) \\
&\quad \times [\mathcal{A}_\lambda(x_1) + \partial \mathcal{A}_\lambda(x_1)(x_2 - x_1) + \dots] \\
&\quad \times [\psi_\lambda(x_1) + \partial \psi_\lambda(x_1)(x_3 - x_1) + \dots]. \quad (65)
\end{aligned}$$

The three dots indicate terms with higher derivatives. If all effective particles move slowly and the derivative terms are small, the Hamiltonian can be approximated by the first term in the expansion

$$H_{I\lambda} = \int d^3x_1 d^3x_2 d^3x_3 h_{I\lambda} \quad (66)$$

$$\sim g_\lambda \int d^3x \bar{\psi}_\lambda(x) \mathcal{A}_\lambda(x) \psi_\lambda(x) \int d^3y d^3z f_\lambda(y, z) + \dots \quad (67)$$

This means that the non-local effective interaction in a slowly moving, NR system still looks like a local one except that its strength is determined not solely by the coupling constant  $g_\lambda$  but also by the integral of the non-local form factor on the LF hyperplane

$$\tilde{g}_\lambda = g_\lambda \int d^3y d^3z f_\lambda(y, z). \quad (68)$$

The point is that the effective Hamiltonian density at  $\lambda \gtrsim \lambda_c$  may partly resemble a Hamiltonian of local gauge theory in its terms that couple quarks with gluons even though the actual effective interactions are non-local, provided that one limits the domain of the Hamiltonian to slowly moving hadrons. In this case, construction of approximate candidates for the LF Hamiltonian density of coupling between quarks and gluons, before they are corrected using RGPEP, may proceed in analogy to QED [51]. This means that one uses the gauge  $A^+ = 0$ . The derivative  $i\partial^\perp$  in the quark kinetic energy in Eq. (60) is supplied with the minimal coupling addition of  $gA^\perp$  (the product of derivatives is separated by the inverse of  $i\partial^+$ , which is not altered). In addition, one includes terms dictated by constraints that imply the result of Eq. (62) for the quark–gluon coupling.

Much less is understood about the gluon part of the effective Hamiltonian density for light hadrons at  $\lambda \gtrsim \lambda_c$ . The lack of understanding of the Hamiltonian is reflected in the lack of understanding of the gluon components in its eigenstates. One would have to calculate the operator  $H_\lambda$  precisely in order to uncover the information it contains about how to build a model approximating light hadrons in LF QCD. While RGPEP provides tools for such calculations, the remaining part of the paper is only devoted to deriving the model that can be treated as a first approximation.

A simple candidate for an approximate Hamiltonian for light hadrons in LF QCD is constructed in the next section assuming that: (1) the result of action of  $W$  in RGPEP can be represented by inclusion of the color-transport factors  $T$  that maintain local gauge symmetry of hadronic states, (2) the operator  $G$  represents the condensation of gluons inside hadrons, (3) a mean field approximation can be applied to the gluon field operator  $A$  in the minimal coupling of effective quarks to the gluons condensed in a hadron, and (4) the effects of quark and gluon binding that are not treatable in the mean field approximation can be included by an *ad hoc* Gaussian approximation to the wave function of relative motion of constituent quarks with respect to the condensed gluons.

5.4. Mass squared with minimal coupling at  $\lambda \gtrsim \lambda_c$

The leading idea is that the approximate LF Hamiltonian for 2 or 3 quarks at scale  $\lambda \gtrsim \lambda_c$  in light hadrons should have a form compatible with minimal coupling between quarks and gluons and it should respect Poincaré symmetry. Realization of the idea employs the four assumptions listed at the end of the previous section in the following way.

The LF Hamiltonian eigenvalue problem at  $\lambda \gtrsim \lambda_c$  for a light hadron built from 2 or 3 quarks and condensed gluons,

$$H_\lambda |\psi\rangle = \frac{M^2 + P_h^{\perp 2}}{P_h^+} |\psi\rangle, \tag{69}$$

can be written in terms of the eigenvalue equation for the effective invariant mass operator of the hadron,  $\mathcal{M}_\lambda^2$ , in the form

$$\mathcal{M}_\lambda^2 |\psi\rangle = M^2 |\psi\rangle. \tag{70}$$

One can think about the operator on the left-hand side of this equation in terms of the invariant mass squared of free particles plus interaction terms.

For a free quark of mass  $m_1$  and a free anti-quark of mass  $m_2$ , one has

$$\mathcal{M}_{\text{free}}^2 = \left( \sqrt{m_1^2 + \vec{p}_1^2} + \sqrt{m_2^2 + \vec{p}_2^2} \right)^2 - (\vec{p}_1 + \vec{p}_2)^2. \tag{71}$$

For slowly moving particles, neglecting terms smaller than the ones that are quadratic in momenta, the NR approximation renders

$$\mathcal{M}_{\text{free}}^2 = (m_1 + m_2)^2 + \frac{m_1 + m_2}{m_1} \vec{p}_1^2 + \frac{m_1 + m_2}{m_2} \vec{p}_2^2 - \vec{P}_{q\bar{q}}^2, \tag{72}$$

where  $\vec{P}$  denotes the total momentum of the two quarks. Using

$$\beta_1 = \frac{m_1}{m_1 + m_2}, \tag{73}$$

$$\beta_2 = \frac{m_2}{m_1 + m_2} \tag{74}$$

one has

$$\mathcal{M}_{\text{free}}^2 = (m_1 + m_2)^2 + \frac{1}{\beta_1} \vec{p}_1^2 + \frac{1}{\beta_2} \vec{p}_2^2 - \vec{P}_{q\bar{q}}^2. \tag{75}$$

According to the gauge rule of minimal coupling, the interaction of quarks with the condensed gluons should be described by

$$\mathcal{M}_{\text{minimal}}^2 = (m_1 + m_2)^2 + \frac{(\vec{p}_1 - g\vec{A}_1)^2}{\beta_1} + \frac{(\vec{p}_2 - g\vec{A}_2)^2}{\beta_2} - \vec{P}_{q\bar{q}}^2, \tag{76}$$

where  $\vec{A}_i$  is an abbreviation for  $\vec{A}(x_i)$ . Analogous reasoning in the case of baryons yields

$$\mathcal{M}_{\text{minimal}}^2 = \left( \sum_{i=1}^3 m_i \right)^2 + \sum_{j=1}^3 \frac{(\vec{p}_j - g\vec{A}_j)^2}{\beta_j} - \vec{P}_{3q}^2, \tag{77}$$

where  $\beta_i = m_i / (m_1 + m_2 + m_3)$ .

For all quarks having the same mass  $m$ ,  $\beta_M = \beta_1 = \beta_2 = 1/2$  in mesons and  $\beta_B = \beta_1 = \beta_2 = \beta_3 = 1/3$  in baryons. Assuming these simplifications, the minimal coupling terms in Eqs. (76) and (77) can be interpreted following Ref. [17] in the context of standard Hamiltonian dynamics as resulting from the Hamiltonian operator with a proper  $\beta$ , *i.e.*,  $\beta_M$  in mesons and  $\beta_B$  in baryons

$$H_{\text{min}} = \frac{1}{\beta} \int d^3x : \bar{\psi}(\vec{x}) \left[ -i\vec{\nabla}_x - g\vec{A}(\vec{x}) \right]^2 \psi(\vec{x}) : . \tag{78}$$

### 5.5. Reinterpretation of the gluon condensate

Using Eqs. (57), (58), and (78), one can consider the eigenvalue problems

$$H_{\text{min}} \sum_{12G} \psi(12G) |12G\rangle = M_{q\bar{q}}^2 \sum_{12G} \psi(12G) |12G\rangle, \tag{79}$$

$$H_{\text{min}} \sum_{123G} \psi(123G) |123G\rangle = M_{3q}^2 \sum_{123G} \psi(123G) |123G\rangle, \tag{80}$$

for quark subsystems in mesons and baryons and project them on the corresponding basis states. The calculation proceeds in a similar way to the one in Ref. [17], with three exceptions.

The first difference is that the eigenvalues  $M_{q\bar{q}}^2$  and  $M_{3q}^2$  refer only to the  $q\bar{q}$  subsystem in mesons, instead of the entire meson mass squared, and to the  $3q$  subsystem in a baryon, instead of the entire baryon mass squared. Note also that the eigenvalue equations considered in Ref. [17] were for usual energies while we now consider the invariant mass squared operators that include the quark free invariant mass squared and minimal coupling interactions. The second difference is that the vacuum state  $|\Omega\rangle$  is replaced by the state of gluons condensed in a hadron,

$$|G\rangle = G^\dagger(\vec{x}_G)|0\rangle. \tag{81}$$

The vacuum state  $|0\rangle$  does not develop any condensate. The third difference is that in the mean-field approximation for the gluon field<sup>21</sup>,

$$\vec{A}(\vec{x}) = \frac{1}{2}\vec{B}(\vec{x}_G) \times (\vec{x} - \vec{x}_G), \tag{82}$$

one introduces the color magnetic field operator at the center of the gluon component  $G$  of a hadron,  $\vec{B}(\vec{x}_G)$ , instead of the vacuum operator  $\vec{B}$  at an arbitrary point  $\vec{x}_0$ . However, these differences do not change the formal calculation of the matrix elements. The field  $\vec{B}(\vec{x}_G)$  is assumed factorized into a product of vectors, one with space and the other with color components as in Ref. [17]. This assumption is meant to reflect the assumed lack of correlation between position space and color space directions in the mean-field approximation and it renders a simple, Abelian form of the approximate model for the effective Hamiltonian. Matrix elements of terms linear in  $A$  are set to zero because they change under gauge transformations whereas the white component of a hadron does not.

The matrix elements one obtains for a  $q\bar{q}$  pair in a meson and  $3q$  in a baryon read<sup>22</sup>

$$\langle 12G|H_{\min}|\psi\rangle_M = \frac{1}{\beta_M} \sum_{i=1}^2 \left\{ -\Delta_i + \frac{g^2}{3} \text{Tr} \frac{\langle G|(A_1 - A_2)^2|G'\rangle}{\langle G|G'\rangle} \right\} \langle 12G|\psi\rangle, \tag{83}$$

$$\begin{aligned} \langle 123G|H_{\min}|\psi\rangle_B &= \frac{1}{\beta_B} \sum_{i=1}^3 (-\Delta_i) \langle 123G|\psi\rangle_B + \frac{1}{\beta_B} \sum_{i=1}^3 \frac{g^2}{3} \\ &\times \text{Tr} \frac{\langle G|(A_i - A_{\frac{j+k}{2}})^2 + \frac{1}{12}(A_j - A_k)^2|G'\rangle}{\langle G|G'\rangle} \langle 123G|\psi\rangle_B. \end{aligned} \tag{84}$$

<sup>21</sup> This approximation replaces the Schwinger gauge formula for the background gluon field that reproduces QCD sum rules for heavy quarkonia in the LF Hamiltonian formulation of the theory [52].

<sup>22</sup> Eq. (14) in Ref. [17] misses the factor 1/3 in front of the condensate term that is correctly printed here in Eq. (83).

All gluon field matrix elements contain squares of differences of the effective gluon field operator at different points. Therefore, the position of the gluon body,  $\vec{x}_G$ , drops out. In the mean-field approximation, one has

$$\vec{A}(\vec{x}) - \vec{A}(\vec{y}) = \frac{1}{2} \vec{B} \times (\vec{x} - \vec{y}), \quad (85)$$

where  $\vec{B} = \vec{B}(\vec{x}_G)$ . Consequently,

$$\text{Tr} \langle G | \frac{g^2}{4\pi^2} (\vec{A}_x - \vec{A}_y)^2 | G' \rangle = 2 (\vec{x} - \vec{y})^2 C_{\text{glue}} \langle G | G' \rangle \quad (86)$$

with

$$\langle G | G' \rangle = \delta_{GG'}. \quad (87)$$

The expectation value  $C_{\text{glue}}$  plays here the same role that the vacuum gluon condensate value  $C_{\text{vacuum}} = \varphi_{\text{vacuum}}^2/96$  with

$$\varphi_{\text{vacuum}}^2 = \langle \Omega | (\alpha/\pi) G^{\mu\nu c} G_{\mu\nu}^c | \Omega \rangle, \quad (88)$$

plays in Ref. [17]. Replacement of the constant  $C_{\text{vacuum}}$  by the constant  $C_{\text{glue}} = \varphi_{\text{glue}}^2/96$ , implies our reinterpretation of the phenomenologically useful quantity  $\varphi_{\text{vacuum}}$  on the order of  $\Lambda_{\text{QCD}}$  as coming from the quantity  $\varphi_{\text{glue}}$  that originates according to the RGPEP model in the gluon condensation only inside a hadron instead of the entire space. Hence,

$$\begin{aligned} \langle 12G | H_{\text{min}} | \psi \rangle_M &= \frac{1}{\beta_M} (-\Delta_1 - \Delta_2) \langle 12G | \psi \rangle_M \\ &+ \frac{1}{\beta_M} \left( \frac{\pi \varphi_{\text{glue}}}{3} \right)^2 \frac{r_{12}^2}{2} \langle 12G | \psi \rangle_M, \end{aligned} \quad (89)$$

$$\begin{aligned} \langle 123G | H_{\text{min}} | \psi \rangle_B &= \frac{1}{\beta_B} (-\Delta_1 - \Delta_2 - \Delta_3) \langle 123G | \psi \rangle_B \\ &+ \frac{1}{\beta_B} \frac{5}{8} \left( \frac{\pi \varphi_{\text{glue}}}{3} \right)^2 \left( \frac{r_{12}^2}{2} + \frac{2r_3^2}{3} \right) \langle 123G | \psi \rangle_B, \end{aligned} \quad (90)$$

where

$$r_{12} = x_1 - x_2, \quad (91)$$

$$r_3 = x_3 - (x_1 + x_2)/2. \quad (92)$$

The results for  $H_{\text{min}}$  with new interpretation of the gluon condensate inside hadrons are used in the next section to construct candidates for effective LF Hamiltonians of light mesons and baryons as first approximations to solutions of RGPEP in QCD.

5.6. Approximate LF Hamiltonian for quarks at  $\lambda \gtrsim \lambda_c$  in light hadrons

Candidates for approximate Hamiltonians for light hadrons can be obtained starting from inclusion of results in Eqs. (83) and (84), or (89) and (90), in Eqs. (76) and (77), correspondingly. The meson case is simpler than the baryon case and explains the leading idea of constructing the effective Hamiltonians using the minimal coupling dictated by gauge symmetry. We have

$$\begin{aligned} \mathcal{M}_{q\bar{q}}^2 &= (m_1 + m_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 \\ &+ \frac{1}{\beta_1} \left[ \vec{p}_1^2 + \left\langle \frac{g^2}{3} \text{Tr} \left( \vec{A}_1 - \vec{A}_2 \right)^2 \right\rangle_G \right] \\ &+ \frac{1}{\beta_2} \left[ \vec{p}_2^2 + \left\langle \frac{g^2}{3} \text{Tr} \left( \vec{A}_1 - \vec{A}_2 \right)^2 \right\rangle_G \right]. \end{aligned} \quad (93)$$

Using the variables

$$\vec{P}_{12} = \vec{p}_1 + \vec{p}_2, \quad (94)$$

$$\vec{k} = \beta_2 \vec{p}_1 - \beta_1 \vec{p}_2, \quad (95)$$

$$\vec{p}_1 = \beta_1 \vec{P}_{12} + \vec{k}, \quad (96)$$

$$\vec{p}_2 = \beta_2 \vec{P}_{12} - \vec{k} \quad (97)$$

one arrives at

$$\mathcal{M}_{q\bar{q}}^2 = (m_1 + m_2)^2 + \frac{1}{\beta_1 \beta_2} \left[ \vec{k}^2 + \left\langle \frac{g^2}{3} \text{Tr} \left( \vec{A}_1 - \vec{A}_2 \right)^2 \right\rangle_G \right]. \quad (98)$$

The LF counterpart of this result is obtained by considering three-vectors with + and  $\perp$  components instead of  $z$  and  $\perp$ . So, one writes

$$P_{12}^{+,\perp} = p_1^{+,\perp} + p_2^{+,\perp}, \quad (99)$$

$$x_1 = p_1^+ / P_{12}^+, \quad x_2 = p_2^+ / P_{12}^+, \quad (100)$$

$$\kappa^\perp = x_2 p_1^\perp - x_1 p_2^\perp, \quad (101)$$

$$p_1^\perp = x_1 P_{12}^\perp + \kappa^\perp, \quad (102)$$

$$p_2^\perp = x_2 P_{12}^\perp - \kappa^\perp. \quad (103)$$

These are standard expressions in LF description of relative motion of two constituents carrying the total momentum  $P$ . Using these variables for two free particles named 1 and 2, one obtains their free invariant mass squared in the form

$$\mathcal{M}_{12}^2 = \frac{\kappa^{\perp 2} + m_1^2}{x_1} + \frac{\kappa^{\perp 2} + m_2^2}{x_2} \quad (104)$$

$$= (m_1 + m_2)^2 + \frac{1}{x_1 x_2} \left[ \kappa^{\perp 2} + (m_2 x_1 - m_1 x_2)^2 \right]. \quad (105)$$

Comparison with Eq. (98) identifies the relative momentum  $\vec{k}$

$$k^\perp = \sqrt{\frac{\beta_1\beta_2}{x_1x_2}} \kappa^\perp, \quad (106)$$

$$k^z = \sqrt{\frac{\beta_1\beta_2}{x_1x_2}} [m_2x_1 - m_1x_2]. \quad (107)$$

This is a new way of parameterizing relative motion of two constituents in LF approach to quantum mechanics and field theory. The new variables match non-relativistic relative momenta that include effects of particle masses. On the other hand, it is known that in the relativistic relative motion of two constituents the mass parameters appear not significant. For example, the standard relative momentum, in which  $k^\perp = \kappa^\perp$ , has length

$$\vec{k}_{\text{standard}}^2 = \frac{1}{4\mathcal{M}_{12}^2} [\mathcal{M}_{12}^2 - (m_1 + m_2)^2] [\mathcal{M}_{12}^2 - (m_1 - m_2)^2], \quad (108)$$

and a relatively complicated expression is obtained for  $k^z$  if one insists on the identification  $k^\perp = \kappa^\perp$ . But when  $\mathcal{M}_{12}$  is large, one has  $\vec{k}_{\text{standard}}^2 = \mathcal{M}_{12}^2/4$  independently of the quark masses. At the same time, the variable  $k_{\text{standard}}^z$  depends on  $\kappa^\perp$ . The length of the standard relative three-momentum in the constituent rest frame differs from the length of the new one,

$$\vec{k}_{\text{standard}}^2 = \frac{\vec{k}^2}{4\beta_1\beta_2} \left[ 1 - \frac{(m_1 - m_2)^2}{\mathcal{M}_{12}^2} \right], \quad (109)$$

and the angular orientations of the three-vectors  $\vec{k}_{\text{standard}}$  and  $\vec{k}$  also differ. This means that the new choice of LF three-momentum variables in relativistic cases involves rotation by some polar angle (the azimuthal angles are the same). Analysis of rotational symmetry is influenced in the sense that in order to recover a simple NR quantum mechanical picture of hadrons from quantum field theory one is motivated by the effective picture in RGPEP to use the new variable  $\vec{k}$  rather than  $\vec{k}_{\text{standard}}$  as arguments of potentials for constituent quarks. One should also remember that when a change of variables is made in the expressions involving effective quark wave functions, the new variables produce Jacobian factors in phase-space integration.

Using Eq. (89), one obtains

$$\left\langle \frac{g^2}{3} \text{Tr} \left( \vec{A}_1 - \vec{A}_2 \right)^2 \right\rangle_G = \frac{1}{2} \left( \frac{\pi\varphi_{\text{glue}}}{3} \right)^2 \frac{r_{12}^2}{2}, \quad (110)$$

where  $\vec{r}_{12}$  should be the relative position variable that is canonically conjugated to  $\vec{k}$ . This means in quantum mechanics that

$$\vec{r}_{12} = i \frac{\partial}{\partial \vec{k}}. \quad (111)$$

Thus, the LF mass squared for a constituent quark–anti-quark pair at scale  $\lambda \gtrsim \lambda_c$  interacting in a gauge minimal way with gluons condensed inside a meson and treated in a mean-field approximation, has the form

$$\mathcal{M}_{q\bar{q}}^2 = (m_1 + m_2)^2 + \frac{1}{\beta_1\beta_2} \left[ \vec{k}^2 + \frac{1}{2} \left( \frac{\pi\varphi_{\text{glue}}}{3} \right)^2 \frac{1}{2} \left( i \frac{\partial}{\partial \vec{k}} \right)^2 \right], \quad (112)$$

where the vector  $\vec{k}$  is defined by Eqs. (106) and (107).

In particular, the effective quarks  $u$  and  $d$  are expected to have practically the same mass  $m$  order  $\Lambda_{\text{QCD}}$  in the RGPEP scheme. For them,  $\beta_1 = \beta_2 = 1/2$  and the associated Jacobi variables are

$$\vec{\rho} = \vec{r}_{12}/\sqrt{2}, \quad (113)$$

$$\vec{p}_\rho = \vec{k}\sqrt{2}. \quad (114)$$

In terms of these variables,

$$\mathcal{M}_{q\bar{q}}^2 = 4m^2 + 2p_\rho^2 + 2 \left( \frac{\pi\varphi_{\text{glue}}}{3} \right)^2 \rho^2. \quad (115)$$

For small relative momenta and corresponding distances, this result approximately matches the NR oscillator dynamics,

$$\mathcal{M}_{q\bar{q}} = 2m + \frac{p_\rho^2}{2m} + \frac{1}{2} m \left( \frac{\pi\varphi_{\text{glue}}}{3m} \right)^2 \rho^2, \quad (116)$$

with frequency  $\omega_M = \pi\varphi_{\text{glue}}/(3m)$ .

On the one hand, this result relates the RGPEP reasoning to the CQM phenomenology since the reinterpreted value of the gluon condensate produces physically reasonable frequency  $\omega_M$  [17]. On the other hand, the variables  $\vec{k}$  and  $\vec{r}$  identified here include the characteristic factor  $\sqrt{x(1-x)}$  that is required in AdS/QCD holographic variables [24, 25, 26]. Thus, it becomes plausible that also the RGPEP reasoning regarding the  $\lambda$ -dependence of effective interactions may provide insight into the significance of soft wall (SW) models [27] of hadron spectrum.

As a result of analogous reasoning in the baryon case, we obtain

$$\mathcal{M}_{123}^2 - (m_1 + m_2 + m_3)^2 = \frac{1}{\beta_3(1-\beta_3)} \vec{Q}^2 + \frac{1-\beta_3}{\beta_1\beta_2} \vec{K}^2, \quad (117)$$

where

$$x_i = p_i^+ / P_{123}^+, \quad (118)$$

$$\beta_i = \frac{m_i}{m_1 + m_2 + m_3}, \tag{119}$$

$$p_3^\perp = x_3 P_{123}^\perp + q^\perp, \tag{120}$$

$$p_2^\perp = x_2 P_{123}^\perp - \frac{x_2}{1 - x_3} q^\perp - k^\perp, \tag{121}$$

$$p_1^\perp = x_1 P_{123}^\perp - \frac{x_1}{1 - x_3} q^\perp + k^\perp, \tag{122}$$

$$Q^\perp = \sqrt{\frac{\beta_3(1 - \beta_3)}{x_3(1 - x_3)}} q^\perp, \tag{123}$$

$$Q^z = \sqrt{\frac{\beta_3(1 - \beta_3)}{x_3(1 - x_3)}} [(m_1 + m_2)x_3 - m_3(x_1 + x_2)], \tag{124}$$

$$K^\perp = \sqrt{\frac{\beta_1\beta_2(1 - x_3)}{x_1x_2(1 - \beta_3)}} k^\perp, \tag{125}$$

$$K^z = \sqrt{\frac{\beta_1\beta_2(1 - x_3)}{x_1x_2(1 - \beta_3)}} \left( m_2 \frac{x_1}{1 - x_3} - m_1 \frac{x_2}{1 - x_3} \right). \tag{126}$$

For slow relative motion of constituents, the new momentum variables  $\vec{K}$  and  $\vec{Q}$  match the non-relativistic three-momenta used in quark models, but in fact they apply in the entire range of relativistic kinematics of motion of constituents inside baryons, and for arbitrary motion of baryons as a whole, thanks to the kinematical symmetries of LF formulation of the theory. However, one needs to remember that the lengths and angular orientations of the new momentum variables differ from the standard LF variables among which, in particular,  $q_{12}^\perp$  and  $q_3^\perp$  (see below) could be thought directly usable for obtaining some effective constituent picture for baryons from LF QCD.

The LF mass squared for 3 constituent quarks of one and the same mass  $m$  at scale  $\lambda \gtrsim \lambda_c$  interacting in a gauge minimal way with gluons condensed inside a baryon in a mean-field approximation, has the form

$$\mathcal{M}_{3q}^2 = 9m^2 + 6\vec{K}^2 + \frac{9}{2}\vec{Q}^2 - 3m^2 \left( \frac{\pi\varphi}{3m} \right)^2 \frac{5}{8} \frac{\Delta_K^2}{2} + \frac{2\Delta_Q}{3}. \tag{127}$$

We identify relations

$$\vec{K} = \vec{q}_{12} = \vec{p}_\rho / \sqrt{2}, \tag{128}$$

$$-i \frac{\partial}{\partial \vec{K}} = \vec{r}_{12} = \sqrt{2} \vec{\rho}, \tag{129}$$

$$\vec{Q} = \vec{q}_3 = -\sqrt{2/3} \vec{p}_\lambda, \tag{130}$$

$$-i \frac{\partial}{\partial \vec{Q}} = \vec{r}_3 = -\sqrt{3/2} \vec{\lambda}, \tag{131}$$

in terms of the Jacobi co-ordinates for 3 quarks with equal masses,  $\vec{\rho}$ ,  $\vec{p}_\rho$ ,  $\vec{\lambda}$ , and  $\vec{p}_\lambda$ . One has

$$P_{123} = p_1 + p_2 + p_3, \quad q_{12} = (p_1 - p_2)/2, \quad (132)$$

$$P_{12} = p_1 + p_2, \quad q_3 = (2p_3 - P_{12})/3, \quad (133)$$

$$P_{12} = 2P_{123}/3 - q_3, \quad p_3 = P_{123}/3 + q_3, \quad (134)$$

$$p_1 = P_{12}/2 + q_{12} = P_{123}/3 + q_{12} - q_3/2, \quad (135)$$

$$p_2 = P_{12}/2 - q_{12} = P_{123}/3 - q_{12} - q_3/2, \quad (136)$$

$$p_3 = P/3 + q_3. \quad (137)$$

$$p_1^2 + p_2^2 + p_3^2 = P_{123}^2/3 + 2q_{12}^2 + 3q_3^2/2. \quad (138)$$

$$R_{123} = (x_1 + x_2 + x_3)/3, \quad r_{12} = x_1 - x_2, \quad r_3 = x_3 - R_{12}, \quad (139)$$

$$x_3 = R_{123} + 2r_3/3, \quad R_{12} = (x_1 + x_2)/2 = R_{123} - r_3/3, \quad (140)$$

$$x_1 = R_{12} + r_{12}/2 = R_{123} - r_3/3 + r_{12}/2, \quad (141)$$

$$x_2 = R_{12} - r_{12}/2 = R_{123} - r_3/3 - r_{12}/2. \quad (142)$$

$$\sum_{i=1}^3 p_i x_i = P_{123} R_{12} + q_{12} r_{12} + q_3 r_3. \quad (143)$$

In the case of the same masses, the relativistic LF result for three quarks in a baryon reads,

$$\mathcal{M}_{3q}^2 = 9m^2 + 3\vec{p}_\rho^2 + 3\vec{p}_\lambda^2 + 3m^2 \left(\frac{\pi\varphi}{3m}\right)^2 \frac{5}{8} (\vec{\rho}^2 + \vec{\lambda}^2). \quad (144)$$

In the NR limit,

$$\mathcal{M}_{3q} = 3m + \frac{\vec{p}_\rho^2}{2m} + \frac{\vec{p}_\lambda^2}{2m} + \frac{1}{2} m \frac{5}{8} \left(\frac{\pi\varphi}{3m}\right)^2 (\vec{\rho}^2 + \vec{\lambda}^2), \quad (145)$$

which matches precisely the result of Ref. [17], and

$$\omega_B^2 = \frac{5}{8} \omega_M^2. \quad (146)$$

This numerical relation is in a reasonable agreement with phenomenology of constituent quark models [21, 22, 23], as noticed already in Ref. [17] (see also Sec. 6.4).

### 5.7. Dynamics of quarks and glue in hadrons at $\lambda \gtrsim \lambda_c$

The constituent models do not, however, include the gluon component  $G$ . Model potentials are also not assigned energy or momentum, while gluons do carry energy and momentum. Bag models do include a bag in terms of boundary conditions on quark wave functions on the bag walls and a constant contribution of the bag to the total hadron energy, but the bag is not placed yet in the context of QCD [53]. The author does not know how to interpret results of lattice formulation of QCD regarding the constituent model and the gluon component in terms of hadronic wave functions. Summarizing, it is not clear how the addition of  $G$  will affect phenomenology familiar through the CQMs without  $G$ , although some foreseeable changes appear welcome (see the next section).

Regarding the contribution of mass of  $G$  to the mass squared of the whole hadron, one cannot say anything but hypothesize that the mass of  $G$  may depend on  $\lambda$ . It can be small, due to interactions that hold gluons among themselves in the form of  $G$ . The mass of  $G$  may even decrease when  $\lambda$  drops down to  $\lambda_c$ . A graphical picture that makes this idea plausible as a result of overlapping of quarks of size  $1/\lambda$  (a quark and an anti-quark in a meson or three quarks in a baryon) and  $G$  in position space, is described in Appendix D. However, a simple concept of a constituent  $G$  may be insufficient in the case of  $\pi$ -mesons, considering that  $\pi$ -mesons may differ from other hadrons as a consequence of their relationship to chiral symmetry and its breaking. The symmetry breaking can be addressed in LF QCD [16] but it is not addressed in this article. Such discussion requires reinterpretation of the quark condensate and  $G$  built from gluons alone is not sufficient.

Regarding the motion of the glue component  $G$  inside a hadron, one may observe that if the state  $G^\dagger|0\rangle$  in RGPEP is to correspond to the vacuum in the instant dynamics way of thinking and  $G$  is associated through RGPEP with a hadron in an effective theory of scale  $\lambda \gtrsim \lambda_c$ , the effective quarks at this scale should not be free to move farther away from  $x_G$  than about  $1/\lambda$  (see also Appendix D). The mean field approximation misses the fact that the gluon component has a size comparable with a hadron. The Schwinger gauge expression for the gluon field potential  $A_\lambda(x)$  with only one term linear in  $\vec{x} - \vec{x}_G$  and  $\vec{B}(\vec{x}_G)$ , Eq. (82), must be missing important effects at distances comparable with the diameter of a hadron, *cf.* [52]. This means that the gauge transport factors  $T(x, \underline{x})$  cannot be approximated well in the entire volume of a hadron using the Schwinger gauge with a linear term alone. Thus, the mean-field calculation does not report well on what happens at the boundary of the glue or the boundary of a hadron. While one cannot determine in advance what will eventually result from RGPEP calculations of effective Hamiltonians and eigenvalue problems for these Hamiltonians, one should certainly expect that there will be forces that keep the center of the quarks and the glue together.

There is no simpler choice for the needed interaction term between the quarks and  $G$  than a quadratic function of the distance between  $\underline{x}$  and  $x_G$ . Such distance is easy to write in NR quantum mechanics. However, one has to carefully define its analog in the hadron mass squared operator on the LF. Solution of this technical issue described here, is found by looking at hadrons as built only from two constituents. One constituent is the effective quarks treated as one particle of mass resulting from eigenvalues of  $\mathcal{M}_{q\bar{q}}^2$  in mesons or  $\mathcal{M}_{3q}^2$  in baryons. The other constituent is  $G$  treated as a particle. The mass of  $G$  is unknown, may depend on  $\lambda$ , and is free to adjust in a process of defining a first approximation to hadrons at  $\lambda \gtrsim \lambda_c$ . With this setup, one applies the same method for constructing the LF mass squared operator that we used for a quark and an anti-quark treated as two constituents in a harmonically bound state. There is no minimal gauge coupling to use for the colorless state of quarks interacting with a white  $G$ , and a large degree of ambiguity must be confronted without clear guidance from QCD<sup>23</sup>. But there is no problem now with introducing the appropriate oscillator potential operator using the distance that is quantum-mechanically conjugated to a relative momentum.

The first-approximation to LF Hamiltonian for light hadrons at  $\lambda \gtrsim \lambda_c$  is suggested to have the form

$$H_{\lambda \gtrsim \lambda_c} = \frac{\mathcal{M}_{\text{quarks}}^2 + P_{\text{quarks}}^{\perp 2}}{P_{\text{quarks}}^+} + \frac{\mathcal{M}_G^2 + P_G^{\perp 2}}{P_G^+} + \frac{\mathcal{M}_{qG}^2}{P_{\text{hadron}}^+}. \quad (147)$$

In order to define a suitable candidate for the last term, one can introduce

$$P_{\text{hadron}}^{+, \perp} = P_{\text{quarks}}^{+, \perp} + P_G^{+, \perp}, \quad (148)$$

$$x_q = P_{\text{quarks}}^+ / P_{\text{hadron}}^+, \quad (149)$$

$$x_G = P_G^+ / P_{\text{hadron}}^+, \quad (150)$$

$$\kappa_q^{\perp} = x_G P_{\text{quarks}}^{\perp} - x_q P_G^{\perp}, \quad (151)$$

$$P_{\text{quarks}}^{\perp} = x_q P_{\text{hadron}}^{\perp} + \kappa_q^{\perp}, \quad (152)$$

$$P_G^{\perp} = x_G P_{\text{hadron}}^{\perp} - \kappa_q^{\perp}. \quad (153)$$

In terms of these variables, the hadron mass squared reads

$$\mathcal{M}_{\text{hadron}}^2 = \frac{\mathcal{M}_{\text{quarks}}^2 + \kappa_q^{\perp 2}}{x_q} + \frac{\mathcal{M}_G^2 + \kappa_q^{\perp 2}}{x_G} + \mathcal{M}_{qG}^2, \quad (154)$$

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<sup>23</sup> It is precisely this type of ambiguity at small energies that RGPEP is meant to eventually help in resolving.

where  $\mathcal{M}_{\text{quarks}}^2$  is now to be identified with the eigenvalue  $M_q^2$  for the mass squared of the quark subsystem in a hadron and  $\mathcal{M}_G$  is set to the corresponding eigenvalue  $M_G^2$ , which could be considered to have a ground state value or an excited value, if the condensed gluons were in an excited state. The constant  $M_q^2$  is either  $4m^2$  or  $9m^2$  plus an appropriate number times  $m\omega$  with  $\omega = \omega_M$  or  $\omega = \omega_B$ , respectively. Having introduced

$$\beta_q = \frac{M_q}{M_q + M_G}, \tag{155}$$

$$\beta_G = \frac{M_G}{M_q + M_G}, \tag{156}$$

one can define

$$k_h^\perp = \sqrt{\frac{\beta_q \beta_G}{x_q x_G}} \kappa_q^\perp, \tag{157}$$

$$k_h^z = \sqrt{\frac{\beta_q \beta_G}{x_q x_G}} [M_G x_q - M_q x_G]. \tag{158}$$

Proceeding as in the case of two quarks in a meson, Eq. (112), one establishes

$$\mathcal{M}_{qG}^2 = \frac{1}{\beta_q \beta_G} \frac{1}{2} \left( \frac{\pi \varphi_h}{3} \right)^2 \frac{1}{2} \left( i \frac{\partial}{\partial \vec{k}_h} \right)^2 - M_\varphi^2, \tag{159}$$

$$\begin{aligned} \mathcal{M}_{\text{hadron}}^2 &= (M_q + M_G)^2 - M_\varphi^2 \\ &+ \frac{1}{\beta_q \beta_G} \left[ \vec{k}_h^2 + \frac{1}{2} \left( \frac{\pi \varphi_h}{3} \right)^2 \frac{1}{2} \left( i \frac{\partial}{\partial \vec{k}_h} \right)^2 \right], \end{aligned} \tag{160}$$

where two new parameters,  $\varphi_h$  and  $M_\varphi$ , are introduced.

The parameter  $\varphi_h$  is used in the similar way to how  $\varphi_{\text{glue}}$  is used in the invariant mass of quarks in mesons. However,  $\varphi_h$  serves merely the purpose of notation for the unknown quantity of binding between quarks and  $G$  whose value must be adjusted in the process of correcting the candidate for the first approximation. In RGPEP,  $\varphi_h$  can be expected to depend on  $\lambda$ . Intuitive arguments are offered in Appendix D. This candidate for a first approximation will only be satisfying if it turns out in future calculations in RGPEP that some optimal value of  $\varphi_h$  can be adjusted as a function of  $\lambda/\Lambda_{\text{QCD}}$  and the corresponding coupling constant, assuming that the light quark mass parameters do not matter at small  $\lambda$  where the effective quark masses are on the order of  $\Lambda_{\text{QCD}}$  anyway and do not vary with  $\lambda$  so rapidly that no average constituent picture can correspond to QCD. Assuming that

$\varphi_h$  does not vary from hadron to hadron, one can introduce

$$\omega_h = \frac{\pi\varphi_h}{3\mu_h}, \tag{161}$$

$$\mu_h = \frac{M_q M_G}{M_q + M_G}. \tag{162}$$

The parameter  $\mu_h$  is the reduced mass for the two-body system quarks- $G$ . It depends on a hadron as far as  $M_q$  depends on a hadron, assuming  $M_G$  can be considered universal in the first approximation. Thus, the mass-squared eigenvalues for the whole hadron takes the form

$$M_h^2 = (M_q + M_G)^2 - M_\varphi^2 + \frac{\mu_h}{\beta_q \beta_G} (n_h + 3/2)\omega_h, \tag{163}$$

where  $n_h$  denotes the excitation quantum number in relative motion of quarks with respect to the gluon condensate in a hadron, being zero in a ground state. Thus the momentum width of relative motion of quarks with respect to  $G$  is of the order of  $\varphi_h$ .

The mass parameter  $M_\varphi$  is introduced by fiat and brings in another considerable degree of ambiguity. Physical motivation for  $M_\varphi$  is that for the gluon condensation to occur in hadrons it must be favorable energetically. If one just added the free energy of  $G$  and a harmonic quarks-gluon potential energy to the energy of quarks, the condensation of gluons would only add energy to the system of quarks. The effect of condensation should rather be opposite: the concept of condensation of gluons inside hadrons implies thinking that the mass of a hadron state is lowered by condensation of gluons in comparison with a state in which condensation is absent. Since the kinematical minimum of energy of quarks and glue is  $M_q + M_G$ , which corresponds to  $\kappa_q^\perp = 0$  and  $x_q = \beta_q$ , the interaction mass parameter  $M_\varphi$  may be estimated by inspecting the condition

$$M_q^2 > (M_q + M_G)^2 + \frac{3}{2}(M_q + M_G)\omega_h - M_\varphi^2. \tag{164}$$

Assuming that the resulting lower bound on  $M_\varphi$  provides an estimate of its likely magnitude in QCD, we obtain

$$M_\varphi^2 \sim M_q M_G \left( 2 + \frac{M_G}{M_q} + \frac{3\omega_h}{2\mu_h} \right), \tag{165}$$

where 3 signifies the number of spatial dimensions. Assuming that  $M_q$  equals nucleon mass  $m_N$ , and  $M_G \sim \omega_h \sim m_N/3$ , one obtains  $M_\varphi \sim 1.2 m_N$ . If  $M_G \sim M_q \sim m_N$  and  $\omega_h \sim m_N$ , we have  $M_\varphi \sim 2.4 m_N$ . This means that  $M_\varphi$  can be adjusted to obtain a hadron mass as coming mainly from the

quark mass eigenvalue  $M_q$  provided that  $M_\varphi$  is quite sizable. Such sizable energy benefit from gluon condensation eliminates a large contribution of  $G$  to a hadron mass and sustains the possibility that a simple oscillator quark model without any glue can reproduce masses of light hadrons assuming that the effective quark masses are on the order of  $\Lambda_{\text{QCD}}$ <sup>24</sup>.

We wish to stress that Eq. (159) does not imply one and the same interaction between quarks and glue  $G$  in all hadrons even if the parameters concerning  $G$  are assumed the same for all hadrons. Different hadrons are actually having different Hamiltonians characterized by different quantum numbers of their quark component, such as radial and orbital excitations (elementary spin effects are discussed in the next section) and the mass eigenvalue for the quarks,  $M_q$ , depends on these quantum numbers. This mass enters the definition of  $\vec{k}_h$  and thus also the definition of harmonic potential between the quarks and  $G$  that is defined in terms of  $\partial/\partial\vec{k}_h$ . As a result, potentials between quarks and  $G$  vary from hadron to hadron in a well-defined pattern: the higher excitation of quarks, the greater their mass and the more momentum of a hadron carried by quarks<sup>25</sup>. For example, quarks in excited nucleons are expected to carry a larger fraction of the resonance momentum than quarks carry in a ground-state nucleon. And vice versa, an excitation of the gluons condensed in a hadron increases  $M_G$  and the fraction of a hadron momentum they thus carry. Nevertheless, a CQM for light hadrons should be viewed as corresponding to  $n_h = 0$  when  $\lambda \gtrsim \lambda_c$  is actually lowered to  $\lambda_c$ . As suggested in Appendix D, when  $\lambda$  is lowered to  $\lambda_c$ , the role of  $G$  may be imagined taken over entirely by the content of extended constituent quarks that overlap each other heavily, so that no separate glue component can exist in hadron besides what is already contained in the extended effective quarks. Instead of the speculation, however, the proper problem is what will result from attempts to solve the RGPEP equation for the effective Hamiltonian at  $\lambda = \lambda_c$ . Strictly speaking, nothing is known currently about the solution.

## 6. RGPEP in QCD and phenomenology

The discussion that follows is limited to the case of all effective quarks having the same mass  $m$ . The common mass is expected to be a reasonable approximation for quarks  $u$  and  $d$  at  $\lambda \sim \lambda_c$ . Computational complications due to differences in mass between these quarks are not discussed.

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<sup>24</sup> If the condensation mass advantage,  $-M_\varphi^2$ , were associated also with condensation of quark-anti-quark pairs, one might expect a pronounced reduction in masses of the hadron states in which a bilinear effective quark field expectation value plays a significant role in the dynamics, with  $\pi$  mesons being the primary candidates.

<sup>25</sup> Provided that the first-approximation quantity  $M_\varphi$  is kept constant.

Assuming that  $\varphi_{\text{glue}}$  is on the order of  $\Lambda_{\text{QCD}}^2$ , which means that it has a similar value to the values obtained in QCD sum rules for  $\varphi_{\text{vacuum}}$ , one obtains phenomenologically attractive values of  $\omega_M$  and  $\omega_B$  [17]. Eqs. (112), (127), (154), (159), and (160), imply together eigenvalues and wave functions for light hadrons in the form of harmonic oscillator solutions that include the glue component  $G$ . The generic oscillator eigenvalue problem for two particles has the form

$$\left( \frac{\vec{p}^2}{2m} + \frac{1}{2}m\omega^2\vec{r}^2 \right) \psi = E\psi \quad (166)$$

with eigenvalues  $E_n = (n + 3/2)\omega$  and the ground-state wave function  $\psi_0 = N \exp[-\vec{p}^2/(2m\omega)]$  for  $n = 0$ , where  $\vec{p}$  and  $\vec{r}$  are canonically conjugated variables.

For two effective constituent quarks in mesons, using Eq. (116) that explains what happens in the LF eigenvalue equation for  $\mathcal{M}_{q\bar{q}}^2$  in terms of a NR approximation to  $\mathcal{M}_{q\bar{q}}$ , Eq. (115) implies the first LF approximation of the form

$$M_{q\bar{q}n}^2 = 4m^2 + 4(n + 3/2)m\omega_M, \quad (167)$$

$$\psi_{q\bar{q}0} = N_{q\bar{q}0} \exp\left[-\vec{k}^2/(m\omega_M)\right] \quad (168)$$

$$= N_{q\bar{q}0} \exp\left\{-\frac{1}{4m\omega_M} \left[ \frac{\kappa^{\perp 2} + m^2}{x(1-x)} - 4m^2 \right]\right\}, \quad (169)$$

with frequency  $\omega_M = \pi\varphi_{\text{glue}}/(3m)$  and wave functions of excited states,  $\psi_{q\bar{q}n}$ , generated by building harmonic oscillator excitation operators from the vector  $\vec{k}$  and gradient  $\partial/\partial\vec{k}$  in a standard way and applying them to the ground state wave function  $\psi_{q\bar{q}0}$ .

For three effective constituent quarks in baryons, using Eq. (145) that explains what happens in the LF eigenvalue equation for  $\mathcal{M}_{3q}^2$  in terms of a NR approximation to  $\mathcal{M}_{3q}$ , Eq. (144) implies the first LF approximation of the form

$$M_{3qn_1n_2}^3 = 9m^2 + 6(n_1 + n_2 + 3)m\omega_B, \quad (170)$$

$$\psi_{3q0} = N_{3q0} \exp\left\{-\frac{1}{6m\omega_B} \left[ \frac{9}{2}\vec{Q}^2 + 6\vec{K}^2 \right]\right\} \quad (171)$$

$$= N_{3q0} \exp\left\{-\frac{1}{6m\omega_B} \left[ \frac{(1-x_3)k^{\perp 2}}{x_1x_2} + \frac{q^{\perp 2}}{x_3(1-x_3)} + m^2 \left( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - 9 \right) \right]\right\}, \quad (172)$$

with baryon oscillator frequency  $\omega_B = \sqrt{5/8} \omega_M$  and wave functions of excited states,  $\psi_{3q n_1 n_2}$ , generated in a standard way by building harmonic oscillator excitation operators from the vectors  $\vec{K}$  and  $\vec{Q}$  and gradients  $\partial/\partial\vec{K}$  and  $\partial/\partial\vec{Q}$  and applying them to the ground state wave function  $\psi_{3q0}$ . The LF momentum variables are defined in Eqs. (118), (120), (121) and (122).

In summary, the factor in a hadron ground-state wave function that depends on the relative motion of  $n$  quarks has the form

$$\psi_{nq0} = N_{nq0} \exp \left\{ -\frac{1}{2nm\omega_n} \left[ \left( \sum_{i=1}^n p_i \right)^2 - (nm)^2 \right] \right\}, \quad (173)$$

with  $n = 2$  for mesons,  $n = 3$  for baryons,  $\omega_2 = \omega_M$ ,  $\omega_3 = \omega_B$ , and  $p_i$  the on-mass-shell four-momentum for quark number  $i$ .

Exponentials of an invariant mass squared of quarks are popular as wave functions in phenomenological studies. Here such exponentials are related to gauge symmetry and reinterpretation of the gluon condensate as a part of a hadron.

The spectrum of light hadron masses in Eq. (163), *i.e.*, the spectrum corresponding to the ground state of relative motion of quarks with respect to the glue  $G$ , becomes equal to the spectrum of masses of the quark component alone when the estimate (165) is adopted for the benefit of the gluon condensation in a hadron. This result reproduces success of the constituent quark models<sup>26</sup>.

At the same time, when the relative motion of quarks with respect to the glue part is described by Eq. (160), the hadron mass is given by Eq. (163) and the corresponding wave function is generated from the ground-state wave function

$$\psi_h = \psi_{nq0} \psi_{qG0}, \quad (174)$$

$$\psi_{qG0} = N_{qG0} \exp \left[ -\frac{(P_q + P_G)^2 - (M_q + M_G)^2}{2(M_q + M_G)\omega_h} \right], \quad (175)$$

in a standard way for harmonic oscillators, first for the quark component and then, using the quark component mass eigenvalue, for the whole hadron

<sup>26</sup> The oscillator functions of relative quark motion are independent of quark spins. A Coulomb part of the wave function must depend on spins. For small  $\lambda$ , effective quark spin wave function can be treated as a separate factor in the oscillator part of the wave function. Spin factors can be constructed in a boost-invariant way using LF spinors and following the example of heavy quarkonia [18] in the case of mesons or Ioffe currents [34] in the case of baryons. However, in contrast to models not related to QCD, RGPEP provides a scheme for systematic study of spin-dependent corrections to the first oscillator approximation.

if the quarks are excited in their motion with respect to  $G$ . Thus, the invariant mass squared of quarks and  $G$  together,  $(P_q + P_G)^2$  is evaluated in a standard way using their LF momenta and their minus components calculated using quark mass eigenvalue  $M_q$  and the glue mass  $M_G$ , respectively. Normalization factors are fixed by normalizing probability to 1, or fixing the charge to the appropriate value (if the charge is not zero these normalization conditions are the same).

*6.1. Form factors*

The relative motion of quarks with respect to the glue  $G$  smears the quark observables that follow from the quark factor in the hadron wave function alone. Consider the ground state of quarks' motion with respect to  $G$ . Calculation of the baryon form factor at a small momentum transfer  $q$ ,  $q^2 = -Q^2$ ,  $0 \leq Q^2 \leq \Lambda_{\text{QCD}}^2$ , is represented graphically in Fig. 1. Meson form factor calculation involves 2 instead of 3 quarks but otherwise proceeds in the same way.

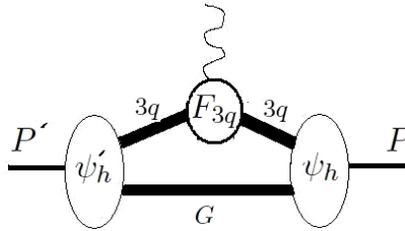


Fig. 1. The RGPEP calculation of a baryon form factor at  $\lambda \sim \lambda_c$ .

Using the Breit frame with  $q^+ = 0$ , which makes  $q^-$  that depends on masses irrelevant, one has  $Q^2 = q^{\perp 2}$  and

$$F_h(Q^2) = \int \frac{dx_q d^2\kappa_q^\perp}{16\pi^3 x_q (1-x_q)} \times \psi_{qG0}^* [x_q, \kappa_q + (1-x_q)q^\perp] F_{3q}(Q^2) \psi_{qG0}(x_q, \kappa_q), \tag{176}$$

with normalization condition

$$1 = \int \frac{dx_q d^2\kappa_q^\perp}{16\pi^3 x_q (1-x_q)} |\psi_{qG0}(x_q, \kappa_q)|^2. \tag{177}$$

Using the wave function in Eq. (174) and changing variables to  $x = x_q$  and  $\kappa^\perp = \kappa_q^\perp + (1-x)q^\perp/2$ , one obtains

$$F_h(Q^2) = F_{3q}(Q^2) f(Q), \tag{178}$$

$$f(Q) = \int \frac{dx d^2\kappa^\perp}{16\pi^3 x(1-x)} \left| \psi_{qG0}^*(x, \kappa^\perp) \right|^2 e^{-\frac{1-x}{x} \frac{Q^2}{4(M_q + M_G)\omega_h}}. \tag{179}$$

After integration over  $\kappa^\perp$ , one is left with

$$f(Q) = \frac{\int_0^1 dx e^{-a\left(\frac{M_q^2}{x} + \frac{M_G^2}{1-x} + \frac{1-x}{x} \frac{Q^2}{4}\right)}}{\int_0^1 dx e^{-a\left(\frac{M_q^2}{x} + \frac{M_G^2}{1-x}\right)}}, \quad (180)$$

where  $a^{-1} = (M_q + M_G)\omega_h$ . It is now visible that  $f(Q^2)$  is hardly different from 1 for small  $Q^2$ : the inclusion of the glue component  $G$  does not significantly alter the result for a form factor calculated as if the glue component was absent. Moreover, in the picture discussed in Appendix D, the glue component may disappear in favor of overlapping constituent quarks when  $\lambda$  is lowered below  $\lambda_c$ . The momentum fraction carried by the quarks,  $x = x_q$ , becomes very close to 1 and the factor  $f(Q^2)$  becomes 1<sup>27</sup>.

At large  $Q$ , much greater than a hadron mass, the hadron state needs to be represented in terms of the Fock components created by quark and gluon operators at  $\lambda$  comparable with  $Q$  itself, in order to obtain a simple picture based on the smallness of an asymptotically small coupling constant that determines the strength of interactions which are responsible for distributing the large momentum transfer  $q$  to a minimal set of constituents required to build a hadron. The minimal quark component may carry practically the whole momentum of a hadron,  $x_q \rightarrow 1$ . When this configuration dominates the transition amplitude, the whole hadron is turned from the total momentum  $P$  to  $P' = P + q$  in a combination of two mechanisms. One is  $x_q \rightarrow 1$ , and the other is the distribution of  $q$  in the quark component. The former contributes to soft effects, and the latter is responsible for hard exclusive processes, *cf.* [45].

## 6.2. Structure functions

The transition amplitude for deep inelastic lepton–hadron scattering is illustrated in Fig. 2<sup>28</sup>. A hard photon or other boson is suddenly absorbed by a constituent characterized by the scale  $Q$ . The new element RGPEP introduces is the possibility of using the transformation  $W_{Q\lambda}$  of Sec. 4 to calculate the probability amplitude for finding such constituent in a hadron.

<sup>27</sup> Even if the component  $G$  contained quark–anti-quark pairs, and were used to account for the quark condensate inside hadrons [20], its neutrality would imply a small contribution to form factors of charged hadrons, such as proton. It might, however, contribute a detectable piece for chargeless hadrons, such as neutron.

<sup>28</sup> The lepton is not shown. Also, Fig. 2 ignores the possibility that the impinging boson is absorbed by a constituent inside the component  $G$ , which may be thought here to be only made of gluons. If  $G$  included quark–anti-quark pairs, it would contribute to the sea parton distributions as well as the quark condensate in hadrons, *cf.* [20].

The scale  $\lambda$  refers to the particle operators in terms of which the hadron wave function is obtained from the eigenvalue problem. The scale  $Q$  refers to the hard boson. The final state in Fig. 2 is made of many constituents produced by  $W_{Q\lambda}$ . Fig. 2 does not show that  $W_{Q\lambda}$  actually acts on all constituents.

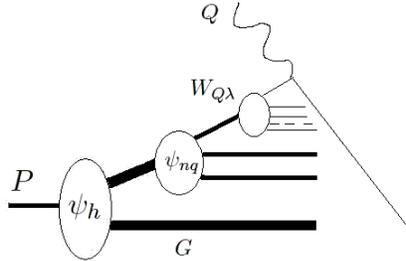


Fig. 2. The RGPEP calculation of a baryon structure function.

The final state in Fig. 2 is in fact a virtual state whose evolution factor into observable particles is assumed to amount to unity. The energy, or  $P^-$  of the constituents that appear in the final state in Fig. 2, is dominated by the constituent at scale  $Q$  that absorbs the boson. This situation corresponds to the leading operator expansion terms, *i.e.*, hand-bag diagrams that are more important than cat-ears diagrams. One can use the Hamiltonian  $H_Q(b_Q)$  to account for  $P^-$  of the virtual state. This Hamiltonian is actually the same as  $H_\lambda(b_\lambda)$ . Therefore,  $H_Q(Q)$  may count the energy of particles in the final state including the interactions that are responsible for grouping constituents at scale  $Q$  into spectators at scale  $\lambda$ . Thus, the inclusive sum over final states may be replaced by summing over a relatively small number of spectators at scale  $\lambda$  and a potentially large number of constituents at scales between  $\lambda$  and  $Q$ . The operator  $W_{Q\lambda}$  in RGPEP is hence expected to describe the evolution of structure functions with  $Q^2$ .

The above qualitative reasoning must be verified by new type of calculations using RGPEP evolution equations in place of other evolution equations in  $Q^2$  [54, 55, 56]. Since it is known that RGPEP incorporates the well-known splitting functions in QCD [29], there is no obvious reason for expecting that RGPEP evolution will significantly differ from known results where they apply. On the other hand, RGPEP provides the framework for combining the perturbative evolution with a non-perturbative hadron wave functions obtained by solving eigenvalue problem for  $H_\lambda(b_\lambda)$ . The eigenvalue problem describes saturation at small  $\lambda$ . But the evolution parameter  $\lambda$  determines dependence of wave functions on invariant masses of constituent states. The invariant masses depend on transverse momenta and fractions  $x$  of longitudinal momentum in a specific way. One can thus expect that the evolution in  $x$  [57, 58] is uniquely related in RGPEP with the evolution in  $Q^2$ .

### 6.3. Other processes

The approximate picture of a hadron at scale  $\lambda \gtrsim \lambda_c$  as built from the effective quark component and from  $G$  treated as a scalar boson suggests thinking that the dynamics of  $G$  can be further approximated by suitable effective Hamiltonian interaction terms. These terms would result in processes in which quanta of type  $G$  participate in strong interactions, softened by RGPEP form factors in vertices. Nothing can be said at this point about physical relevance of the approximate concept of quanta of the type  $G$  being exchanged between hadrons. In particular, it is not excluded that an exchange of such quanta may contribute to diffractive processes in hadron–hadron collisions [59], or even in photon–hadron scattering [60] if a photon were allowed to contain its own glue component coupled to quarks.

### 6.4. Connection with AdS/QCD

The NR form of minimal coupling in LF Hamiltonians at small  $\lambda$  apparently leads to variables and harmonic oscillator potentials that are similar to the ones that Brodsky and Teramond [24, 25, 26] identified as providing a correspondence between formulae for hadronic form factors in LF QCD and in field theory in 5-dimensional AdS space [61, 62]. The new variables  $\vec{k}$  for mesons, Eqs. (106) and (107), and  $\vec{K}$  and  $\vec{Q}$  for baryons, Eqs. (123), (124), (125), (126), differ from standard LF relative three-momenta of constituents in the CRF<sup>29</sup>. For example,  $\vec{k}_{\text{standard}}$  in Eq. (108) differs in length and direction from  $\vec{k}$ . But there is no difference between these momentum variables when they are small in comparison to masses. Thus, the new variables are identical to standard variables in the NR dynamics at  $\lambda \gtrsim \lambda_c$ . In addition, they fully describe a relativistic relative motion of constituents, always providing an exact representation of their free invariant mass, but in a different way than the standard momentum variables do. Standard variables  $k_{\text{standard}}^z$ ,  $K_{\text{standard}}^z$ , and  $Q_{\text{standard}}^z$ <sup>30</sup>, depend on the transverse relative momenta. The new variables  $k^z$ ,  $K^z$ , and  $Q^z$ , are defined using the constituent masses and fractions  $x$  of their total  $P^+$ , independently of the transverse relative momenta. At the same time, the transverse relative momenta are scaled by mass ratios and  $\sqrt{x(1-x)}$ . The square-root factor is precisely the one that appears in Brodsky–Teramond holography [26].

To illustrate the possibility of correspondence between the reinterpreted gluon-condensate induced harmonic potential in LF QCD and a soft-wall potential in Brodsky–Teramond AdS/QCD holography, we use example of

<sup>29</sup> In LF dynamics, CRF differs from the bound state LF rest frame because conservation of  $P^+$  implies that  $P^3$  is not conserved by interaction terms in LF Hamiltonians.

<sup>30</sup> *E.g.*, see Ref. [34].

a meson form factor. The discussion is far from complete and raises many questions regarding interpretation of simple equations. However, it is needed for showing similarities and differences among different approaches.

Let the struck quark and the spectator quark carry nearly whole hadron momentum. The glue component  $G$  provides additional smearing in the form factor expression, Eq. (176), unless it is totally absorbed in the constituent quarks<sup>31</sup>. Let us assume here that the latter option holds and all one needs to consider is two constituent quarks.

In terms of the new variables,

$$k^\perp = \kappa^\perp / \sqrt{4x(1-x)}, \tag{181}$$

$$k^z = (x - 1/2) 2m / \sqrt{4x(1-x)}, \tag{182}$$

the LF eigenvalue equation for a meson of mass  $M$  reads

$$\left( 4m^2 + 4\vec{k}^2 + \kappa^4 \vec{r}^2 \right) \psi = M^2 \psi, \tag{183}$$

where  $\kappa^2 = m\omega_M$  and  $\vec{r} = i\partial/\partial\vec{k}$ . Using a factorized wave function,  $\psi(k^\perp, k^z) = N\phi(k^\perp) f_\eta(x)$ , where  $f_\eta$  is an eigenfunction of the oscillator in  $z$ -direction,

$$f_\eta(x) = H_\eta(k_z) e^{-\frac{k_z^2}{\kappa^2}} = H_\eta \left[ \frac{m(x-1/2)}{\kappa\sqrt{x(1-x)}} \right] e^{-\frac{m^2(x-1/2)^2}{\kappa^2 x(1-x)}}, \tag{184}$$

one arrives at the eigenvalue equation for  $\phi(k^\perp)$ ,

$$\left[ 4m^2 + 4k^{\perp 2} + \kappa^4 r^{\perp 2} + (4\eta + 2)\kappa^2 \right] \phi = M^2 \phi, \tag{185}$$

which is the eigenvalue problem to compare with the Brodsky–Teramond holographic eigenvalue problem, *e.g.*, Eq. (10) in [26]. We denote quantities used by Brodsky and Teramond with subscript ‘BT’, except for their  $\zeta$ ,

$$k^\perp = k_{\text{BT}}^\perp / 2, \tag{186}$$

$$r^\perp = 2\zeta^\perp. \tag{187}$$

Eq. (185) can be written using  $\kappa_{\text{BT}} = \sqrt{2}\kappa$  as

$$\left( k_{\text{BT}}^{\perp 2} + \kappa_{\text{BT}}^4 \zeta^{\perp 2} \right) \phi = M_{\text{BT}}^2 \phi, \tag{188}$$

$$M_{\text{BT}}^2 = M^2 - 4m^2 - (2\eta + 1)\kappa_{\text{BT}}^2, \tag{189}$$

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<sup>31</sup> See Appendix D.

where  $k_{\text{BT}}^{\perp 2} = -(\partial/\partial\zeta^{\perp})^2$ . For angular momentum around  $z$ -axis equal  $l_z$ , using  $\zeta = |\zeta^{\perp}|$ , one has

$$\left[ -\frac{\partial^2}{\partial\zeta^2} - \frac{1}{\zeta} \frac{\partial}{\partial\zeta} + \frac{l_z^2}{\zeta^2} + \kappa_{\text{BT}}^4 \zeta^2 \right] \phi = M_{\text{BT}}^2 \phi. \tag{190}$$

Defining  $\phi = \phi_{\text{BT}}/\sqrt{\zeta}$ , we arrive at

$$\left[ -\frac{\partial^2}{\partial\zeta^2} - \frac{1-4l_z^2}{4\zeta^2} + \kappa_{\text{BT}}^4 \zeta^2 + (2\eta+1)\kappa_{\text{BT}}^2 \right] \phi_{\text{BT}} = (M^2 - 4m^2) \phi_{\text{BT}}, \tag{191}$$

which by comparison with Eq. (11) in [26] for massless quarks renders

$$U(\zeta) = \kappa_{\text{BT}}^4 \zeta^2 + (2\eta+1)\kappa_{\text{BT}}^2, \tag{192}$$

as a counterpart of the potential  $U(\zeta)$  in Ref. [26], with

$$\kappa_{\text{BT}}^2 = 2m\omega_M = \frac{2\pi}{3} \varphi_{\text{glue}}. \tag{193}$$

The role of quark masses on the right-hand side of Eq. (191) requires explanation. A sound explanation is currently not available in the sense that it is not clear why the quark mass is set to 0 in Eqs. (10) and (11) in Ref. [26]. But it is plausible for constituent quarks that the condensate mass-advantage constant  $M_{\varphi}$  in Eq. (163) can be considered a result of incorporation of  $G$  in the constituent quarks entirely.  $M_{\varphi}$  may be so large that it reduces the contribution of the quark masses to the hadron mass and thus reduces the term  $-4m^2$  in the eigenvalue.

On the other hand, the SW model eigenvalue Eq. (12) in Ref. [27] reads

$$-\psi'' + \left[ z^2 + \frac{\lambda_z^2 - 1/4}{z^2} \right] \psi = E\psi, \tag{194}$$

$$E = 4n_z + 2\lambda_z + 2. \tag{195}$$

By dividing Eq. (191) by  $\kappa_{\text{BT}}^2$ , and using variable  $z = \kappa_{\text{BT}}\zeta$ , one obtains

$$-\phi_{\text{BT}}'' + \left[ W(z) + \frac{l_z^2 - 1/4}{z^2} \right] \phi_{\text{BT}} = E \phi_{\text{BT}}, \tag{196}$$

$$E = \frac{M^2 - 4m^2 - (2\eta+1)\kappa_{\text{BT}}^2}{\kappa_{\text{BT}}^2}. \tag{197}$$

Comparing this result with the SW model result, Eq. (194), and identifying  $\lambda_z$  with  $l_z$ , one obtains

$$W(z) = z^2. \quad (198)$$

The SW model quadratic potential appears to have a coefficient given by the gluon condensate inside hadrons, Eq. (193).

If  $\eta = 0$ , which means no excitation along the LF, only  $-\kappa_{\text{BT}}^2$  enters the eigenvalues. This is a sizable term (see below) and its compensation is not guaranteed by  $M_\varphi$ .

Baryon form factors are described in terms of an active constituent and spectators. The analysis proceeds in a similar way to the case of a meson built from two constituents of different masses. This simplification is available because the free LF invariant mass can be written in terms of the active constituent and the rest of a hadron as if the rest of a hadron was a single particle with its mass equal to the invariant mass of the rest of a hadron. The closest analogy to consider is that a baryon form factor and eigenvalue problem can be represented in terms of a quark and a di-quark.

It should be stressed, however, that it is far too early for drawing a firm conclusion regarding connection between gluon condensation in hadrons and SW models on the basis of a pure RGPEP calculation. The reason can be seen using power counting [16]. Besides uncertainties concerning  $x^-$  and inverse of  $i\partial^+$ , spin-independent quark–anti-quark interaction term in a LF Hamiltonian density with a quadratic potential, which is absent in canonical QCD, may have a coefficient proportional to  $\Lambda_{\text{QCD}}^2$ , to compensate the dimension of  $x^{\perp 2}$ . There also exists a possibility [31] that a dimensionless square of a product  $r_\mu P^\mu$  is used instead of  $x^{\perp 2}$ , where  $r$  is the relative position of arguments of two field operators on the LF,  $r^+ = 0$ , and  $P$  is the momentum carried by a product of two quark fields, *i.e.*, two constituents interacting by the potential. This possibility implies that the harmonic potential does not necessarily requires a factor  $\Lambda_{\text{QCD}}^2$  because a gradient cancels dimension of a distance. Another  $\Lambda_{\text{QCD}}^2$  may be needed to cancel the dimension of additional integration measure  $d^2x^\perp$ . Still, the potential term must vanish when  $\lambda \rightarrow \infty$ . If it vanishes as  $1/\lambda^2$ , another  $\Lambda_{\text{QCD}}^2$  may be expected in the coefficient. There always exists a possibility to use quark masses as dimensionfull quantities. Together, one may need powers of  $\Lambda_{\text{QCD}}$  as high as 6. Such terms may be hard to establish quickly in any renormalization group procedure.

In any case, the above results certainly allow one to entertain the possibility that the small- $\lambda$  RGPEP picture of hadrons, at strong coupling, can actually be the one that corresponds to the AdS picture, including the SW model [27]. On the other hand, even if the SW model does correspond to the Brodsky–Teramond LF holographic picture with a harmonic oscillator

potential between an active constituent and spectators (in transverse directions), and even if these pictures can be related to the gluon condensation in hadrons, how is RGPEP to explain emergence of the 5-th dimension in the AdS/QCD analogy on the QCD side?

It has been pointed out that the 5-th dimension in AdS may correspond to a renormalization group scale in gauge theory [63, 64]. More recent calculations, such as [65, 66], also point in this direction. We observe in this context that RGPEP provides an opportunity to explicitly introduce a 5-th dimension in QCD with a simple physical interpretation. Namely, Eqs. (39) and (40) for effective fields can be rewritten using the observation that  $\lambda$  corresponds to the inverse size of effective particles. The effective particles exhibit this size in their interactions; the effective interactions contain vertex form factors whose momentum width is determined by the RGPEP parameter  $\lambda$ . It is convenient to think about the size of the effective particles using variable  $s = 1/\lambda$ <sup>32</sup>.

So, instead of labeling the field and particle operators with  $\lambda$ , we can label them with  $s$  that corresponds to the size of effective particles. The effective fields on the LF can be rewritten according to a rule

$$\psi_s(x) = \int [k] b_s(k) e^{ikx} = \psi(x, s). \quad (199)$$

In this rule, the 5-th argument of the quantum field is the RGPEP size of effective particles. This size plays the role of a renormalization group parameter in RGPEP.

While the AdS dual picture involves its own renormalization issues [67, 68, 69, 70], and RGPEP methodology may eventually find applications in these issues, too, the 5-th dimension of particle size makes QCD *a priori* a 5-dimensional theory, with scale invariance broken by  $\Lambda_{\text{QCD}}$ . The structure of such a theory may or may not be similar to an AdS field theory. On the other hand, RGPEP is certainly available for studying the 5-dimensional QCD.

The issue that emerges immediately is that in order to evaluate observables in renormalized Hamiltonian formulation of LF QCD one in principle can use just one value of the RGPEP parameter  $s$ , or  $\lambda$ . This means that there is no need for integration over the renormalization group parameter in QCD. Instead, to evaluate observables, one can use a Hamiltonian that corresponds to an arbitrarily chosen value of  $s$ , or  $\lambda$ . So, if the RGPEP parameter were to correspond to the 5-th dimension in the AdS picture, what would it mean in LF QCD that one integrates over this parameter?

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<sup>32</sup> The particle size parameter  $s$  enters the non-perturbative RGPEP equations through replacement of the derivative  $d/d\lambda^{-4}$  on the left-hand side of Eq. (C.1) by  $d/ds^4$ .

The Brodsky–Teramond LF holography for hadronic form factors suggests that the integration over the 5-th dimension in AdS field theory corresponds to integration over relative transverse momentum of an active hadronic constituent with respect to spectators. But RGPEP suggests that the LF wave functions have the form shown in Eq. (174), which is Gaussian in invariant mass of the hadronic constituents. Integration over the transverse relative momenta in form factors amounts to integration over invariant mass, with a proper rescaling by  $\sqrt{x(1-x)}$ . The invariant mass appears in ratio to a square of the width parameter  $\lambda$  or, equivalently, in product with the square of the effective particle size parameter  $s = 1/\lambda$ . The challenge of understanding integration over the AdS 5-th dimension using RGPEP is to explain how the integration over  $\mathcal{M}$  can be turned into integration over  $\lambda$  or  $s = 1/\lambda$  in renormalized LF QCD.

Although we do not provide here the ultimate response to this challenge, we offer a qualitative argument that suggests where one can look for the solution in future studies. Consider a fixed scale of an observable, such as  $Q$  in a form factor. For a given  $\lambda = 1/s$ , one can evaluate the observable in QCD by integrating wave functions over  $\mathcal{M}$ . But if the wave functions are functions of  $\mathcal{M}/\lambda = s\mathcal{M}$ , they are constant on hyperbolas defined by the condition that  $s\mathcal{M}$  is fixed. One can consider a plane of variables  $\mathcal{M}$  and  $s$  and plot a wave function in a perpendicular direction, forming a 3-dimensional plot of the wave function. Select a value of  $s = 1/Q$  and draw a profile of the 3-dimensional plot for the selected value of  $s$  as a function of  $\mathcal{M}$ . Draw another profile as a function of  $s$  for  $\mathcal{M} = Q$ . These two profiles are identical for all wave functions that only depend on the variables  $\mathcal{M}$  and  $s$  through their product  $t$ . In such circumstances, integration over relative motion of quarks of fixed size  $Q$  is equivalent to integration over all sizes  $s$  of quarks that have fixed invariant mass  $Q$ . In QCD, this pure scaling picture for lightest quarks must be broken by the scale set through  $\Lambda_{\text{QCD}}$ .

The small- $\lambda$ , or large- $s$  picture of hadrons in RGPEP appears in accordance with expectations that there exists a low-energy regime with a stable strong coupling constant [71]<sup>33</sup>. In the stabilization region, the gluon condensation parameter  $\varphi_{\text{glue}}$  inside hadrons is considered constant as a function of  $\lambda$ . The hadronic wave functions are Gaussian in the invariant mass with the width provided by quark masses and  $\varphi_{\text{glue}}$ . The overall result is that at small  $\lambda$  the effective parameters are frozen at values comparable with  $\Lambda_{\text{QCD}}$ . On the other hand, when  $\lambda$  increases, and  $s$  decreases, the Hamiltonian changes and it is expected to eventually simplify to the QCD canonical form with counterterms at  $\lambda \rightarrow \infty$ , *i.e.*, for point-like quarks and gluons,  $s \rightarrow 0$ . Therefore, the wave functions of hadronic eigensolutions change

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<sup>33</sup> RGPEP may be used to seek a precise definition of such coupling constant using quark–gluon, three-gluon, and inter-quark potential terms in the corresponding Hamiltonians, *cf.* [42, 29].

with  $\lambda$ , or  $s$ . One may expect that when the wave function width in momentum space increases and the sizes of effective quarks and gluons decrease, the invariant mass dependence turns from Gaussian to a power-law behavior at large virtuality, perhaps as indicated by perturbation theory for exclusive processes in a collinear approximation [45].

Finally, regarding the Brodsky–Teramond holography and SW models, we wish to mention that one considers different values of the SW parameter  $\kappa^4$  in front of  $\zeta^2$  in the AdS equations for masses of mesons ( $M$ ) and baryons ( $B$ ) [72]. Namely,  $\kappa_M \sim 0.54\text{--}0.59$  GeV and  $\kappa_B \sim 0.49$  GeV, with

$$\frac{\kappa_M}{\kappa_B} \sim 1.15 \pm 0.5. \quad (200)$$

In the reinterpreted gluon condensate image for hadrons, assuming a universal value of mass for  $u$  and  $d$  constituent quarks, one has

$$\frac{\kappa_M}{\kappa_B} = \left(\frac{8}{5}\right)^{1/4} \sim 1.125. \quad (201)$$

This agrees quite well with the phenomenological result.

## 7. Conclusion

Reinterpretation of the gluon condensate in RGPEP has several implications that matter in theory of hadrons. Instead of the entire vacuum, the gluons condense only inside hadrons. If this option were actually realized in QCD, there would be no need anymore to construct the quantum vacuum state in Minkowski space that satisfies fundamental requirements of invariance with respect to Poincaré transformations, a construction that so far eluded theoreticians. At the same time, the LF Hamiltonian formulation of QCD would become free from any obligation to produce a non-trivial vacuum that is commonly assumed to provide physically important expectation values. When reinterpreted, the same expectation values come from distribution of matter that is limited to the interior of hadrons. Simultaneously, it becomes not clear how one should interpret expectation values that are measured in lattice formulation of gauge theories; they might also correspond to the hadron interior rather than a state of infinite volume. In any case, the reinterpretation is available within the RGPEP that provides tools for a close inspection what actually happens. Thus, the picture of hadrons that is sketched here in the context of reinterpretation of gluon condensate, provides an idea about how the CQM may emerge as an approximate solution to QCD when one applies RGPEP to it and, simultaneously, the RGPEP itself is developed here as a method to the extent that is sufficient for undertaking attempts at verifying this idea numerically.

The constituent picture based on gauge symmetry dictates Gaussian LF wave functions that are exponentials of free invariant masses of constituents. The associated oscillator frequencies are in agreement with quark models phenomenology when the gluon condensate inside hadrons is set to the value found in phenomenology using QCD sum rules. The corresponding eigenvalue problems appear naturally in terms of variables in which AdS/QCD models are developed, especially the SW model. The ratio of baryon and meson oscillator frequencies in SW models agrees with the ratio implied by the reinterpreted gluon condensate in RGPEP.

This article provides no comparable evidence for reinterpretation of the quark condensate. But one may hope that RGPEP can generate constituent quark masses when  $\lambda$  is lowered to values comparable with  $\Lambda_{\text{QCD}}$  and shed this way some light on the mechanism of absence of chiral symmetry in hadronic spectrum and the nature of  $\pi$ -mesons as nearly Goldstone particles.

The author's opinion that RGPEP provides a method for verifying the reinterpretation of the gluon condensate and solving for hadronic structure in QCD is mainly based on the fact that RGPEP is now available in both perturbative and non-perturbative versions, both being boost invariant. The step beyond perturbation theory maintaining boost invariance is seen as a chance for simultaneously taking advantage of knowledge about hadrons in the constituent picture and in the parton picture, while RGPEP appears capable of generating required scale dependence dynamically. Seen this way, RGPEP provides an operator calculus for deriving quark and gluon wave functions of hadrons in the LF Fock space and unify calculations of the hadron mass spectrum and partonic distributions inside hadrons. At the same time, new variables identified in the reinterpretation of the gluon condensate, relative momentum three-vectors  $\vec{k}$  in mesons and  $\vec{K}$  and  $\vec{Q}$  in baryons, provide a frame of reference in which LF solutions can be classified in terms of well-known angular momentum classification of constituent wave functions. The same transverse variables occur in LF AdS/QCD holography. Longitudinal variables are new and require further testing.

A summary of RGPEP program for verifying the reinterpretation of gluon condensate is following. Start with canonical QCD. Set up regulated  $H_{\text{QCD}}$  on the LF. Apply RGPEP to lower the scale parameter  $\lambda$  toward  $\Lambda_{\text{QCD}}$ . Find counterterms in  $H_{\text{QCD}}$  and evaluate effective  $H_\lambda$ . Solve eigenvalue problem of  $H_\lambda$  and see if the gluon-condensate induced oscillator picture is reproduced by gluon components in hadronic states. So far, the RGPEP scheme is known to work only in some crudely approximate calculations in QCD for heavy quarkonia. Studies of hadrons built in QCD from light quarks require breaking an entirely new ground.

## Appendix A

### *Universality in RGPEP*

This appendix explains the RGPEP scheme in which CQMs can be sought as limited representatives of the universality class which QCD is thought to belong to as a quantum field theory with a Hamiltonian acting in a Hilbert space, *cf.* [8]. The scheme is composed of two interrelated parts.

In the first part of RGPEP, one constructs a sequence of rotations  $U_\lambda$  for creation and annihilation operators in order to discover counterterms in  $H_\infty$  and obtain the effective Hamiltonian  $H_\lambda$ . The second part involves solving the eigenvalue problem of  $H_\lambda$ . Solutions to the eigenvalue problem of  $H_\lambda$  are needed to fix finite parts of counterterms in  $H_\infty$  [73,74].

The first part of the RGPEP calculation involves dealing with ultraviolet divergences. Instead of standard renormalization group procedure based on Gaussian elimination in linear problems [75,76], RGPEP is based on a unitary transformation of field or particle operators. In terms of matrix elements of a Hamiltonian, this means that one rotates the basis states like in the similarity renormalization group approach (SRG) [73,74].

Formulation of RGPEP in terms of operators for effective particles organizes otherwise un-intelligible variety of Hamiltonian matrix elements in a physically motivated way; one traces coefficients of specific operators instead of only calculating matrix elements to which many operators may contribute. It also provides the concept of effective quantum field; see Eqs. (39), (40), and below. The operator structure of RGPEP results in unitarity features in the calculated interactions that are absent in the general formulation of SRG, since SRG applies also to Hamiltonians that cannot be written in terms of creation and annihilation operators. For example,  $U_\lambda(b_\infty) = U_\lambda(b_\lambda)$ . The operator structure is also helpful in demonstrating a connection between the momentum space formulation of the theory in terms of quanta (effective particles) and position space formulation in terms of quantum fields (effective fields). The interaction terms in effective Hamiltonian densities are non-local. RGPEP provides a method for calculating the effective non-local interactions that correspond to renormalized canonical theories [31].

In contrast to the first part of the RGPEP scheme that is akin to the SRG scheme, the second part resembles the standard Wilsonian procedure because it involves Gaussian elimination, in the mathematical sense as part of solving a linear problem. However, in sharp distinction from the standard procedure, the second step of RGPEP does not involve ultraviolet divergences. These are eliminated in the first step.

The RGPEP differential equation for  $H_\lambda$  in its generic form reads

$$\frac{d}{d\lambda} \mathcal{H}_\lambda = \mathcal{F}_\lambda [\mathcal{H}_\lambda], \quad (\text{A.1})$$

where  $\mathcal{H}_\lambda$  denotes  $H_\lambda(b_\infty)$  and  $\mathcal{F}_\lambda$  is a suitable functional<sup>34</sup>. The solution is thus also of the generic form

$$\mathcal{H}_\lambda = \mathcal{H}_\infty + \int_\infty^\lambda d\lambda' \mathcal{F}_{\lambda'} [\mathcal{H}_{\lambda'}]. \quad (\text{A.2})$$

$H_\lambda(b_\lambda)$  is obtained by replacing  $b_\infty$  in  $\mathcal{H}_\lambda$  by  $b_\lambda$ .

Irrespective of the theory one considers, for  $\lambda$  much greater than the regularization parameter  $\Delta^2/\epsilon$  (see Sec. 2) times the largest allowed number of creation or annihilation operators in any product of them in the initial interaction Hamiltonian,  $\mathcal{F}_\lambda[\mathcal{H}_\lambda]$  is exponentially close to zero. The reason is that the regulated interactions are exponentially close to zero when the invariant mass of interacting particles is greater than allowed by the regularization.

So, one can always introduce some  $\Lambda \gg \Delta/\sqrt{\epsilon}$  and arrange a series of cutoffs  $\lambda_0 = \Lambda$ ,  $\lambda_1 = \Lambda/2$ ,  $\lambda_2 = \Lambda/4$ , etc., until one reaches  $\lambda = \lambda_n = \Lambda/2^n$  for some large  $n$ <sup>35</sup>. This is done in analogy to Refs. [75,76] for the purpose of sequencing the RGPEP procedure of evaluating  $H_\lambda$  into manageable steps that are free from huge terms resulting from ultraviolet divergences of the local theory. Namely, step number  $k$  in the sequence that builds up the integral in Eq. (A.2), contains only integration over scales between  $\lambda_{k+1}$  to  $\lambda_k$ . When one evaluates effects of interactions that allow changes of invariant masses only in the range between  $\lambda_{k+1}$  and  $\lambda_k$ , the ratio  $\lambda_0/\lambda_n \rightarrow \infty$  does not contribute. Moreover, one can carry out each and every one of the finite steps using well-known techniques, such as perturbation theory or numerical methods.

Although the sequence one obtains is analogous to the sequence considered in Wilsonian renormalization group procedure [76], it differs considerably because no states are eliminated from the domain of the Hamiltonian.

<sup>34</sup> *E.g.*, see [77]. Appendix C provides a non-perturbative definition of  $\mathcal{F}_\lambda[\mathcal{H}_\lambda]$ . One needs to adjust the dimension of parameter  $\lambda$  to dimensions of the Hamiltonian  $\mathcal{H}_\lambda$  and functional  $\mathcal{F}_\lambda[\mathcal{H}_\lambda]$ . The parameter  $\lambda$  is defined in terms of a momentum scale that plays the role of width in momentum-space vertex form factors, denoted also by  $\lambda$ , or by the inverse of the width. The inverse of  $\lambda$  corresponds to the size of effective particles that they exhibit in the interactions. The inverse is denoted by letter  $s$  that refers to the word *size*.

<sup>35</sup> Instead of the factor 2, one can choose  $e$  or any other similar number. The exponential spacing of  $\lambda$ s in the sequence of cutoffs is introduced for the purpose of handling logarithmic singularities. In the case of simple models in which one can easily introduce an exponential grid in momentum space, such as the NR Schrödinger equation for one particle in a  $\delta$ -function potential, it is most convenient to use a grid in which the chosen factor for cutoff reduction, similar to 2 or  $e$  above, is also equal to an integer power of the constant used in the grid, *i.e.*, if  $p_k = p_0 a^k$  in the grid, then  $\Lambda_n/\Lambda_{n+1} = a^l$  with an integer  $l$  such that  $a^l$  equals 2,  $e$ , or some other convenient factor chosen for the cutoff reduction. This relates the value of  $a$  chosen for the exponential grid in momentum variables with the RGPEP cutoff reduction factor in terms of an integer  $l$ .

The RGPEP steps are automatically free from the dependence on eigenvalues that appear in Gaussian elimination in matrix notation for eigenvalue problems and there is no need for an additional (and in principle determined only up to a nearly arbitrary rotation of basis) procedure required for maintaining formal hermiticity of effective Hamiltonians<sup>36</sup>.

The ratio of subsequent parameters,  $\lambda_{k+1}/\lambda_k$  does not depend on  $k$  and one can seek regularity in how successive integrations transform  $H_{\lambda_k}$  to  $H_{\lambda_{k+1}}$ . Every element in the resulting sequence of Hamiltonians can be written in terms of dimensionless momentum variables  $y^\perp = p^\perp/\lambda_k$  instead of momenta  $p^\perp$ . Creation and annihilation operators also require rescaling in order to keep their commutation relations in terms of variables  $y^\perp$  independent of the step number  $k$ . In principle, such rescaling corresponds to scale-dependent renormalization constants for fields in standard perturbative formulation of quantum field theory. The coefficients  $c_{\lambda_k}$  of products of creation and annihilation operators  $b_{\lambda_k}$  in  $H_{\lambda_k}$  can be studied as functions of  $y^\perp$  and  $x$ . These functions evolve in  $k$ . Their evolution can be classified in terms of characteristic dominant behavior associated with universality classes, presumably associated with fixed points, limit cycles, and even chaotic behavior. Such analysis is also expected to help in identifying dominant effects due to small- $x$  cutoff parameter  $\delta$  in gauge theories quantized on the LF (see Sec. 2). No farther comments are offered here regarding the evaluation of  $H_\lambda$ , except for stressing that the establishment of finite parts of ultraviolet counterterms eventually requires a reference to the second part of the RGPEP scheme.

The second part involves solving for the spectrum of  $H_{\lambda_n}$ . This is facilitated using techniques similar to the ones used in Ref. [78] with the following qualitative distinction. The Hamiltonian  $H_{\lambda_n} = H_{0\lambda_n} + H_{I\lambda_n}$  is so narrow on energy (invariant mass) scale (defined by eigenvalues of  $H_{0\lambda_n}$ ), that the wave functions of its eigenstates are typically exponentially suppressed outside the range of energies (invariant masses) that are similar in size to the observable energies (invariant masses) one is interested in [79]. Therefore, one is no longer in need to solve the severe ultraviolet renormalization problem when carrying out calculation that removes residual (exponentially small) cutoff effects using a procedure such as in [78]. This suggests, for example, that the non-linear operators that produce corrections to scaling in Wegner's sense [80] only lead to small corrections to the scaling properties (if any are obtained) that result already from the first step of the RGPEP scheme.

The expectation of lack of significant sensitivity of the second part of RGPEP scheme to the actual values of smallest eigenvalues can be forecast by analogy with (and it is confirmed in) an elementary model of a harmonic oscillator with an additional potential proportional to the fourth power of

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<sup>36</sup> In Ref. [76], the operation chosen to maintain Hermiticity is denoted by  $R$ , Eq. (4.1).

distance (a Higgs-like field dynamics in zero dimensions) [81]. Excited states of the pure oscillator (excited by multiples of  $\hbar\omega$ ) serve as a model of the Fock space basis with different numbers of effective particles, each of mass  $m = \hbar\omega$ . The Hamiltonian with the quartic term models  $H_{\lambda_n}$  after the entire first part of RGPEP scheme is completed. Namely, the interaction is narrow on the energy scale because it is able to change the number of particles only by 0, 2, or 4, which means that the size of energy changes due to interactions is limited from above by  $4\hbar\omega$ , or  $4m$ . The result of numerical computation of eigenstates in that model [81] is that for obtaining smallest eigenvalues accurately it is sufficient to use Gaussian elimination assuming that the smallest eigenvalues are simply zero. One can come down this way to cutoffs on energy of basis states not much greater than about 10 times  $\hbar\omega$ , or  $10m$ . This holds even for quite large values of the coupling constant that characterizes the size of the quartic coupling.

Since the entire RGPEP procedure is in fact a sequence of successive approximations with an increasing space of variables when regularization is lifted, both parts of it, the rotation and solution, need to be iterated, in the sense of Wilsonian triangle of renormalization, until a stable result is established. The iteration involves finite parts of counterterms. They are fixed by comparison with data in the solution part. But the finite parts influence the rotation in the first part. This is why the two parts of the procedure are interrelated and neither the part using unitary rotation of particle operators nor the solution part with elimination of states can be distinguished as determining the path of successive approximations for the Hamiltonian on the triangle. As a result, the evolution of a Hamiltonian on the triangle is neither purely unitary nor a plain solving of an *a priori* specified eigenvalue problem. In other words, the ultimate self-consistent solution to the triangle of renormalization defines the theory one is interested in solving to explain observables.

RGPEP requires extensive studies in order to verify if it can produce a universal low-energy theory that explains success of the CQMs starting from the Lagrangian for QCD. In addition, all of these models could also be subjected to RGPEP or SRG as long as they can be written in a Hamiltonian form<sup>37</sup>. Therefore, there arises a question if all CQMs that do reproduce a considerable number of observables for smallest-mass states in hadronic spectrum actually converge on a single, universal model with some specific shape of quark potential. Then, the question would be if a universal model, if it is obtained, matches the effective theory obtained from QCD using RGPEP.

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<sup>37</sup> This is not immediately obvious in the case of approaches based on the Dyson–Schwinger and Bethe–Salpeter equations.

It should be stressed, however, that the procedure outlined here is not limited to RGPEP for quarks and gluons in LF QCD. It can be applied to studies of universality in all Hamiltonians that can be expressed in terms of creation and annihilation operators acting in some Fock space and, even more generally, to all Hamiltonians that require renormalization and can be subject to the SRG procedure [73,74]. In particular, one can apply the same procedure when using beautiful Wegner's flow equation [82,83,84,85], and the latter can also be suitably altered in order to improve weak-coupling expansion that may apply in the case of asymptotically free theories [49,50].

## Appendix B

### *Calculations of $W$*

Assuming that  $\Lambda_{\text{QCD}}$  formally tends to 0 in comparison to quark masses, so that  $g_\lambda$  can be treated as extremely small when  $\lambda$  is comparable with the masses,  $W$  could be calculated in RGPEP using perturbative formulae given in Sec. 4.3 and expanding LF quantum fields into bare creation and annihilation operators. In this case, the dominant term besides 1 is  $g_\lambda W_1$ . However, for a realistic value of  $\Lambda_{\text{QCD}}$ , which is much larger than  $u$  and  $d$  quark mass parameters in the SM, and for  $\lambda \sim \lambda_c$  that is expected to correspond to the CQMs, perturbative expression cannot be considered reliable on the basis of smallness of  $g_\lambda$ . Nevertheless, perturbative calculations described here illustrate the structure of  $W$  that carries over to values of  $\lambda$  comparable with  $\lambda_c$  for as long as  $\mathcal{H}_\infty$  as an initial condition in RGPEP equations is replaced by the corresponding  $\mathcal{H}_\lambda$  and  $\lambda$  is sufficiently near in magnitude to  $\lambda_c$  so that  $W$  is not violently different from 1. The key argument is that the strength of the interaction is not fully determined by  $g_\lambda$  alone. Namely, when the RGPEP form factor  $f_\lambda$  is narrow as function of momentum variables, the net strength of the interaction is suppressed and the interaction can be weak even if  $g_\lambda$  is sizable. Thus, the calculation given below illustrates not only how the lowest-order perturbative  $W$  looks like in the case of small  $g_\lambda$  but also how the structure of  $W$  is related to the structure of  $\mathcal{H}_\lambda$  that is weak because  $\lambda$  is small, even though  $\mathcal{H}_\lambda$  is expected to significantly differ from  $\mathcal{H}_\infty$  when  $\lambda$  approaches  $\lambda_c \sim \Lambda_{\text{QCD}}$  for light quarks.

For the canonical quark field, we have on the LF

$$\psi = \sum_{\sigma cf} \int [k] \left[ \chi_c u_{fk\sigma} b_{k\sigma cf} e^{-ikx} + \chi_c v_{fk\sigma} d_{k\sigma cf}^\dagger e^{ikx} \right]_{x^+=0}. \quad (\text{B.1})$$

The momentum and spinor notation involves (*e.g.*, see Ref. [18])

$$[k] = \theta(k^+) dk^+ d^2 k^\perp / (16\pi^3 k^+), \quad (\text{B.2})$$

$$u_{fk\sigma} = B(k, m_f)u_{f\sigma}, \quad (\text{B.3})$$

$$v_{fk\sigma} = B(k, m_f)v_{f\sigma}, \quad (\text{B.4})$$

$$v_{f\sigma} = Cu_{f\sigma}^*, \quad (\text{B.5})$$

$$u_{f\sigma} = \sqrt{2m_f} \zeta_f \chi_\sigma, \quad (\text{B.6})$$

where  $\zeta_f$  is a normalized vector in flavor space,  $\chi_\sigma$  is a spinor that corresponds to one of two spin states of a fermion at rest, both normalized to 1 and orthogonal to each other,  $C$  is the charge conjugation matrix,

$$B(k, m) = \frac{1}{\sqrt{k^+m}} \left[ \Lambda_+ k^+ + \Lambda_- (m + k^\perp \alpha^\perp) \right] \quad (\text{B.7})$$

is the LF boost matrix, and  $\Lambda_\pm = \frac{1}{2}\gamma_0\gamma^\pm$ . For the canonical gluon field, we have

$$A^\mu = \sum_{\sigma c} \int [k] \left[ t^c \varepsilon_{k\sigma}^\mu a_{k\sigma c} e^{-ikx} + t^c \varepsilon_{k\sigma}^{\mu*} a_{k\sigma c}^\dagger e^{ikx} \right]_{x^+=0}, \quad (\text{B.8})$$

$$\varepsilon_{k\sigma}^\mu = \left( \varepsilon_{k\sigma}^+ = 0, \varepsilon_{k\sigma}^- = 2k^\perp \varepsilon_\sigma^\perp / k^+, \varepsilon_\sigma^\perp \right). \quad (\text{B.9})$$

The same expansions can be used for effective quantum fields in which creation and annihilation operators correspond to some scale  $\lambda$  [31]<sup>38</sup>, see Eqs. (39) and (40).

To evaluate  $W_1$  using Eq. (31), one first needs to evaluate  $\mathcal{H}_{I\infty 1}$  and  $\mathcal{H}_0 = \mathcal{H}_{0\infty}$ . According to Eq. (32),

$$\mathcal{H}_{0\infty} = \int dx^- d^2x^\perp h_{0\infty}, \quad (\text{B.10})$$

$$\mathcal{H}_{I\infty 1} = \int dx^- d^2x^\perp h_{I\infty 1}. \quad (\text{B.11})$$

Using Eq. (33) and the field expansions, one obtains the normal-ordered expression for  $\mathcal{H}_0$ ,

$$\begin{aligned} \mathcal{H}_0 = & \sum_{\sigma c f} \int [k] \frac{k^{\perp 2} + m_f^2}{k^+} \left[ b_{k\sigma c f}^\dagger b_{k\sigma c f} + d_{k\sigma c f}^\dagger d_{k\sigma c f} \right] \\ & + \sum_{\sigma c} \int [k] \frac{k^{\perp 2}}{k^+} a_{k\sigma c}^\dagger a_{k\sigma c}. \end{aligned} \quad (\text{B.12})$$

<sup>38</sup> In order to relate the above notation to the one introduced in Ref. [31], one needs to treat  $d^\dagger$  with positive  $k^+$  as  $b$  with negative  $k^+$ , and  $a^\dagger$  with positive  $k^+$  as  $a$  with negative  $k^+$ , taking into account the well-known constraint relations for fermion field  $\psi_- = \Lambda_- \psi$  and gauge boson fields  $A^-$  [51, 86, 87, 88, 16].

Similarly, using Eq. (34), one obtains  $\mathcal{H}_{I\infty 1}$ ,

$$\begin{aligned} \mathcal{H}_{I\infty 1} = & \sum_{123} \int [123] 2(2\pi)^3 \delta^3(P_c - P_a) r_{\Delta\delta}(1, 2, 3) \left[ \bar{u}_2 \not{\epsilon}_1^* u_3 t_{23}^1 b_2^\dagger a_1^\dagger b_3 \right. \\ & - \bar{v}_3 \not{\epsilon}_1^* v_2 \cdot t_{32}^1 \cdot d_2^\dagger a_1^\dagger d_3 + \bar{u}_1 \not{\epsilon}_3 v_2 \cdot t_{12}^3 \cdot b_1^\dagger d_2^\dagger a_3 \\ & \left. + Y_{123} a_1^\dagger a_2^\dagger a_3 + \text{h.c.} \right], \end{aligned} \tag{B.13}$$

where  $P_c$  denotes the total momentum of created particles,  $P_a$  denotes the total momentum of annihilated particles,  $r_{\Delta\delta}(1, 2, 3)$  denotes the regularization factors described in Sec. 2,  $t_{ij}^a = \chi_{ic}^\dagger t^a \chi_{jc}$ ,

$$Y_{123} = if^{c_1 c_2 c_3} \left[ \varepsilon_1^* \varepsilon_2^* \cdot \varepsilon_3 \kappa - \varepsilon_1^* \varepsilon_3 \cdot \varepsilon_2^* \kappa \frac{1}{x_{2/3}} - \varepsilon_2^* \varepsilon_3 \cdot \varepsilon_1^* \kappa \frac{1}{x_{1/3}} \right], \tag{B.14}$$

with  $\varepsilon \equiv \varepsilon^\perp$  and  $\kappa = k_1^\perp - k_1^\dagger k_3^\perp / k_3^\dagger$ , and h.c. denotes Hermitian conjugation of the 4 explicitly written terms in the square bracket in Eq. (B.13).

The result for  $\mathcal{H}_{I\infty 1}$  needed in Eq. (31), can be written in an abbreviated form

$$\mathcal{H}_{I\infty 1} = \sum_{123} \int [123] 2(2\pi)^3 \delta^3(P_c - P_a) r_{\Delta\delta}(1, 2, 3) \{1, 2, 3\}, \tag{B.15}$$

where  $\{1, 2, 3\}$  denotes the 4 + h.c. terms with  $b_\infty$ ,  $d_\infty$ , and  $a_\infty$  replaced by  $b_\lambda$ ,  $d_\lambda$ , and  $a_\lambda$ , respectively. Using this notation, the exact perturbative result for  $W_1$  in RGPEP reads

$$W_1 = \sum_{123} \int [123] 2(2\pi)^3 \delta^3(P_c - P_a) r_{\Delta\delta}(1, 2, 3) \frac{f_\lambda - f_{\lambda_c}}{P_a^- - P_c^-} \{1, 2, 3\}, \tag{B.16}$$

where  $P_a^-$  and  $P_c^-$  are eigenvalues of  $\mathcal{H}_0$  and  $f$  denotes the RGPEP form factor,

$$f_\lambda = e^{-(P_c^2 - P_a^2)^2 / \lambda^4}, \tag{B.17}$$

in which the squares of invariant masses of created particles,  $\mathcal{M}_c^2 = P_c^2$ , and annihilated particles,  $\mathcal{M}_a^2 = P_a^2$ , are evaluated using eigenvalues of  $\mathcal{H}_0$  associated with the particle momentum components  $k_i^\perp$  and  $k_i^\dagger$  for  $i = 1, 2, 3$  in every term [31].

The structure of  $\{1, 2, 3\}$  implies that  $W_1$  can replace a quark by two particles: a quark and a gluon. It can replace a gluon by a quark–anti-quark pair, or by two gluons. Reversed changes are also possible.

According to Eq. (30), evaluation of  $W_2$  involves action of  $\mathcal{H}_{I_{\infty 1}}$  twice and  $\mathcal{H}_{I_{\infty 2}}$  once. This means that  $W_2$  can replace one virtual particle by 3, or vice versa, change the number of virtual particles by 1, or do not change their number at all but only alter their individual momenta and other quantum numbers as dictated by  $\mathcal{H}_{I_{\infty}}$ .

When one evaluates  $W$  for  $\lambda$  and  $\lambda_c$  using expansion in powers of  $g_\lambda$ , creation and annihilation operators in Eq. (B.16) being those corresponding to  $\lambda$ , the coefficient  $(f_\lambda - f_{\lambda_c})/(P_a^- - P_c^-)$  only appears with first power of  $g_\lambda$ . Higher powers involve different coefficients and the entire sum of the series in powers of  $g_\lambda$  is not known. Moreover, a complete answer may contain dependence on  $\Lambda_{\text{QCD}}$  that perturbation theory cannot identify. Therefore, the question is what structure of  $W$  one can expect for  $\lambda \sim \lambda_c \sim \Lambda_{\text{QCD}}$ . In particular, seeking a connection between CQMs and QCD, one needs to estimate the structure of  $W$  that is responsible for changing the number of effective particles, since CQMs do not include such interactions.

When  $\lambda$  is near  $\lambda_c$ , one knows that  $W$  is near 1 irrespective of the size of both  $\lambda$ s and  $g_\lambda$ , because  $W$  is 1 when  $\lambda = \lambda_c$ . However, the deviation of  $W$  from 1 is not the same for  $\lambda \sim \lambda_c \sim \Lambda_{\text{QCD}}$  as in the perturbative case discussed above for two  $\lambda$ s that are much greater than  $\Lambda_{\text{QCD}}$ . The significant change in  $W$  in comparison to the perturbative result for large  $\lambda$ s comes from the difference between  $\mathcal{H}_{\infty}$  and full solution for  $\mathcal{H}_\lambda$ . Nevertheless, if one assumes certain particle-number changing interaction in  $\mathcal{H}_\lambda$  for  $\lambda$  near  $\lambda_c$ ,  $W$  will put this structure of  $\mathcal{H}_\lambda$  to action in a similar way to the one described above. The reason is not the smallness of  $g_\lambda$  but the presence of the form factors  $f_\lambda$ . They weaken the interaction by limiting the range of momenta that interacting particles may have. Thus, one can look at a candidate for  $\mathcal{H}_\lambda$  for some value of  $\lambda$  near  $\lambda_c$  as an initial condition for integration over a relatively small range of  $\lambda$  near  $\lambda_c$  and treat the whole interaction that changes the number of particles as weak even if  $g_\lambda$  appears large.

The weakness of particle-number changing interactions with small  $\lambda$  is reinforced by the particle masses as described in Sec. 2. Both quark,  $m$ , and gluon,  $m_g$ , mass parameters in the region of  $\lambda \sim \lambda_c$  may be sizable. Gluon mass terms for effective gluons are not excluded by local gauge invariance of Lagrangian for QCD because the interaction terms in LF QCD Hamiltonians with finite  $\lambda$  are not local and their non-locality is not of canonical type. Since regularization of LF QCD Hamiltonian requires gluon mass counterterms and the latter have finite parts that are expected to depend on  $\Lambda_{\text{QCD}}$ , perturbation theory for scattering in femtouniverse [89] cannot be used to argue that effective gluons cannot have sizable mass terms for  $\lambda \sim \lambda_c$ . The spectrum of masses of lightest hadrons, which is quite sparse in comparison to the QED near-threshold atomic spectra with massless photons, suggests instead that there is a considerable mass gap involved in excitation of gluon

degrees of freedom. Since QCD is assumed to describe these data, it seems reasonable to expect that RGPEP equations in QCD lead to massive effective gluons, instead of massless ones.

As an illustration of what one might expect of  $W$  in QCD in the range of  $\lambda \sim \lambda_c \sim \Lambda_{\text{QCD}}$  when one reasons by analogy with the perturbative solution at large  $\lambda$ , consider the structure of local three-gluon interaction term that originates from the gauge-invariant Lagrangian density  $-\text{Tr}F^{\mu\nu}F_{\mu\nu}/2$  in canonical theory. The interaction causes splitting of a bare gluon into two bare gluons. Its structure is given in Eq. (B.13), as the term with  $Y_{123}$ . In distinction from this local canonical term (in which  $\lambda = \infty$ ), the corresponding term in an effective Hamiltonian at small  $\lambda \sim \Lambda_{\text{QCD}}$ , is non-local [31]. Suppose the complete non-local three-gluon vertex includes the same structure  $Y_{123}$  and a vertex factor  $V_\lambda(1, 2, 3)$  times the form factor  $f_\lambda$ . Such structure is known to emerge through order  $g_\lambda^3$  [29]. The corresponding 1-to-2-gluon term in  $W$  of lowest order in the effective interaction reads

$$W_{Y_{123}} = \sum_{123} \int [123] 2(2\pi)^3 \delta^3(P_c - P_a) \frac{f_\lambda - f_{\lambda_c}}{P_a^- - P_c^-} \times Y_{123} V_\lambda(1, 2, 3) a_{\lambda 1}^\dagger a_{\lambda 2}^\dagger a_{\lambda 3}, \tag{B.18}$$

where  $V_\lambda(1, 2, 3)$  plays the role of initial condition, instead of  $V_\infty(1, 2, 3)$  that provided the initial condition in  $\mathcal{H}_\infty$ . Note that now the regularization factor  $r_{\Delta\delta}(1, 2, 3)$  is absent. This follows since the form factors  $f_\lambda$  and  $f_{\lambda_c}$  entirely eliminate the ultraviolet divergences and they eliminate the small- $x$  divergences when one adopts the assumption that effective gluons are heavy [16]. In these circumstances, the regularization factors are immaterial.

Suppose  $\lambda_c$  is smaller than the gluon mass  $m_g$  at  $\lambda_c$ . In this case, for  $\lambda \sim \lambda_c$ , one can approximate  $P_a^-$  by  $m_g^2/p_3^+$  and  $P_c^-$  by  $4m_g^2/p_3^+$  in the difference in denominator while the form factors can be approximated using (see [31] for details)

$$f_\lambda = e^{-[\mathcal{M}_{12}^2 - m_g^2]^2/\lambda^4}, \tag{B.19}$$

with  $\mathcal{M}_{12}^2 = 4(m_g^2 + \vec{k}^2)$ , rewritten as

$$f_\lambda = e^{-(3m_g^2/\lambda^2)^2} e^{-\frac{\vec{k}^2 + 3m_g^2/2}{(\lambda/2)^2} \frac{\vec{k}^2}{(\lambda/2)^2}}. \tag{B.20}$$

For  $\lambda$  smaller than  $m_g$ , the relative momentum  $|\vec{k}|$  of gluons 1 and 2 is smaller than  $\lambda/2 < m_g/2$  and the form factor can be very well approximated by Gaussian

$$f_\lambda = e^{-9m_g^4/\lambda^4} e^{-24m_g^2\vec{k}^2/\lambda^4}. \tag{B.21}$$

The exponential in front causes the form factor to vanish when  $\lambda$  is much smaller than  $m_g$ , as discussed in Sec. 2. This exponential is a part of the mechanism by which the presence of RGPEP form factors suppresses interactions that change the number of effective particles. The other part of the mechanism is the Gaussian factor whose width decreases when  $\lambda$  decreases. This factor causes the range of interaction in momentum space to decrease and thus to diminish the net effect of it. For the form factor suppression mechanism to work, the coupling constant  $g_\lambda$  must not blow up to huge values and compensate for the smallness of the form factor. However, such compensation should not be expected in QCD if the theory is to explain the phenomenological success of the CQMs.

For  $\lambda_c$  much smaller than  $m_g$ , one could literally set  $f_{\lambda_c}$  to 0 in Eq. (B.18). The resulting estimate for the structure of  $W_{Y121}$  reads

$$\begin{aligned}
 W_{Y121} = & e^{-9m_g^4/\lambda^4} \sum_{123} \int [123] 2(2\pi)^3 \delta^3(P_{12} - p_3) \frac{e^{-24m_g^2 \vec{k}^2/\lambda^4}}{3m_g^2/p_3^+} \\
 & \times Y_{123} V_\lambda(1, 2, 3) a_{\lambda 1}^\dagger a_{\lambda 2}^\dagger a_{\lambda 3}. \tag{B.22}
 \end{aligned}$$

This term in  $W_1$  replaces a gluon with a pair of gluons. The relative motion of the two emerging gluons corresponds to a Gaussian wave function of the form  $e^{-c\vec{k}^2/\lambda^2}$  with  $c = 24m_g^2/\lambda^2$ . Assuming that  $\lambda$  is comparable with  $m_g \sim 1$  GeV, the width of the Gaussian wave function is about  $\lambda/5 \sim 200$  MeV, which is not unreasonable. Nevertheless, one has to remember that the full vertex contains also the function  $V_\lambda(1, 2, 3)$  which causes that the non-locality of  $W_{Y121}$  is not determined solely by the Gaussian factor in Eq. (B.22).

Another illustration is provided by consideration of the first term in  $\{1, 2, 3\}$ , *i.e.*, the term that replaces a quark of scale  $\lambda$  by a quark and a gluon of the same scale. The only new elements in the analysis of this term, in comparison to the analysis described above in the case of gluons, are different values of masses and spinor factors instead of  $Y$ . However, the conclusion is similar:  $W$  changes a quark into a quark and a gluon.

The larger the difference between  $\lambda$  and  $\lambda_c$ , the more important particle-changing interactions in  $H_\lambda$ . The full solution for  $W$  may replace a state of two or three constituent quarks by a superposition of states with many different numbers of gluons (and additional quark–anti-quark pairs created from additional gluons) at  $\lambda$ , as dictated by the interaction terms in  $H_\lambda$ .

The perturbative calculation of  $W$ , and calculations for small difference between  $\lambda$  and  $\lambda_c$  using assumed simple picture at  $\lambda_c$  and guesses concerning interactions at  $\lambda \gtrsim \lambda_c$ , can be replaced by non-perturbative solutions of RGPEP equations, see Appendix C.

## Appendix C

### *RGPEP beyond perturbation theory*

For the principles of renormalization group procedure for Hamiltonians [73] to apply in a boost-invariant LF dynamics, the right-hand side in Eq. (A.1) must guarantee that matrix elements of  $H_\lambda$  vanish when changes of the invariant masses exceed  $\lambda$  and become comparable with the invariant masses themselves. This feature will be called narrowness in invariant mass of interacting effective particles of width  $\lambda$ , or just narrowness. One way of securing narrowness of width  $\lambda$  in perturbation theory is to work with vertex form factors in RGPEP as described in Appendix B. While this option is welcome in perturbative calculations [50], it involves a derivative of the Hamiltonian in the definition of functional  $\mathcal{F}_\lambda$  and the derivative is not easy to calculate outside perturbation theory. The perturbative approach of Ref. [74] defines differential equations in which the functional  $\mathcal{F}_\lambda$  guarantees narrowness of width  $\lambda$  without depending on derivatives of the Hamiltonian on the right-hand side, but the generator of the transformation is again defined recursively in terms of a commutator of the generator with the Hamiltonian. The recursion formally secures an expansion to all orders in the coupling constants, but it is difficult to solve beyond perturbation theory.

Non-perturbative formulation of RGPEP for operators in the Fock space can be defined in the form of a double commutator [90, 91, 92] equation that reads

$$\frac{d}{d\lambda^{-4}} \mathcal{H}_\lambda = [[\mathcal{H}_{0\lambda}, \mathcal{H}_{P\lambda}], \mathcal{H}_\lambda] . \quad (\text{C.1})$$

The Hamiltonian  $H_\lambda$  is obtained when  $b_\infty$  in  $\mathcal{H}_\lambda$  is replaced by  $b_\lambda$ . The required transformation  $U_\lambda$  is thus meant to be always incorporated with accuracy dictated by the accuracy of solving Eq. (C.1). Strictly speaking, nothing is known yet about the accuracy one can actually achieve this way beyond perturbative RGPEP in QCD.

The structure of Eq. (C.1) resembles the beautiful equation for Hamiltonian matrices that Wegner proposed for solving theoretical problems in condensed matter physics [82, 83, 84]<sup>39</sup>. In Wegner's proposal,  $\mathcal{H}_{0\lambda}$  is a diagonal part of the Hamiltonian matrix,  $\mathcal{H}_{I\lambda}$  is the off-diagonal part, and  $\mathcal{H}_{P\lambda}$  is the matrix  $\mathcal{H}_{I\lambda}$  itself. Attempts have been made to apply the Wegner equation to Hamiltonian matrices obtained from quantum field theory for the purpose of eliminating matrix elements that involve a change in the

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<sup>39</sup> To avoid a misunderstanding, it should be clarified that in his original work Wegner did not relate his equation to renormalization group procedures in condensed matter physics or quantum field theory.

number of virtual particles. Such matrix elements are found in the initial conditions set using interaction terms from canonical Hamiltonians in quantum field theory. The Tamm–Dancoff truncation to a few Fock components was used to define a sufficiently small Hamiltonian matrix for using Wegner’s equation [93, 94, 95, 96].

Wegner’s equation with matrix  $\mathcal{H}_{P\lambda} = \mathcal{H}_{I\lambda}$  implies that the matrix  $\mathcal{H}_\lambda$  with small  $\lambda$  is narrow with a width order  $\lambda$  on energy scale (it would be  $P^-$  scale in LF dynamics). However, Wegner’s equation for Hamiltonian matrices does not agree with kinematical LF symmetries, including the Lorentz boosts that are essential for explaining a connection between the constituent picture and the parton model picture of hadrons. Also, one desires an equation for a Hamiltonian operator that in principle may act in the entire Fock space rather than only in a severely truncated space in a Tamm–Dancoff approach. These two issues will be addressed below by defining the operator  $\mathcal{H}_{P\lambda}$  and showing that the resulting equation leads to narrowness of  $\mathcal{H}_\lambda$  that respects all kinematical LF symmetries and is narrow in the entire Fock space.

### *Boost invariance*

The origin of the difficulty with LF boost symmetry in Wegner’s equation is that the left-hand side in his equation is supposed to be a derivative of a Hamiltonian with respect to a parameter, while the right-hand side is trilinear in a Hamiltonian, an object clearly depending on a frame of reference in a different way than a Hamiltonian does. LF power counting [16] for Hamiltonian terms suggests that a straightforward application of Wegner’s equation in the SRG scheme may create a host of complex counterterms required for restoring kinematical LF symmetries. An attempt to cure the situation was undertaken [97, 98] by assuming a double-commutator equation for matrix elements of the invariant mass squared rather than a Hamiltonian. It was assumed that the LF boost invariance could be maintained because powers of the invariant mass are invariant with respect to boosts if the mass itself is. However, invariant masses of Fock states depend on spectators, resulting in effective interactions that depend on spectators. This means that the resulting effective interactions require extra care to recover the cluster decomposition principle [99] and it is not clear that they can help in literally solving quantum field theory where this principle is expected to be valid in effective theories, including effects of a complete renormalization procedure.

The issue of LF symmetries in RGPEP was considered before [28]. Here, the operator  $\mathcal{H}_{P\lambda}$  is defined using the Hamiltonian  $\mathcal{H}_\lambda$ . Assuming that  $\mathcal{H}_\lambda$  conserves momentum and is of the form

$$\mathcal{H}_\lambda = \sum_{n=2}^{\infty} \int \! \! \! \int [1 \dots n] h_\lambda(1, \dots, n) q_1^\dagger \cdots q_n, \quad (\text{C.2})$$

where  $q$  denotes annihilation operators  $b_\infty$ , the operator  $\mathcal{H}_{P\lambda}$  is defined to be of the form

$$\mathcal{H}_{P\lambda} = \sum_{n=2}^{\infty} \int \! \! \! \int [1 \dots n] h_\lambda(1, \dots, n) \left( \frac{1}{2} \sum_{k=1}^n p_k^+ \right)^2 q_1^\dagger \cdots q_n, \quad (\text{C.3})$$

and the symbol  $\int \! \! \! \int$  denotes integration over momenta and summation over discrete quantum numbers that label creation and annihilation operators while  $p_k^+$  denotes the  $+$ -component of momentum that labels the creation or annihilation operator number  $k$ . In words,  $\mathcal{H}_{P\lambda}$  differs from  $\mathcal{H}_\lambda$  by multiplication of its terms by the square of  $+$ -component of total momentum carried by the particles that enter a term, which is the same as the  $+$ -component of total momentum carried by the particles that leave the term. In the resulting Eq. (C.1), both sides behave in the same way with respect to operations of kinematical LF symmetries, and the RGPEP width parameter  $\lambda$  is invariant with respect to boosts of kinematical LF symmetry.

The operator  $\mathcal{H}_{0\lambda}$  in Eq. (C.1) is now set equal to a free-particle Hamiltonian  $\mathcal{H}_0$ , *i.e.*, an operator built from products of one creation operator and one annihilation operator per particle species. The notation  $\mathcal{H}_{0\lambda} = \mathcal{H}_0$  also indicates that  $\mathcal{H}_{0\lambda}$  in Eq. (C.1) is chosen here to be independent of  $\lambda$ <sup>40</sup>. It is a diagonal operator in the Fock space spanned by states created by operators  $b_\infty$  from the LF vacuum state. Diagonal matrix elements of  $\mathcal{H}_0$  are not equal to the diagonal matrix elements of the full Hamiltonian. Typical Hamiltonians include interactions that contribute to diagonal matrix elements<sup>41</sup>.

#### *Non-perturbative narrowness in the Fock space*

Eq. (C.1) is designed for operators that can act in the entire Fock space (action on every state is well-defined). Eq. (C.1) with a constant  $\mathcal{H}_0$  renders  $\lambda$ -dependent interaction terms that die out when the change of the invariant mass squared (evaluated using the kinematical momenta and eigenvalues of  $\mathcal{H}_0$ ) of particles involved in an interaction exceeds  $\lambda$  (spectators do not influence Hamiltonian interaction terms). The narrowness feature is obtained through a universal mechanism for double commutator evolution equations. The factors of  $P^{+2}$  in  $\mathcal{H}_{P\lambda}$  do not change the general mechanism.

<sup>40</sup> This assumption simplifies the analysis that follows.

<sup>41</sup> It is possible to include interactions in  $\mathcal{H}_{0\lambda}$  in Eq. (C.1) in the form of mass effects (self-interactions) and beyond (potentials), but these options are ignored here for simplicity.

We start from the observation that Eq. (C.1) can be solved using a method of successive approximations. An approximation is defined by how many terms are kept in the Hamiltonian. In principle, a Hamiltonian with finite  $\lambda$  contains infinitely many terms. The terms that are kept in an approximation can be defined by the condition that they contain no more than a prescribed number  $N$  of creation and annihilation operators in a product. The number  $N$  labels the approximation. Terms that contain more operators in a product are ignored in the approximation. Successive approximations are labeled by increasing numbers  $N$ .

Suppose one wants  $\mathcal{H}$  to contain at most  $N$  creators and at most  $N$  annihilators<sup>42</sup>. Such Hamiltonian will be denoted by  $\mathcal{H}_{\lambda N}$ . Using notation corresponding to Eq. (C.2), this means that

$$\mathcal{H}_{\lambda N} = \sum_{n_c=1}^N \sum_{n_a=1}^N \not\int [1 \dots n_c + n_a] h_{\lambda N n_c n_a}(1, \dots, n_c + n_a) t_{n_c n_a}, \quad (\text{C.4})$$

where

$$t_{n_c n_a} = \prod_{k=1}^{n_c} q_k^\dagger \prod_{l=n_c+1}^{n_c+n_a} q_l, \quad (\text{C.5})$$

$n_c$  is the number of creation operators,  $n_a$  is the number of annihilation operators, and subscripts denote also all relevant quantum numbers of the operators they label.

Suppose one wants to know the terms in  $\mathcal{H}_{\lambda N}$  that contribute to the dynamics of a physical state that certainly contains a specified Fock component  $|\psi\rangle$  of the form

$$|\psi\rangle = \not\int [1 \dots n] \psi_P(1, \dots, n) \prod_{k=1}^n q_k^\dagger |0\rangle, \quad (\text{C.6})$$

where  $P$  denotes the fixed total kinematical momentum of the state and otherwise the wave function  $\psi_P(1, \dots, n)$  is not known. For example, in the case of mesons one may select the state  $|\psi\rangle$  that contains a quark and an anti-quark, and in the case of a baryon a state  $|\psi\rangle$  that contains three quarks. But one could also consider a state that contains 1182 quarks and many gluons and quark–anti-quark pairs in order to describe a collision of two nuclei of gold neglecting all interactions but the strong.

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<sup>42</sup> One can limit the number of particle operators for different species of particles differently. In this case  $N$  becomes a vector with natural components corresponding to species.

Having specified the set of operators that are included in  $\mathcal{H}_{\lambda N}$ , with  $n_c \leq N$ ,  $n_a \leq N$ , and unknown coefficients  $h_{\lambda N n_c n_a}$ , one can enumerate forms of all possible states that can be generated from  $|\psi\rangle$  by action of  $\mathcal{H}_{\lambda N}$  on it once. All these states are eigenstates of the total kinematical momentum operator with one and the same eigenvalue  $P$  that characterizes  $|\psi\rangle$ . Knowing the set of states that are obtained from  $|\psi\rangle$  by acting on it with  $\mathcal{H}_{\lambda N}$  once, one can construct the set of all states that can be obtained by acting with  $\mathcal{H}_{\lambda N}$  on  $|\psi\rangle$  twice, and so on.

We introduce the set of states, denoted by  $R$ , that can be generated from  $|\psi\rangle$  by acting with  $\mathcal{H}_{\lambda N}$  on it  $\tau$  times and are normalized in a limit of infinite volume. Then we introduce a minimal subspace in the Fock space (a space of smallest possible basis) that is sufficient to build the set  $R$ . This subspace in the Fock space is denoted by  $R_{\tau\psi N}$ . For example, if one starts with a state of 3 quarks, action just one time by the  $\mathcal{H}_{\lambda N}$  that is equal to regulated canonical Hamiltonian of LF QCD,  $\tau = 1$  and  $N = 3$ , on  $|3q\rangle$  produces states with 3 quarks, 3 quarks and a gluon, 3 quarks and a quark–anti-quark pair, and 3 quarks and 2 gluons. The space  $R_{13q3}$  is built from the Fock space basis states that are needed to construct states with these particles. Action twice,  $\tau = 2$ , generates additional gluons and quark–anti-quark pairs. The corresponding space  $R_{23q3}$  is built by adding the Fock space basis states that are needed to construct the additional states. When one assumes that  $\mathcal{H}_{\lambda N}$  has more terms than the canonical LF Hamiltonian for QCD [16], the space  $R_{\tau\psi N}$  is greater than in the canonical case. States created from  $|\psi\rangle$  with  $k$  particles by  $\tau$  actions of  $\mathcal{H}_{\lambda N}$  may contain up to  $k + \tau(N - 2)$  particles.

The set  $R$  of states one obtains for any  $\tau$  is *a priori* infinite, because one can have an arbitrary relative motion of particles in a state and the range of relative momentum is infinite. The corresponding subspace in the Fock space is also unlimited. This happens despite that a regulated Hamiltonian cannot change a relative momentum by an arbitrarily large amount. The reason is that the unknown wave function  $\psi_P(1, \dots, n)$  in Eq. (C.6) does not limit the magnitude of relative momenta. For example, a square-integrable function, such as Gaussian, quickly falls off as a function of relative momentum of two particles but the range of relative momentum is infinite.

In order to introduce a space of states with a finite range of relative momenta, we impose a cutoff  $\Delta$  on the invariant mass. Namely, we define the space of states that are in  $R_{\tau\psi N}$  and whose invariant mass  $\mathcal{M} < \Delta$ .  $\mathcal{M}$  is calculated using kinematical momenta and eigenvalues of  $\mathcal{H}_0$ . The resulting space with this cutoff is denoted by  $R_{\Delta\tau\psi N}$ , or shortly  $R$ . We introduce the projection operator on the space  $R$  and denote this projector also by  $R$ . Hamiltonian  $H_{\lambda N}$  projected on the space  $R$  is denoted by

$$\mathcal{H}_{\lambda NR} = R\mathcal{H}_{\lambda N}R. \quad (\text{C.7})$$

The successive approximation to  $\mathcal{H}_\lambda$  order  $N$  in application to states of the type  $|\psi\rangle$  with accuracy to  $\tau$  actions of  $\mathcal{H}_\lambda$  on  $|\psi\rangle$  with invariant mass cutoff  $\Delta$  is obtained by solving equation

$$\mathcal{H}'_{\lambda NR} = [[\mathcal{H}_0, R \mathcal{H}_{\lambda NP} R], R \mathcal{H}_{\lambda N} R], \quad (\text{C.8})$$

with initial conditions specified by the canonical Hamiltonian of LF QCD with counterterms.

Structure of the counterterms is found by demanding that all matrix elements of the effective Hamiltonian  $\mathcal{H}_{\lambda NR}$  for a large ratio  $\Delta/\lambda$  among basis states with invariant masses much smaller than  $\Delta$  do not depend on the regularization applied in the canonical Hamiltonian in the limit of this regularization being removed [73].

The degree of dependence of the counterterms on  $N$  and  $\tau$  that are used in defining an approximation order  $N$  to  $\mathcal{H}_\lambda$  using  $\tau$  actions on various states  $|\psi\rangle$  requires studies. Specific choices of states  $|\psi\rangle$  may simplify identification of structure of counterterms in the canonical LF Hamiltonian for QCD. However, fixing all finite parts of the counterterms will require systematic studies of symmetries that are expected to relate finite parts of different counterterms, and input from phenomenology in order to fix mass parameters as required, in addition to the value of  $\Lambda_{\text{QCD}}$ . We proceed to showing narrowness of  $\mathcal{H}_{\lambda NR}$  that satisfies Eq. (C.8).

For notational brevity, Eq. (C.9) is written as

$$H' = [[H_0, H_{IP}], H], \quad (\text{C.9})$$

where

$$H = R \mathcal{H}_{\lambda N} R, \quad (\text{C.10})$$

$$H_0 = R \mathcal{H}_0 R, \quad (\text{C.11})$$

$$H_I = H - H_0 = R \mathcal{H}_{I\lambda N} R, \quad (\text{C.12})$$

$$H_{IP} = R \mathcal{H}_{I\lambda NP} R. \quad (\text{C.13})$$

Evolution in  $\lambda$  according to Eq. (C.8), or (C.9), preserves traces of powers of  $H$ . Trace is defined by summing diagonal matrix elements in the set  $R$ . The diagonal matrix elements are proportional to  $\delta^3(0)$  in momentum space which has interpretation of volume and can be divided out. The argument 0 corresponds to the conservation of total momentum.

Using the condition that the trace of  $H^2$  does not depend on  $\lambda$ , one has

$$0 = \text{Tr} (H_0^2 + 2H_0 H_I + H_I^2)'. \quad (\text{C.14})$$

Since  $H_0$  is fixed,

$$(\text{Tr} H_I^2)' = -2 \text{Tr} H_0 H_I'. \quad (\text{C.15})$$

Thus, using the basis in space  $R$  that is built from normalized eigenstates  $|m\rangle$  of  $H_0$ , with eigenvalues  $E_m$ , so that  $H_0|m\rangle = E_m|m\rangle$  and  $H_{mn} = \langle m|H|n\rangle$ , etc., we have

$$\left( \sum_{mn} |H_{Imn}|^2 \right)' = -2 \sum_m E_m H'_{Imm}. \quad (\text{C.16})$$

But Eq. (C.9) implies that

$$H'_{Imm} = [[H_0, H_{IP}], H_0]_{mn} + [[H_0, H_{IP}], H_I]_{mn} \quad (\text{C.17})$$

$$= -(E_m - E_n)^2 P_{mn}^{+2} H_{Imn} + \sum_k [(E_m - E_k) P_{mk}^{+2} + (E_n - E_k) P_{kn}^{+2}] H_{Imk} H_{Ikn}. \quad (\text{C.18})$$

The momentum  $P_{ij}^+$  is the + component of the total momentum of the particles that undergo interaction in the relevant matrix element  $\langle i|H_I|j\rangle$ . Momenta of the spectators of the interaction do not contribute to  $P_{ij}^+$ . The result needed on the right-hand side of Eq. (C.16) is

$$H'_{Imm} = \sum_k (E_m - E_k) (P_{mk}^{+2} + P_{km}^{+2}) H_{Imk} H_{Ikm}. \quad (\text{C.19})$$

Therefore, the sum on the right-hand side of Eq. (C.16) is

$$\sum_m E_m H'_{Imm} = \sum_{km} (E_k - E_m)^2 |H_{Ikm}|^2 (P_{mk}^{+2} + P_{km}^{+2}) / 2, \quad (\text{C.20})$$

and

$$\left( \sum_{mn} |H_{Imn}|^2 \right)' = - \sum_{km} (E_k - E_m)^2 |H_{Ikm}|^2 (P_{mk}^{+2} + P_{km}^{+2}) \leq 0. \quad (\text{C.21})$$

This result leads to the following conclusion: *The sum of moduli squared of all matrix elements of the interaction Hamiltonian decreases with  $\lambda$  until all off-diagonal matrix elements of the interaction Hamiltonian between states with different free invariant masses vanish. Thus also: Matrix elements of the interaction Hamiltonian on the diagonal and between states of equal invariant masses and between states of small invariant masses as measured by  $H_0$  may stay constant or even increase when  $\lambda$  decreases but only at the expense of still faster decrease of the off-diagonal and large invariant mass matrix elements and only until all off diagonal matrix elements between non-degenerate states are eliminated.*

Another way of writing relation (C.21) is

$$\left( \sum_{mn} |\mathcal{H}_{Imn}|^2 \right)' = -2 \sum_{km} (\mathcal{M}_{km}^2 - \mathcal{M}_{mk}^2)^2 |\mathcal{H}_{Ikm}|^2 \leq 0, \quad (\text{C.22})$$

where  $\mathcal{M}_{ab}$  denotes an invariant mass of particles in state  $a$  that interact in an action of  $\mathcal{H}_I$  once on particles in state  $b$ . This form concludes our demonstration that RGPEP provides effective Hamiltonians which are narrow in the invariant mass of interacting particles beyond perturbative calculus. The invariant masses result from cancellation of spectator contributions to  $P^-$  and multiplication only by  $P^+$  of particles that are involved in an interaction, *i.e.*, not including spectators.

Initial comparison between the non-perturbative RGPEP calculus described in this appendix and perturbative RGPEP calculus described earlier, can be done by solving Eq. (C.1) using expansion in powers of a coupling constant, such as  $g_\lambda$  in the case of QCD, and observing how results of one way of calculating compare with the other for terms order  $g_\lambda$  and  $g_\lambda^2$ . Using notation introduced earlier (*e.g.*, see [29], Sec. III.C, or [100], Sec. II.B), one rewrites Eq. (C.1) as

$$\mathcal{H}'_{ac} = -ac^2 [\mathcal{H}_I]_{ac} + \sum_b (p_{ab} ab + p_{cb} cb) [\mathcal{H}_I \mathcal{H}_I]_{ac}, \quad (\text{C.23})$$

where letters  $a$ ,  $b$ , and  $c$ , denote configurations of particles. Using expansion

$$\mathcal{H} = \mathcal{H}_0 + g\mathcal{H}_1 + g^2\mathcal{H}_2 + \dots, \quad (\text{C.24})$$

one obtains for the first two terms equations

$$\mathcal{H}'_{1ac} = -ac^2 \mathcal{H}_{1ac}, \quad (\text{C.25})$$

$$\mathcal{H}'_{2ac} = -ac^2 \mathcal{H}_{2ac} + \sum_b (p_{ab} ab + p_{cb} cb) [\mathcal{H}_1 \mathcal{H}_1]_{ac}. \quad (\text{C.26})$$

Using the particle size parameter  $s = 1/\lambda$ , one obtains the solutions

$$\mathcal{H}_{1ac} = e^{-ac^2 s^4} \mathcal{H}_{1ac}(0), \quad (\text{C.27})$$

$$\begin{aligned} \mathcal{H}_{2ac} = & e^{-ac^2 s^4} \mathcal{H}_{2ac}(0) + \sum_b [\mathcal{H}_1(0) \mathcal{H}_1(0)]_{ac} \\ & \times e^{-ac^2 s^4} \frac{p_{ba} ba + p_{bc} bc}{ba^2 + bc^2 - ac^2} \left[ e^{-(ab^2 + bc^2 - ac^2)s^4} - 1 \right]. \end{aligned} \quad (\text{C.28})$$

The first-order result is Gaussian in the change of invariant masses squared, identical to the first-order result in perturbatively defined RGPEP. The

second-order result matches the perturbatively defined RGPEP, *e.g.*, see Eq. (2.22) in Ref. [100], when  $ac$  can be neglected in comparison to  $ab$  or  $bc$ . When  $ac$  refers to a dominant Fock component and  $ba$  and  $bc$  refer to a component with additional constituents, this condition may be satisfied very well if the additional constituents introduce a significant contribution to the invariant mass of the subsystem that counts. Such situation is encountered in the dynamics of heavy quarkonia [18], where the additional constituent is an effective gluon while kinetic energies of quarks cannot change by much because quarks move slowly with respect to each other and their masses are large in comparison to  $\Lambda_{\text{QCD}}$ . This means that the non-perturbative formulation of RGPEP can be directly applied to heavy quarkonia with arbitrary motion as a bound state in QCD, and implies that the non-perturbative formulation can be used to derive corrections to the harmonic oscillator force in Ref. [18] beyond perturbation theory to describe the spectrum of charmonium and bottomonium states that results from gluon dynamics neglecting light quarks.

The procedure described here differs from truncation in powers of a bare coupling constant  $g$ , truncation in the number of Fock sectors, and combinations of both these truncations. Although we use the Fock space subspace  $R$  for solving Eq. (C.8), and it is useful to use an expansion for  $\mathcal{H}_\lambda$  in powers of the coupling constant  $g_\lambda$  [16], the solutions provide in principle not just the matrix elements of Hamiltonian terms in a limited space of states but the matrix elements of Hamiltonian terms from which one extracts the coefficients  $h_\lambda$  of particle operators in the Hamiltonian terms that *a priori* act in the entire Fock space. Neither the Fock space nor perturbation theory limitations prevent us from seeking a non-perturbative solution for the coefficients  $h_\lambda$ .

Some comments are in order here. One can alter the generator in Eq. (C.8) by introducing convergence-improving factors that are required for perturbative approximations [49, 50]. For bound states of low-mass hadrons alone, the relative motion of constituents can be described using other basis functions than plane waves, such as the oscillator basis. Scattering states of hadrons may require a more elaborate basis in coupled channels. Reasoning described in this appendix justifies methods used in nuclear physics with only kinetic energy in SRG generator [101, 102, 103, 104, 105]<sup>43</sup>, including the

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<sup>43</sup> Note added in proof: The author has also shown that in the Wegner generator,  $[D, V]$ , which generates a flow of the Hamiltonian matrix  $H = D + V$ , the diagonal part,  $D$ , can be replaced not only by a constant matrix  $H_0 = H - H_I$ , such as the kinetic energy matrix  $T$  used in the nuclear physics applications, but also by any monotonically increasing function of  $H_0$ , including examples described in Ref. [106], and the mechanism that narrows Hamiltonian matrices still works. In the case of a positive power  $p$  of  $H_0$  in the LF dynamics, a corresponding factor of  $(P^+)^{p+1}$  in  $H_{IP}$  renders powers of the invariant mass squared, but the contribution of spectators does not drop out for  $p \neq 1$ .

impact of bound states [107]. The impact can only lead to increase of interaction matrix elements that are small since the combined strength of all interaction matrix elements must only decrease until it reaches a minimum. This result extends the range of applicability of SRG with a kinetic energy or a similar term in the generator. It suggests applications of simpler flow equations than Wegner’s in relevant areas of physics.

### Appendix D

#### *Visualization of RGPEP scale dependence of hadronic structure*

Quantum fields for effective particles of size  $s = 1/\lambda$  in RGPEP, see Eqs. (39), (40), and (199), are the degrees of freedom from which one can build effective Hamiltonian densities. An effective Hamiltonian at scale  $\lambda$  describes a hadron only in terms of quarks of size  $s = 1/\lambda$ . Therefore, when the scale parameter  $\lambda$  approaches  $\lambda_c \sim \Lambda_{\text{QCD}}$ , the size of quarks, as measured by the strong interaction range in interaction vertices [31], becomes comparable with  $1/\Lambda_{\text{QCD}}$ . This in turn means that the effective quarks become as large as the whole hadron. Such quarks must nearly overlap and this is how they form a white mixture. There is some imbalance of color on the boundary, perhaps able to couple to  $\pi$ -mesons that form a cloud around the quarks. Visualization of these circumstances is provided in Figs. 3 and 4. When  $\lambda$  is near but greater than  $\lambda_c$ , the constituent quarks become smaller and part of the color-active medium inside a hadron is no longer fully overlapped by quarks. Therefore, when  $\lambda \gtrsim \lambda_c$ , there must be additional matter density involved in filling the volume of a hadron.

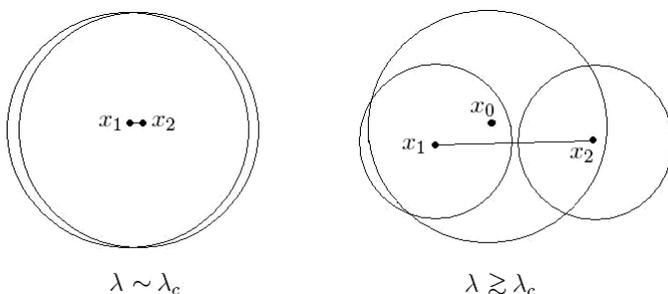


Fig. 3. The RGPEP image of a meson seen at  $\lambda$  near  $\lambda_c$ .

One can use an analogy with swarms of bees. For example, a baryon can be seen at scale  $\lambda_c$  as built from three swarms. One swarm is made of red bees, one of green, and one of blue. One of the three swarms corresponds to one constituent quark. There is also a fourth swarm, built from bi-colored

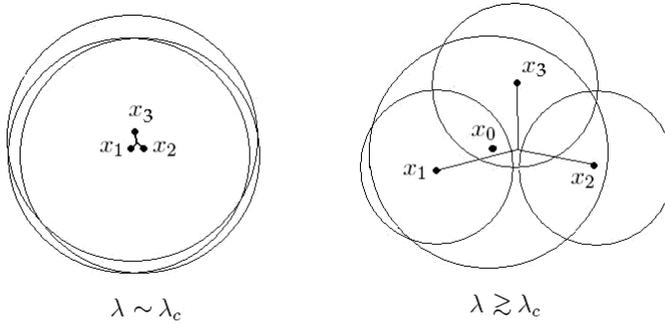


Fig. 4. The RGPEP image of a nucleon seen at  $\lambda$  near  $\lambda_c$ .

glue bees. But the four swarms overlap and in nearly every part of the baryon volume one has one red, one green and one blue bee, or glue bees. The regions where the three quark swarms are overlapping are locally white.

When one changes the scale  $\lambda = \lambda_c$  to a nearby  $\lambda \gtrsim \lambda_c$  as described in Sec. 5.1, the size  $s$  of each of the three swarms becomes smaller than  $s_c = 1/\lambda_c$ , but the medium around the quarks of size  $s$  remains filled with the glue and quark bees that can still balance to white, too. When one decreases  $\lambda$  below  $\lambda_c$ , the mass of the glue component may decrease toward 0 or stabilize while the constituent quarks fully saturate the volume of a hadron.

In distinction from the parton model [11] and related scaling pictures [108, 109], the new element of this visualization is that the constituent swarms are much larger in size than the distances between their centers. It seems to the author that this feature of RGPEP deserves a visualization because none of the visualizations of hadrons known to the author has this feature. Namely, common visualizations represent a baryon as made of three constituent quarks that fly like apples in a bag, sometimes accompanied by some strands of glue in some kind of a cavity of unspecified nature. Images associated with an increase of momentum scale typically include more tiny objects in two ways: either as finer pieces scattered around or as interaction processes among quarks and gluons (such as on the cover of Ref. [110]). It is hard to visualize the origin of constituent quarks and CQM potentials in these ways.

In the context of RGPEP, the analogy with swarms of colored bees provides the following visualization for the mechanism of formation of potentials. We start with a meson built from a quark and an anti-quark. A quark has opposite color charge density to anti-quark. When the effective particles are completely overlapping like two densities of opposite charges, their state is locally neutral. This picture oversimplifies the non-Abelian picture to the Abelian one but still offers some intuition. Imagine now that the centers

of the initially overlapping swarms move a little bit apart. It is well known that the Coulomb force between two spherically symmetric distributions of opposite charges grows linearly with the distance between their centers for as long as this distance is small in comparison to the individual sizes of the distributions. Therefore, the potential energy of two nearly overlapping spheres is described by a harmonic oscillator potential. It should be stressed that the distance between the centers of the swarms is much smaller than the size of each swarm. In the case of baryons, one needs to consider three centers of three overlapping swarms being relatively close to each other.

Of course, this visualization has many drawbacks: it is classical, Abelian, non-relativistic, and does not provide any insight concerning the difference between quarks and gluons. But it does model the transformation  $W$  that increases the number of quarks and gluons of decreasing size  $s$  when  $\lambda$  increases. These smaller quarks and gluons form the medium that provides the expectation value of the gluon field that is interpreted in this article as a gluon condensate.

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