BREAKING THE LIGHT SPEED BARRIER

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As it is well known, classical special relativity allows the existence of three different kinds of particles: bradyons, luxons and tachyons. Bradyons have non-zero mass and hence always travel slower than light. Luxons are particles with zero mass, like the photon, and they always travel with invariant velocity. Tachyons are hypothetical superluminal particles that always move faster than light. The existence of bradyons and luxons is firmly established, while the tachyons were never reliably observed. In quantum field theory, the appearance of tachyonic degrees of freedom indicates vacuum instability rather than a real existence of the faster-than-light particles. However, recent controversial claims of the OPERA experiment about superluminal neutrinos triggered a renewed interest in superluminal particles. Driven by a striking analogy of the old Frenkel–Kontorova model of a dislocation dynamics to the theory of relativity, we conjecture in this note a remarkable possibility of existence of the forth type of particles, elvisebrions, which can be superluminal. The characteristic feature of elvisebrions, distinguishing them from tachyons, is that they are outside the realm of special relativity and their energy remains finite (or may even turn to zero) when the elvisebrion velocity approaches the light velocity.

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1. Introduction

Superluminal sources of radiation were first considered by Heaviside in 1888 and in the following years he derived most of the formalism of what is nowadays called Cherenkov radiation [1–3]. Sommerfeld, being unaware of Heaviside's insights, also considered electromagnetic radiation from superluminal electrons [1, 4]. However, the timing when these works occurred was unfortunate [5] because Einstein's first paper on special relativity has appeared a few months after Sommerfeld's 1905 publication on superluminal electrons and it became clear that electrons and all other particles with non-zero mass cannot be accelerated beyond the light velocity in vacuum. As a result, we had to wait for several decades before the accidental, experimental discovery of the Cherenkov radiation in 1934 [6] and even more so to realize that special relativity does not prohibit superluminal sources of radiation [7].

Of course, these superluminal sources of radiation cannot be individual electrons or other Standard Model charged particles which are ordinary bradyons and hence cannot overcome the light-speed barrier. Nothing precludes though the aggregates of such particles to produce superluminally moving patterns in a coordinated motion [7, 8]. The simplest example of such a superluminally moving pattern is a light spot produced by a rotating source of light on a sufficiently remote screen. One can imagine a three dimensional analog of such a superluminal light spot, namely a radiation pulse with a conical frontal surface as a result of light reflection by a conical mirror. The vertex of this conical frontal surface is a focus which can travel superluminally and the field energy density at this spot is several orders of magnitude higher than in a flat light spot, making this object look like a particle [9].

One may argue that the light spot is not a real object and its propagation in space is not a real process at all since it does not transfer an energy from one point to another on its path [10]. However, already in classical physics it is not easy to give a general definition of what a real thing is, without even speaking about the quantum theory [11]. As a result, our understanding of what kind of velocities are limited by special relativity continues to evolve [12, 13].

Recently, the OPERA experiment reported an alleged evidence for superluminal muon neutrinos [14]. Although it was evident from the beginning that this experimental result contradicts all that we know about neutrinos and weak interactions [15–18], and hence was most probably due to some unaccounted systematic errors [19–23], it has generated a huge interest in our postmodern physics community. Many explanations of this unexpected and surprising result, one more fantastic than another, were proposed in literature. We cite only a few representatives which are potentially interesting but in our opinion improbable [24–30].

The Lorentz invariance is one of the most experimentally well established and tested feature of Nature [31–34]. In light of this impressive experimental evidence, it is not surprising that, finally, no indications of superluminality in the neutrino sector were found out, as expected, and it seems the original anomaly was most probably due to the OPERA equipment malfunctioning [35–37].

However, “There are more things in Heaven and Earth, Horatio, than are dreamt of in your philosophy” [38], and we cannot be “certain that Nature has exhausted her bag of performable tricks” [39]. Therefore, it makes sense to ask in a broader context whether the established unprecedented high accuracy of Lorentz invariance precludes a superluminal energy transfer at moderate energy scales in all conceivable situations. As we will try to argue in this paper, the answer is negative.

2. Tachyons

In 1905, Einstein published his paper on special relativity [40] in which he concluded that “speeds in excess of light have no possibility of existence”. For many years this has become an axiomatic statement, and any assumptions that were contrary to this dogma were perceived with a bias, as unscientific fantasies.

The reason behind the Einstein’s conclusion was that according to the theory of relativity you need an infinite amount of energy to accelerate a particle to the speed of light. Also, the special relativistic relationship between particle’s energy and its mass implies that the mass of a particle moving with velocity $v > c$ would be imaginary and hence “unphysical”. This is also applied to other physical quantities, such as the proper time and the proper length. Finally, it was believed that if such particles exist, the principle of causality would be violated as they can be used to send information in the past (the so-called Tolman antitelephone paradox [41]).

Interestingly, despite being a proponent of the concept of a velocity-dependent electromagnetic mass, Heaviside never acknowledged this limitation on the particle’s velocity [1], and maybe for a good reason. In fact, Einstein’s conclusion is fallacious, even absurd. As eloquently expressed by Sudarshan in 1972, this is the same as asserting “that there are no people North of the Himalayas, since none could climb over the mountain ranges. That would be an absurd conclusion. People of central Asia are born there and live there: they did not have to be born in India and cross the mountain range. So with faster-than-light particles” [42].

Probably Einstein was well aware of the weakness of the infinite energy argument. In fact, Tolman’s antitelephone paradox was invented by him [43], and it is indeed a serious conundrum and basic problem for any theory involving faster-than-light propagation of particles. Its essence is the following.

For events separated by a space-like interval, their relative time order is not invariant but depends on the choice of reference frame. However, the interval between the emission and absorption events of a superluminal particle is just space-like. Therefore, in some inertial reference frames the superluminal particle will be absorbed before it is emitted, and it appears that we have a grave problem with causality.

However, the same problem is already present in quantum field theory that unifies the fundamental ideas of special relativity and quantum mechanics and conforms the modern basis of elementary particle physics. In quantum field theory, the amplitude for a particle to propagate from a space-time point $x=(x_0,\vec{x}\,)$ to a point $y=(y_0,\vec{y}\,)$ is Lorentz invariant and is given by the Wightman propagator ($\hbar=c=1$ is assumed) [44]

$$D(x-y)=\int\frac{d\vec{p}}{(2\pi)^3}\,\frac{e^{-ip\cdot(x-y)}}{2\sqrt{\vec{p}^{\,2}+m^2}}\,. \tag{1}$$

When the difference $x-y=(0,\vec{r}\,)$ is purely in the spatial direction, the integral (1) can be evaluated by

$$D(x-y)=-\frac{1}{4\pi^2 r}\,\frac{\partial}{\partial r}\int\limits_0^\infty\frac{\cos{(pr)}}{\sqrt{p^2+m^2}}=-\frac{1}{4\pi^2 r}\,\frac{\partial}{\partial r}K_0(mr)=\frac{m}{4\pi^2 r}K_1(mr)\,, \tag{2}$$

where K_0 and K_1 are modified Bessel functions of the second kind. Using the well known asymptotics

$$K_1(x)\approx\sqrt{\frac{\pi}{2x}}\,e^{-x}\qquad\text{if}\qquad x\ll 1$$

we see that within its Compton wavelength, m^{-1}, a particle has a significant probability to propagate with infinite velocity (with respect to this particular reference frame in which $x-y=(0,\vec{r}\,)$).

For particles of a very small mass (neutrinos) the Compton wavelength can be macroscopically large. Interestingly, this kind of superluminal propagation of neutrinos within their Compton wavelength was even suggested as a possible explanation of the OPERA anomaly (then not yet falsified) [30, 45], but shown to be non-working [45].

Quantum field theory offers a miraculously clever solution of this superluminal propagation dilemma [44, 46, 47]. Suppose that in the reference frame S a particle propagates superluminally between the points x and y separated by space-like interval $(x-y)^2<0$, and suppose that x is the emission point and y is the absorption point so that $x_0<y_0$. Since the interval is space-like, there exists another reference frame S' such that $x'_0>y'_0$

and, therefore, in this frame the particle propagates backward in time: its absorption precedes its emission in apparent violation of causality. However, in the frame S' the particle's energy is negative as it can be easily checked using the Lorentz transformation properties of the energy-momentum four-vector. But a negative energy particle propagating backward in time is nothing more than a positive energy antiparticle propagating forward in time. This Feynman–Stueckelberg interpretation of antiparticles is at the heart of quantum field theory's resolution of superluminal propagation dilemma. The observer in the frame S' does not see that the particle is absorbed at y before its emission at x, instead he/she sees the antiparticle emitted at y and absorbed at x, therefore, he/she has no apparent reason to worry about causality violation.

On a deeper level, for causality to be restored one needs not to suppress a superluminal propagation of particles but to ensure that any measurement (disturbance) at a space-time point x cannot influence an outcome of another measurement at space-time point y if the points are separated by a space-like interval. Evoking antiparticles, quantum field theory ensures the cancellation of all acasual terms in commutators of two local observables at space-like separation and does not allow information to be transmitted faster than the speed of light.

Many subtleties and open questions remain, however, because it is not a trivial task to merge quantum mechanics, with its notorious non-localities, and special relativity [48–51]. "Relativistic causality — formulate it as you like! — is a subtle matter in relativistic quantum theories" [52]. We just mention two interesting examples, where the alleged superluminal effects can be interpreted as being due to the propagation of virtual photons outside of the light cone. Nevertheless, no one of them allows messages to be transmitted faster than the speed of light.

It can be shown that entanglement and mutual correlations can be generated at space-like separated points [51]. Of course, this problem is as old as the Einstein–Podolsky–Rosen paradox [53].

Another example is the so-called Hartman effect. Quantum mechanics predicts that the transmission time across a potential barrier becomes independent of barrier thickness for very thick barriers [54]. This strange prediction was experimentally confirmed in frustrated total internal reflection, which is an optical analog of quantum mechanical tunneling [55, 56], and in other optical tunneling experiments [57]. Apparent superluminal behavior in such experiments is related to evanescent modes, a kind of classical analog of virtual photons [58].

How real are virtual photons? Sometimes virtual particles are considered as pure mathematical constructions, just a tool to visualize perturbation theory calculations. However, there are many things in modern physics which cannot be observed as separate asymptotic states and, nevertheless, nobody questions their real existence, quarks being the most notorious example. Another example is short-lived particles, like ω and ϕ mesons. Therefore, we cannot deny a kind of existence of virtual particles and hence of the superluminal phenomena associated with them.

Anyway, it seems we have no compelling reason from special relativity against tachyons, alleged superluminal particles, and it is surprising that the first serious papers on tachyons have appeared only in the early sixties of the past century. In 1962, Sudarshan, Bilaniuk and Deshpande, not without the help of personal contacts, published their article *"Meta" Relativity* [59], which became the starting point of serious thinking about tachyons. Quickly enough, this article became famous and has induced many debates and other publications (see [60–62] and references therein). In these publications, it was discussed whether the existence of tachyons is consistent with the theory of relativity and also the formalism for quantum theory of tachyons was developed. The term "tachyon" itself (from the Greek $\tau\alpha\chi\upsilon\varsigma$, meaning "swift") was proposed by Gerald Feinberg in 1967 for particles with a velocity greater than the speed of light [60].

According to these studies, tachyons, bradyons and luxons constitute three independent groups of particles that cannot be converted into each other by Lorentz transformations. Thus, we perceive all particles that move relative to us with a speed lower than the speed of light as bradyons. When accelerating, the velocity of a bradyon increases up to the speed of light but even despite the consumption of any finite amount of energy, never reaches it. Tachyons have their superluminal velocities not due to acceleration but because they are born with $v > c$ velocities, like photons (luxons) are always born with velocity $v = c$. With respect to any system of bradyon observers, tachyons always travel at a speed greater than the speed of light. There is no reference frame, equivalent to our own frame up to a Lorentz transformation, which would be the rest frame for a tachyon, so even in principle, we are not able to make measurements of its mass or proper length. According to the equations of special relativity, the mass and proper length of a tachyon turn out to be imaginary, but this does not contradict the principle that all observable physical quantities must be real because finally we are not able to measure these quantities, and so, they are unobservable.

The principle of causality is also not violated by tachyons much in the same way as it is not violated in quantum field theory thanks to the Feynman–Stueckelberg interpretation of antiparticles. We can conclude then that special relativity does not prohibit tachyons and, therefore, they must exist according to the Gell-Mann's totalitarian principle "everything not forbidden is compulsory" [61] (in fact, this wonderful phrase first appeared in T.H. White's fantasy novel *The Once and Future King* [63]. Sometimes the phrase is erroneously attributed to George Orwell's famous novel *Nineteen Eighty-Four*, see for example [64]. We were unable to find the phrase in the Orwell's novel).

Tachyons were searched but never reliably found [62, 65]. Although there are some observed anomalies in extensive air showers which could be attributed to tachyons [65], the evidence is not conclusive enough. It seems that the Gell-Mann's totalitarian principle fails for tachyons, but why?

The clue for the resolution of this enigma is to realize that the totalitarian principle is about quantum theory and "the break that quantum mechanics introduces in the basic underlying principles that have been working through history in the human thought since immemorial times, is absolute" [66]. The truth is that the Gell-Mann's totalitarian principle does not fail at all and tachyons do exist. However, the meaning of "exist" is quit different from what is usually assumed.

First of all, tachyons exist as virtual particles. In fact, every elementary particle can become tachyonic as a virtual particle. Note that up to now we have emphasized superluminality as a defining property of tachyons. This is justified when we are talking about tachyons in the framework of special relativity, because special relativity is essentially a classical theory, but is no longer justified in quantum theory with its radical distinction from classical concepts. For example, when the evanescent modes in the photon tunneling experiments are considered as virtual photons and claimed that they propagate superluminally, this is not quite correct. Classical concept of propagation velocity is not well-defined for evanescent modes or virtual photons. Nothing well defined and localized propagates through the tunneling barrier passing continuously through every point along the trajectory.

The notion of particles which we have borrowed from the classical physics is also not quite satisfactory. Instead of talking about dubious wave-particle duality which is a concept as incoherent [67] as the devil's pitchfork, a classic impossible figure [68], it is better to accept from the beginning that the objects that we call elementary particles are neither particles nor waves but quantons, some queer objects of the quantum world [69, 70].

The best way to classify elementary quantons is the use of space-time symmetry, where the elementary quantons correspond to the irreducible unitary representations of the Poincaré group [71–73], first given by Wigner [71]. The norm of the energy-momentum four-vector, $P_\mu P^\mu = m^2$, is a Casimir invariant of the Poincaré group and hence its value partially characterizes a given irreducible representation. If $m^2 > 0$, positive energy representations are classified by the mass m and the spin s which comes from the compact stabilizer subgroup SO(3) (or, better, from its double cover SU(2)).

In the massless case, $m = 0$, irreducible representations of the Poincaré group are induced by the Euclidean stabilizer subgroup $E(2)$ which is non-compact and has no finite-dimensional representations other than trivial. The trivial one-dimensional representation of $E(2)$ induces the irreducible representations of the Poincaré group, labeled by the helicity, describing photons and other massless particles.

Usually one discards irreducible representations of the Poincaré group induced by infinite-dimensional representations of $E(2)$ (the so-called continuous spin representations) because the corresponding particles have been never experimentally observed, "but there is no conceptual *a priori* reason not to consider them" [74]. Interestingly, quantons corresponding to continuous spin representations exhibit many tachyonic features though they are not normal tachyons in the sense that they have light-like four-momentum [75]. Wigner's original objection against such "continuous spin tachyons" is that they lead to the infinite heat capacity of the vacuum which can be avoided in the supersymmetric version with its characteristic cancellation between bosons and fermions [76].

Normal tachyonic representations with space-like four-momentum (negative mass squared) appear on the equal footing in the Wigner's classification. However, this fact does not mean that tachyons are as ubiquitous around us as bradyons and luxons. Let us underline that not every quanton (irreducible unitary representation of the Poincaré group) corresponds to localizable objects which can be called particles in the classical sense. Apart from the continuous spin representations mentioned above, we can also refer to the non-trivial vacuum representations of the Poincaré group with zero four-momentum which could correspond to pomerons [77], queer objects in QCD with some particle-like features (one speaks, for example, about pomeron exchange between protons) but, nevertheless, being far away from what is usually meant by a particle.

Superluminality ceases to be a defining property of tachyons in quantum theory. When we realize this, quite a different interpretation of tachyons emerges [78]. In the quantum field theory, to every quanton we associate a field ϕ. The squared mass of the quanton is the second derivative of the self-interaction potential $V(\phi)$ of the field at the origin $\phi = 0$. If the squared mass is negative, then the origin cannot be the minimum of the potential and thus, $\phi = 0$ configuration cannot be a stable vacuum state of the theory. In other words, the system with tachyonic degree of freedom at $\phi = 0$ is unstable and the tachyonic field ϕ will roll down towards the true vacuum. As the true vacuum is the minimum of the self-interaction potential, the squared mass is positive for the true vacuum. Therefore, small excitations of the field ϕ around the true vacuum will appear as ordinary bradyons. In fact, such a scenario is an important ingredient of the Standard Model and is known under the name of Higgs mechanism. The Higgs boson is the most famous would-be tachyon.

Interpreted in such a way, tachyons have an important revival in string theory [78, 79] and in early cosmology [80]. Even the emergence of time in quantum cosmology could be related to tachyons [81].

Summing up, tachyons do exist and play a significant role in modern quantum theory (virtual particles, spontaneous symmetry breaking, string theory). However, tachyons cannot support the true superluminal propagation — the aim of their initial introduction. It can be shown that, even in a rolling state towards the true vacuum, localized disturbances of the tachyonic field never travel superluminally [80]. "Contrary to popular prejudice: *the tachyon is not a tachyon!*" [80].

3. Frenkel–Kontorova solitons

Frenkel–Kontorova model [82, 83] describes a one-dimensional chain of atoms subjected to an external sinusoidal substrate potential. The interactions between the nearest neighbors are assumed to be harmonic. Therefore, the Lagrangian of the model is

$$\mathcal{L} = \sum_n \left\{ \frac{m}{2}\left(\frac{dx_n}{dt}\right)^2 - \frac{k}{2}(x_{n+1} - x_n - l)^2 - \frac{V_0}{2}\left(1 - \cos\left(\frac{2\pi x_n}{l}\right)\right)\right\}, \tag{3}$$

where k is the elastic constant of the interatomic interaction, m is the mass of the atom, V_0 is the amplitude of the substrate potential and l is its spatial period which coincides with the equilibrium distance of the interatomic potential in our assumption. The equation of motion resulting from the Lagrangian (3) is the following

$$m\frac{d^2x_n}{dt^2} - k(x_{n+1} + x_{n-1} - 2x_n) + \frac{\pi V_0}{l}\sin\left(\frac{2\pi x_n}{l}\right) = 0\,. \tag{4}$$

Let us consider the continuum limit of (4) when the length l characterizing the chain discreteness is much smaller in comparison to any relevant length scale under our interest. For this goal, we introduce the continuous variable x instead of the discrete index n with the relation $x = nl$ so that $n \pm 1$ corresponds to $x \pm l$. Besides, let us introduce the displacements of the individual atoms from their equilibrium positions $u_n = x_n - nl$. Note that displacements u_n satisfy the same equation (4) as the coordinates x_n do. In the continuum limit, we can consider u_n as a function of the continuous coordinate x and expand $u_{n\pm1}(t) \equiv u(x \pm l, t)$ in the Taylor series

$$u(x \pm l, t) \approx u(x,t) \pm \frac{\partial u}{\partial x}\, l + \frac{1}{2}\,\frac{\partial^2 u}{\partial x^2}\, l^2\,.$$

Substituting this expansion into equation (4) and introducing the dimensionless field of displacements

$$\Phi = \frac{2\pi u}{l}$$

we get the so-called sine-Gordon equation [84]

$$\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + \frac{1}{\lambda^2}\,\sin\Phi = 0\,, \tag{5}$$

where

$$c = l\sqrt{\frac{k}{m}}\,, \qquad \lambda = \frac{l^2}{\pi}\sqrt{\frac{k}{2V_0}}\,. \tag{6}$$

For small oscillations, $\varPhi \ll 1$, equation (5) turns into the Klein–Gordon equation

$$\frac{1}{c^2}\,\frac{\partial^2 \varPhi}{\partial t^2} - \frac{\partial^2 \varPhi}{\partial x^2} + \frac{1}{\lambda^2}\,\varPhi = 0 \tag{7}$$

describing the relativistic particle with the Compton wavelength λ. If the external potential is switched off, $V_0 \to 0$, then $\lambda \to \infty$ and we get the massless phonons traveling at the speed c. Therefore, c is the sound velocity for the primordial chain of atoms. In presence of the substrate potential, the phonons become massive and move with subsonic velocities (are bradyons).

We can also consider small oscillations around the point $\varPhi = \pi$ which is the point of unstable equilibrium for the substrate potential. Writing $\varPhi = \pi - \varphi$ and assuming $\varphi \ll 1$, we get the equation

$$\frac{1}{c^2}\,\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{\lambda^2}\,\varphi = 0 \tag{8}$$

which has the “wrong” sign of the mass term and describes supersonic phonons (tachyons).

Interestingly, despite the supersonic behavior, (8) does not allow information to be transmitted with the velocity $v > c$. The reason is basically the following [85]: from (8) we have the relation between the frequency ω and the wave number k of the tachyonic excitation

$$\omega = c\sqrt{k^2 - \frac{1}{\lambda^2}}\,.$$

If $k > 1/\lambda$, the tachyonic excitations are stable. But if $k < 1/\lambda$, ω becomes imaginary indicating the onset of instability. However, any sharply localized source of perturbation (information) will have such wave numbers in its Fourier spectrum and, therefore, any local disturbance inevitably will set off instability. Atoms will fall over from their unstable $\varphi = 0$ equilibrium in a domino fashion, the exponentially growing modes of the field φ will quickly make the approximation (8) inadequate and we will have to resort to the full nonlinear equation (5) to understand what is actually happening.

So let us return to equation (5) and try to find its traveling wave solution $\varPhi = f(x - vt)$. Substituting this traveling wave into (5), we find that the function f which determines the profile of the wave satisfies the ordinary differential equation

$$\left(1 - \frac{v^2}{c^2}\right)\frac{d^2 f}{d\xi^2} = \frac{\sin f}{\lambda^2}\,, \tag{9}$$

where $\xi = x - vt$. It is easy to find the first integral of this equation in the form

$$\left(1-\frac{v^2}{c^2}\right)\left(\frac{df}{d\xi}\right)^2=\frac{2}{\lambda^2}(\mu-\cos f)\,, \tag{10}$$

where μ is an arbitrary integration constant. Separation of variables in (10) produces, in general, an elliptic integral

$$\frac{\sigma}{L}\,(x-vt)=\int\limits_{f(0)}^{f}\frac{d\phi}{\sqrt{2(\mu-\cos\phi)}}\,, \tag{11}$$

where $\sigma = \pm 1$ and

$$L=\lambda\sqrt{1-\frac{v^2}{c^2}}\equiv\frac{\lambda}{\gamma}\,. \tag{12}$$

However, if $\mu = 1$, the integral (11) can be calculated in terms of elementary functions and the result is

$$\frac{\sigma}{L}(x-vt)=\ln\tan\frac{f(\xi)}{4}-\ln\tan\frac{f(0)}{4}\,.$$

Introducing x_0 through the relation

$$\tan\frac{f(0)}{4}=\exp\left(-\frac{\sigma}{L}\,x_0\right)$$

we get

$$\frac{\sigma}{L}(x-x_0-vt)=\ln\tan\frac{f(x-vt)}{4}\,.$$

As we see, x_0 can always be eliminated by a suitable choice of the coordinate origin, and so we get the following traveling wave solution of the sine-Gordon equation

$$\varPhi(x,t)=4\arctan\,\exp\left[\frac{\sigma}{L}(x-vt)\right]. \tag{13}$$

It is said that for $\sigma = 1$ we have a kink and for $\sigma = -1$ we have an antikink.

But what about supersonic traveling waves? If $v > c$, then $L = i\tilde{L}$, where

$$\tilde{L}=\lambda\sqrt{\frac{v^2}{c^2}-1}\,, \tag{14}$$

and in the case of $\mu = -1$ relation (11) gives

$$\varPhi(x,t)=\pi-4\arctan\,\exp\left[-\frac{\sigma}{\tilde{L}}(x-vt)\right]. \tag{15}$$

Frank and Merwe call such a tachyonic solution an anti-dislocation [86]. We will call them T-kink (if $\sigma = 1$) and T-antikink (if $\sigma = -1$) to emphasize their tachyonic nature.

In contrast to subsonic kinks, T-kinks are not expected to be stable. The reason is simple to explain [87]. Ground state of the periodic substrate potential of the Frenkel–Kontorova model is degenerated. In fact, we have an infinite number of different vacuum states occurring at $\Phi = 2n\pi$, where n is an integer number. To visualize this situation, imagine a long sheet of slate. Its depressions are just different vacuum states. A kink corresponds to an infinite rope which begins in one depression (vacuum state) and ends up in another neighboring depression (vacuum state). Somewhere in between the rope must climb up the ridge (the maximum of the potential), and then fall again in different valley. The kink is stable because to destroy it you need to throw a rope from one valley to another one so that it ends up completely in one vacuum state. But the rope is infinite and you need infinite amount of energy to perform this task.

The situation with T-kinks is different. T-kink corresponds to a rope which lays on a potential ridge, then somewhere on the ridge it falls in the valley and raises again to the adjacent ridge. It is clear that such a configuration cannot be stable.

4. Emergent relativity

A remarkable fact about the Frenkel–Kontorova solitons is that they exhibit relativistic behavior [84, 88]. For example, it is clear from (13) that the kink is not a point-like object but an extended one and its characteristic length is of the order of L. More precisely, as for a kink ($\sigma = 1$) we have

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\partial\Phi}{\partial x}\,dx = \frac{1}{2\pi}\left(\Phi(\infty,t) - \Phi(-\infty,t)\right) = 1\,,$$

and as $\Phi_x = \frac{\partial\Phi}{\partial x}$ is positive, symmetrically peaked around the center of the kink quantity, we can consider $\Phi_x/2\pi$ as the spatial distribution for the kink [89]. Then center-of-mass coordinate of the kink can be defined as [89]

$$q = \langle x\rangle = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} x\Phi_x\,dx\,, \tag{16}$$

and its length as

$$L_q = \sqrt{\langle x^2\rangle - \langle x\rangle^2}\,. \tag{17}$$

It can be easily found that

$$\Phi_x = \frac{2}{L}\,\frac{1}{\cosh\frac{x-vt}{L}}\,, \tag{18}$$

and

$$\begin{aligned}
\langle x\rangle &= \frac{L}{\pi}\int_{-\infty}^{\infty}\left(y+\frac{vt}{L}\right)\frac{dy}{\cosh y} = vt\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dy}{\cosh y} = vt\,,\\
\langle x^2\rangle &= \frac{L^2}{\pi}\int_{-\infty}^{\infty}\left(y+\frac{vt}{L}\right)^2\frac{dy}{\cosh y} = \frac{L^2}{\pi}\int_{-\infty}^{\infty}\frac{y^2}{\cosh y}\,dy + v^2t^2\,.
\end{aligned} \tag{19}$$

Then, we obtain

$$L_q = L\sqrt{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y^2}{\cosh y}\,dy} \approx 1.57\,L\,, \tag{20}$$

and (12) shows that the kink length, L_q, is submitted to the Lorentz contraction. Interestingly, such length contraction can be observed by naked eyes. See, for example, a strobe photography of a kink traveling in the mechanical model of the sine-Gordon equation in [84], page 244.

Now let us consider the energy of the kink. From

$$E = \sum_n\left\{\frac{m}{2}\left(\frac{dx_n}{dt}\right)^2 + \frac{k}{2}(x_{n+1}-x_n-l)^2 + \frac{V_0}{2}\left(1-\cos\left(\frac{2\pi x_n}{l}\right)\right)\right\}$$

we get in the continuum limit that

$$E = \frac{V_0}{4}\int_{-\infty}^{\infty}\left[\frac{\lambda^2}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2 + \lambda^2\left(\frac{\partial\Phi}{\partial x}\right)^2 + 2(1-\cos\Phi)\right]\frac{dx}{l}\,, \tag{21}$$

where c and λ are given by (6). For kink (13) we have

$$\frac{\partial\Phi}{\partial t} = -\frac{2v}{L}\,\frac{1}{\cosh\frac{x-vt}{L}}\,,\qquad \frac{\partial\Phi}{\partial x} = \frac{2}{L}\,\frac{1}{\cosh\frac{x-vt}{L}} \tag{22}$$

and

$$2(1-\cos\Phi) = \frac{16\tan^2\frac{\Phi}{4}}{\left(1+\tan^2\frac{\Phi}{4}\right)^2} = \frac{4}{\cosh^2\frac{x-vt}{L}}\,. \tag{23}$$

Substituting (22) and (23) into (21) and using the identity

$$1+\frac{\lambda^2}{L^2}+\frac{v^2}{c^2}\,\frac{\lambda^2}{L^2}=2\gamma^2$$

we get

$$E=2\gamma^2\,\frac{L}{l}\,V_0\int\limits_{-\infty}^{\infty}\frac{dy}{\cosh^2 y}=4\,\frac{\lambda}{l}\,V_0\gamma\,. \tag{24}$$

Thus, we end up with the relativistic relationship between the energy and the mass $E=Mc^2\gamma$, where the mass of the kink is

$$M=4\,\frac{\lambda}{l}\,\frac{V_0}{c^2}=\frac{2}{\pi^2}\,\frac{l}{\lambda}\,m=\frac{2m}{\pi}\sqrt{\frac{2V_0}{kl^2}}\,. \tag{25}$$

This striking analogy between the energy of a moving single dislocation and the energy of a particle in relativistic mechanics was first discovered by Frenkel and Kontorova [86].

In the same way, in the case of T-kink we get tachyonic relations for the T-kink length and energy

$$L_q=L_{q0}\sqrt{\frac{v^2}{c^2}-1}\,,\qquad E=\frac{Mc^2}{\sqrt{\frac{v^2}{c^2}-1}}\,, \tag{26}$$

where $L_{q0}\approx 1.57\lambda$ and the mass M is given again by equation (25). As we see, T-kinks can be considered as a mechanical model for tachyons [86]. Note that in (26) it is assumed that the T-kink energy is measured with respect to the potential ridge so that we have

$$E=\frac{V_0}{4}\int\limits_{-\infty}^{\infty}\left[\frac{\lambda^2}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2+\lambda^2\left(\frac{\partial\Phi}{\partial x}\right)^2-2(1+\cos\Phi)\right]\frac{dx}{l} \tag{27}$$

instead of (21).

It is clear from (26) that upon loss of energy a T-kink will accelerate and become wider and wider. This paradoxical property of superluminal particles was deduced already by Sommerfeld just before the advent of special relativity [61], and it points once again towards a transient and unstable nature of T-kinks.

There is nothing particularly unexpected in emergence of the relativistic relationships considered above because the sine-Gordon equation (5) is Lorentz invariant excepting the fact that the light velocity is replaced by the sound velocity c.

It is though really remarkable that this relativistic invariance is an emergent phenomenon. It is absent at the fundamental level (the Lagrangian (3) is not Lorentz invariant) but appears in the long-wavelength limit.

Emergent relativity in the Frenkel–Kontorova model is approximate and holds only insofar as we can neglect discreteness effects. Let us return to equation (4) and rewrite it in the following way

$$\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2}-\frac{\Phi(x+l)+\Phi(x-l)-2\Phi(x)}{l^2}+\frac{1}{\lambda^2}\,\sin\Phi=0\,.$$

Using the Taylor expansion of the form [91]

$$\Phi(x\pm l)=e^{\pm l\partial_x}\,\Phi(x)\,,$$

where

$$\partial_x=\frac{\partial}{\partial x}\,,$$

we get

$$\Phi(x+l)+\Phi(x-l)-2\Phi(x)=2\left[\cosh\left(l\partial_x\right)-1\right]\Phi(x)\,. \tag{28}$$

But

$$\cosh x\approx 1+\frac{x^2}{2}+\frac{x^4}{24}$$

and (28) then gives

$$\Phi(x+l)+\Phi(x-l)-2\Phi(x)=l^2\,\left[1+\frac{l^2}{12}\,\partial_x^2\right]\partial_x^2\Phi(x)\,. \tag{29}$$

Therefore, we get the equation

$$\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2}-\left(1+\frac{l^2}{12}\,\partial_x^2\right)\frac{\partial^2\Phi}{\partial x^2}+\frac{1}{\lambda^2}\,\sin\Phi=0\,. \tag{30}$$

However, this equation is not convenient for considering the discreteness effects [91, 92]. For example, it contains the forth derivative of Φ with respect to the spatial coordinate x and, hence, necessitates additional boundary conditions at the ends of the chain absent in the original discrete formulation or in the zeroth-order continuum approximation. The remedy against this drawback is simple [92]. Let us multiply (30) by

$$\left(1+\frac{l^2}{12}\,\partial_x^2\right)^{-1}\approx 1-\frac{l^2}{12}\,\partial_x^2\,.$$

After some rearranging, we get the equation which correctly and conveniently reproduces the first-order effects produced by the chain discreteness [83]

$$\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2}-\frac{\partial^2\Phi}{\partial x^2}+\frac{1}{\lambda^2}\,\sin\Phi=\frac{l^2}{12\lambda^2}\left[\frac{\lambda^2}{c^2}\,\frac{\partial^4\Phi}{\partial t^2\partial x^2}+\cos\Phi\,\frac{\partial^2\Phi}{\partial x^2}-\sin\Phi\left(\frac{\partial\Phi}{\partial x}\right)^2\right]. \tag{31}$$

From (22), it is clear that every derivative of Φ brings the $2\gamma/\lambda$ factor with it. Therefore, the first term in the r.h.s. of (31) is the leading one in the high energy limit and compared to the first term in the l.h.s., it contains an extra smallness of the order of

$$\frac{l^2\gamma^2}{3\lambda^2}=\frac{1}{3}\left(\frac{\pi^2M\gamma}{2m}\right)^2\sim\left(\frac{\pi Mc^2\gamma}{mc^2}\right)^2.$$

As we see, Lorentz violation remains small if the kink energy $Mc^2\gamma$ is small in comparison to the "Plank energy" $E_{\mathrm{P}}=mc^2/\pi$.

In real life, much more significant Lorentz symmetry violation for mechanical kinks is caused by dissipation what brings the $\beta\Phi_t$ term in the equation of motion. For example, the Lorentz contraction of the kink width is prominent only if $\gamma\ll 1/\beta$ and it saturates at a value proportional to β [89] in the limit of high energies.

Interestingly, the breakdown of Lorentz contraction may happen even in Lorentz invariant theory due to quantum-field theory effects [93–95] and, hence, it alone does not signal a breakdown of special relativity. The size of an object is a classical concept and it cannot be unambiguously extended on quantum domain. In QCD, for example, the size of the region which contains the information necessary to identify a hadron is determined by fast partons and undergoes Lorentz contraction as expected, while the low momentum parton cloud is universal and also determines the reasonable notion of the size of the hadron which however does not Lorentz contract [94]. This leads to a very counterintuitive picture of a fast-moving nucleus being much thinner than any of its constituent nucleons thus grossly violating our classical expectation that the size of a system is always larger than the size of constituents from which the system is built [95].

5. Supersonic solitons

The emergent relativity in the Frenkel–Kontorova model is not universal in the sense that it is applied only to the excitations of the considered chain and does not encompass, for example, the dynamics of the substrate atoms. This fact allows us to arrange solitons whose behavior is not restricted by relativistic laws. Let us consider, for example, a one-dimensional chain of substrate atoms with exponential interatomic interactions so that the Lagrangian of the model is [88, 96]

$$\mathcal{L} = \sum_n \left\{ \frac{m}{2} \left(\frac{du_n}{dt} \right)^2 - \frac{k}{b} \left\{ u_n - u_{n-1} + \frac{1}{b} \left[e^{-b(u_n - u_{n-1})} - 1 \right] \right\} \right\} . \quad (32)$$

Here again $u_n = x_n - nl$ and m, l, k, b are some constants. We use the same notations m, l, k as in the Frenkel–Kontorova model even though numerical values of these physical quantities may be different. The equation of motion that follows from this Lagrangian is then

$$m \frac{d^2 u_n}{dt^2} + \frac{k}{b} \left[e^{-b(u_{n+1} - u_n)} - e^{-b(u_n - u_{n-1})} \right] = 0 . \quad (33)$$

Note that the case of small b corresponds to the harmonic interatomic interactions.

To find a solitonic solution of (33), we can proceed as follows [97]. Let us introduce dimensionless variables w_n and τ through relations

$$\tau = \sqrt{\frac{k}{m}}\, t \quad (34)$$

and

$$1 + \dot{w}_n = e^{-b(u_n - u_{n-1})} , \qquad w_n - w_{n+1} = b\, \dot{u}_n . \quad (35)$$

Here the dot indicates differentiation with respect to τ so that

$$\dot{u}_n = \sqrt{\frac{m}{k}} \frac{du_n}{dt} .$$

By differentiation of the second equation in (35) with respect to time t and with help of the first one, it is easy to check that u_n satisfies indeed the original equation (33). On the other hand, we have an equation for w_n following from (35)

$$\frac{\ddot{w}_n}{1 + \dot{w}_n} = w_{n+1} + w_{n-1} - 2w_n \quad (36)$$

and we can find its solitonic solution by a Bäcklund transformation [97].

In differential geometry, the Bäcklund transformation enables the construction of a new pseudospherical surface (a surface with a constant and negative Gaussian curvature) from a given pseudospherical surface. Technically, the Bäcklund transformation is a pair of first order partial differential equations which relate two different solutions of the second order partial differential equations. This transformation has important applications in soliton theory [98].

Toda and Wadati extended the idea to a differential-difference equations and obtained a discrete analog of Bäcklund transformation for the exponential lattice [99]. For equation (36) the Bäcklund transformation was found in [97] and it has the form

$$\begin{aligned} 1+\dot{w}_n &= \left(\lambda+w'_n-w_n\right)\left(\lambda+w_n-w'_{n+1}\right), \\ 1+\dot{w}'_n &= \left(\lambda+w'_n-w_n\right)\left(\lambda+w_{n-1}-w'_n\right), \end{aligned} \tag{37}$$

where λ is an arbitrary constant. If $w_n(t)$ and $w'_n(t)$ are any two functions related by (37), they both are solutions of the equation (36). For example, we have from (37)

$$\frac{\ddot{w}'_n}{1+\dot{w}'_n}=\frac{d}{d\tau}\ln\left(1+\dot{w}'_n\right)=\frac{\dot{w}'_n-\dot{w}_n}{\lambda+w'_n-w_n}+\frac{\dot{w}_{n-1}-\dot{w}'_n}{\lambda+w_{n-1}-w'_n}\,. \tag{38}$$

However, again from (37)

$$\begin{aligned} \dot{w}'_n-\dot{w}_n &= \left(\lambda+w'_n-w_n\right)\left(w_{n-1}-w'_n-w_n+w'_{n+1}\right), \\ \dot{w}_{n-1}-\dot{w}'_n &= \left(\lambda+w_{n-1}-w'_n\right)\left(w'_{n-1}-w_{n-1}-w'_n+w_n\right). \end{aligned} \tag{39}$$

Substituting (39) into (38), we see that w'_n is indeed a solution of (36)

$$\frac{\ddot{w}'_n}{1+\dot{w}'_n}=w'_{n+1}+w'_{n-1}-2w'_n\,.$$

Let $w'_n=0$ be a trivial solution of (36). Then (37) takes the form

$$1+\dot{w}_n=\lambda^2-w_n^2\,,\qquad 1=(\lambda-w_n)(\lambda+w_{n-1})\,. \tag{40}$$

We further assume that $\lambda^2\geq 1$ so we can write $\lambda=\pm\cosh\phi$ for some ϕ. The first equation in (40) can easily be integrated then

$$w_n=\sinh\phi\,\tanh\left[\tau\,\sinh\phi+\alpha_n\right], \tag{41}$$

where α_n is the integration constant which is the only quantity in (41) that can depend on n. Using (41) and the identity $\tanh x-\tanh y=\tanh\left(x-y\right)\left[1-\tanh x\,\tanh y\right]$, we obtain that the second equation of (40) can be rewritten in the form

$$\begin{aligned} &\sinh^2\phi= \\ &\sinh^2\phi\,\tanh x\,\tanh y-\sinh\phi\,\cosh\phi\,\tanh\left(\alpha_n-\alpha_{n-1}\right)\left[1-\tanh x\,\tanh y\right], \end{aligned}$$

where $x=\tau\,\sinh\phi+\alpha_n$, $y=\tau\,\sinh\phi+\alpha_{n-1}$ and for definiteness we have taken $\lambda=-\cosh\phi$. This identity must be valid for any τ. It is possible only if

$$\cosh\phi\,\tanh\left(\alpha_n-\alpha_{n-1}\right)=-\sinh\phi\,.$$

Consequently, $\alpha_n - \alpha_{n-1} = -\phi$ what implies $\alpha_n = -n\phi + \alpha_0$. Finally, we get the following nontrivial solution of (36)

$$w_n = \sinh\phi\, \tanh\left[\tau\, \sinh\phi - n\phi + \alpha_0\right]. \tag{42}$$

In the continuum limit with $x = nl$, we have

$$w(x,t) = \sinh\frac{l}{L}\, \tanh\frac{vt - x + x_0}{L}\,, \tag{43}$$

where

$$L = \frac{l}{\phi}\,, \qquad x_0 = \alpha_0 L\,, \qquad v = c\,\frac{L}{l}\,\sinh\frac{l}{L}\,, \tag{44}$$

and $c = \sqrt{k/m}\, l$ is the sound velocity for the harmonic chain (in the limit $b \to 0$). Note that the Toda soliton (43) is supersonic, $v > c$, since $\sinh x > x$ for any $x > 0$.

It is clear from (43) that the soliton width is of the order of L. The continuum approximation assumes $L \gg l$, then the soliton is only slightly supersonic

$$\left(\frac{v}{c}\right)^2 = \frac{\sinh^2\phi}{\phi^2} = \frac{1}{2\phi^2}\,(\cosh 2\phi - 1) \approx 1 + \frac{1}{3}\frac{l^2}{L^2}\,,$$

and its width depends on the velocity as follows

$$L = \frac{l/\sqrt{3}}{\sqrt{\frac{v^2}{c^2} - 1}}\,. \tag{45}$$

From (35) we get in the continuum limit when $w \ll 1$

$$\frac{du_n}{dt} \approx -\frac{c}{b}\,\frac{\partial w}{\partial x}\,, \qquad u_n - u_{n-1} \approx -\frac{1}{b}\sqrt{\frac{m}{k}}\,\frac{\partial w}{\partial t} = \frac{v}{c}\,\frac{l}{b}\,\frac{\partial w}{\partial x}\,. \tag{46}$$

Therefore, in the harmonic approximation for the potential energy, the energy of the soliton is

$$\begin{aligned} E &\approx \frac{m}{2b^2}\left(v^2 + c^2\right)\int\limits_{-\infty}^{\infty}\left(\frac{\partial w}{\partial x}\right)^2\frac{dx}{l} \\ &= \frac{m}{2b^2}\left(v^2 + c^2\right)\frac{l}{L^3}\int\limits_{-\infty}^{\infty}\frac{dy}{\cosh^4 y} = \frac{2l}{3b^2L^3}\, m\left(v^2 + c^2\right). \end{aligned} \tag{47}$$

As we see from (45) and (47), when the velocity of the Toda soliton approaches the sound velocity, its energy turns to zero and its width turns to infinity. Such a behavior is opposite to that of a tachyon but the result is the same: the Toda soliton cannot cross the sound barrier and become subsonic. But there are other types of solitons which can: generalized Frenkel–Kontorova model with the special kind of anharmonicity is a specific example [100]. The Lagrangian of the model is [83, 90, 100]

$$\begin{aligned} \mathcal{L} &= \sum_n \left\{ \frac{m}{2}\left(\frac{du_n}{dt}\right)^2 - \frac{k}{2}(u_{n+1}-u_n)^2 \right. \\ &\left. \times \left[1+\frac{\chi}{l^2}\left(u_{n+1}-u_n\right)^2\right] - \frac{V_0}{2}\left(1-\cos\left(\frac{2\pi u_n}{l}\right)\right)\right\}, \end{aligned} \tag{48}$$

where χ is a dimensionless anharmonicity parameter. Correspondingly, the equations of motion are

$$\begin{aligned} & m\frac{d^2u_n}{dt^2} - k(u_{n+1}+u_{n-1}-2u_n) - \frac{2k\chi}{l^2}\left[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3\right] \\ & +\frac{\pi V_0}{l}\sin\left(\frac{2\pi x_n}{l}\right) = 0\,. \end{aligned} \tag{49}$$

To get the continuum limit, let us expand these equations up to terms of the order of l^5 (assuming that u_n and its spatial derivatives are of the order of l)

$$\begin{aligned} u_{n+1}+u_{n-1}-2u_n &\approx l^2\,\frac{\partial^2 u_n}{\partial x^2} + \frac{l^4}{12}\,\frac{\partial^4 u_n}{\partial x^4}\,, \\ \frac{1}{l^2}\left[(u_{n+1}-u_n)^3-(u_n-u_{n-1})^3\right] &\approx 3l^2\left(\frac{\partial u_n}{\partial x}\right)^2\frac{\partial^2 u_n}{\partial x^2}\,. \end{aligned} \tag{50}$$

Therefore, the continuum limit of (49) is given by

$$\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} - \frac{l^2}{12}\,\frac{\partial^4\Phi}{\partial x^4} - \frac{3\chi l^2}{2\pi^2}\left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2} + \frac{1}{\lambda^2}\sin\Phi = 0\,, \tag{51}$$

where, as before, Φ is given by

$$\Phi = \frac{2\pi u}{l}\,.$$

For the special value of the anharmonicity parameter χ, equation (51) has solitonic solutions of the same functional form (13) as in the harmonic case. But the velocity dependence of L is not given by (12) and has a more complicated form. Indeed, (22) and (23) indicate that we have the following relations

$$\frac{\partial^2\Phi}{\partial t^2} = v^2\,\frac{\partial^2\Phi}{\partial x^2}\,, \qquad \frac{\partial^2\Phi}{\partial x^2} = \frac{\sin\Phi}{L^2}\,, \qquad \left(\frac{\partial\Phi}{\partial x}\right)^2 = \frac{2(1-\cos\Phi)}{L^2}\,,$$

from which we get

$$\begin{aligned} \frac{\partial^4 \Phi}{\partial x^4} &= -\frac{2\sin\Phi}{L^4} + \frac{3\sin 2\Phi}{2L^4}\,, \\ \left(\frac{\partial \Phi}{\partial x}\right)^2 \frac{\partial^2 \Phi}{\partial x^2} &= \frac{2\sin\Phi}{L^4} - \frac{\sin 2\Phi}{L^4}\,. \end{aligned} \tag{52}$$

It follows from (52) that if

$$\chi = \frac{\pi^2}{12}\,, \tag{53}$$

then

$$\frac{l^2}{12}\frac{\partial^4 \Phi}{\partial x^4} + \frac{3\chi l^2}{2\pi^2}\left(\frac{\partial \Phi}{\partial x}\right)^2 \frac{\partial^2 \Phi}{\partial x^2} = \frac{l^2}{12}\frac{\sin\Phi}{L^4}\,,$$

and (13) will be a solution of (51) if

$$\frac{v^2/c^2 - 1}{L^2} - \frac{l^2}{12L^4} + \frac{1}{\lambda^2} = 0\,,$$

or

$$L^4 - \left(1 - \frac{v^2}{c^2}\right)\lambda^2 L^2 - \frac{l^2\lambda^2}{12} = 0\,. \tag{54}$$

The positive solution of (54) is

$$L^2 = \frac{\lambda^2}{2}\left[1 - \frac{v^2}{c^2} + \sqrt{\left(1 - \frac{v^2}{c^2}\right)^2 + \frac{1}{3}\left(\frac{l}{\lambda}\right)^2}\,\right]. \tag{55}$$

Note that (55) remains finite and non-zero at $v = c$. Therefore, there is no sonic barrier for this type of solitons (Kosevich–Kovalev solitons). In a sense, Kosevich–Kovalev soliton interpolates between the subsonic Frenkel–Kontorova solitons and supersonic Toda solitons [83, 90]. Indeed, for $v \gg c$ we get from (55)

$$L = \frac{l/2\sqrt{3}}{\sqrt{\frac{v^2}{c^2} - 1}}$$

which is half the width of the Toda soliton. While for $l/\lambda \ll 1$ and $v \ll c$ we have the same width as for the Frenkel–Kontorova solitons

$$L = \lambda\sqrt{1 - \frac{v^2}{c^2}}\,.$$

In contrast to Frenkel–Kontorova and Toda solitons, Kosevich–Kovalev solitons can move with any velocity from zero to infinity.

6. Elvisebrions

We believe “that theory acquires authority by confronting and conforming to experiment, not the other way around” [101]. To quote Richard Feynman, “it does not make any difference how beautiful your guess is. It does not make any difference how smart you are, who made the guess, or what his name is — if it disagrees with experiment it’s wrong” [102]. Special relativity is an idea that was scrutinized experimentally many times and always found to be conforming to experiment. However, “history of physics shows that with the unique exception of current laws and theories, all previous hypotheses have been surpassed by the new order introduced and that, subsequently, they have been proved wrong or limited in some way or another” [66]. Why should special relativity be an exception?

Frenkel–Kontorova model is a simple mechanical example which hints toward a possibility that special relativity might be actually an emergent phenomenon: valid only when things are inspected at relevant scales but disappears at finer scales. In the realistic Frenkel–Kontorova model, relativity disappears both in the short wave-length limit (due to discreteness effects) and in the very long-wave-length limit (due to finiteness of the chain). Interestingly, superfluid ^{3}He–A provides another and even more interesting and realistic example, where the relativistic quantum field theory emerges as the effective theory in the low energy corner but, at the same time, the limiting behaviour for high and ultralow energies contradicts special relativity theory [103].

There are several reasons for why we should take the idea of emergent relativity seriously. The Frenkel–Kontorova model is just only one example of a relativistic behavior which emerges in purely classical-mechanical systems [88]. In quantum world such examples proliferate. All ingredients of the Standard Model, such as chiral fermions, Lorentz symmetry, gauge invariance, chiral anomaly, have their counterparts as emergent phenomena in condensed matter physics [104]. Last but not least, it seems that emergence of hierarchies of laws is the basic principle of Nature’s functionality [101, 105]. All our experience in physics confirms this basic principle, especially in condensed matter physics “where theoretical ideas are forced to immediate and brutal confrontation with experiment by virtue of the latter’s low cost” [101].

However, if the special relativity is indeed an emergent phenomenon then there may exist a “substrate” whose excitations do not belong to the relativistic world and, therefore, can move superluminally. It is clear that such type of superluminal particle-like excitations of the substrate, analogs of Toda or Kosevich–Kovalev solitons, are conceptually different from tachyons and deserve their own name. We name them “elvisebrions” (ელვისებრი — elvisebri in Georgian means “swift as a lightning flash”. Hopefully, admirers of the Elvis Presley music will also appreciate the name).

Giving a name to something already brings it into a kind of existence, "it is made at least virtually real" [106]. However, is this existence more substantial than that of unicorns? Only experiment can tell. We believe that it is at least worthwhile to continue the search of superluminal particles. Probability of success is hard to estimate, but we can refer to Alvarez principle to justify such a research (the argument is taken from [107], where it was applied to the search of tachyons). The Alvarez principle relates the merit of an experiment, μ, to the probability of its success, P, and to the significance of the result, σ, in the following way: $\mu = \sigma \cdot P$. We suspect that most physicists, in their sound mind, will insist that for elvisebrions $P = 0$. Nevertheless, they probably will agree that in the case of positive result, $\sigma = \infty$, and in Calculus $0 \cdot \infty$ is indeterminate. In the case of indeterminate μ, everything rests on "the gumption of the experimenters" [107].

To be a bit more specific what we have in mind when speaking about elvisebrions, let us consider the dynamics of interacting Frenkel–Kontorova and Kosevich–Kovalev chains given by the Lagrangian

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\rm int}\,, \tag{56}$$

where $\mathcal{L}_1$ is given by (3) with changes $m \to m_1$, $k \to k_1$, $V_0 \to V_1$ and $\mathcal{L}_2$ is given be (48) with changes $m \to m_2$, $k \to k_2$, $V_0 \to V_2$ and $u_n \to v_n = y_n - nl$, while

$$\mathcal{L}_{\rm int} = -\frac{V_0}{2}\left(1 - \cos\left(\frac{2\pi(x_n - y_n)}{l}\right)\right). \tag{57}$$

In the long-wavelength limit, and assuming that

$$\frac{k_1}{k_2} = \frac{m_1}{m_2} = \frac{V_1}{V_2} = \frac{V_0}{V_1}\,, \tag{58}$$

we get the following system of coupled equations

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} + \frac{1}{\lambda^2}\sin\Phi = \frac{1}{\lambda^2}\frac{m_1}{m_2}\sin\left(\Psi - \Phi\right),\\
&\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} - \frac{l^2}{12}\frac{\partial^4\Psi}{\partial x^4} - \frac{l^2}{8}\left(\frac{\partial\Psi}{\partial x}\right)^2\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{\lambda^2}\sin\Psi\\
&= \frac{1}{\lambda^2}\left(\frac{m_1}{m_2}\right)^2\sin\left(\Phi - \Psi\right),
\end{aligned} \tag{59}$$

where Ψ is the dimensionless field of y-displacements defined analogously to Φ, and

$$c = l\sqrt{\frac{k_1}{m_1}}\,, \qquad \lambda = \frac{l^2}{\pi}\sqrt{\frac{k_1}{2V_1}}\,. \tag{60}$$

Now suppose $m_1 \ll m_2$. Then, in the zeroth approximation the dynamics of chains decouple and Φ-excitations live in a relativistic sine-Gordon world unaware of the existence of a hidden elvisebrion Ψ-sector which is not Lorentz invariant. In the first approximation, however, the decoupling is not complete and while we do still have the Lorentz invariant $\Psi = 0$ sector and no possibility for Φ-inhabitants of this sector to excite (in this approximation) Ψ degrees of freedom, the opposite is not correct: Ψ-excitations are coupled (albeit weakly) to the Φ-sector and, therefore, their presence can be detected by Φ-inhabitants.

Note that the system (59) still has a supersonic solution

$$\begin{aligned}\Psi(x,t) &= 4\arctan\exp\left[\frac{\sigma}{L}(x-vt)\right],\\ \Phi(x,t) &= \pi - 4\arctan\exp\left[-\frac{\sigma}{L}(x-vt)\right]\\ &= -\pi + 4\arctan\exp\left[\frac{\sigma}{L}(x-vt)\right]\end{aligned} \tag{61}$$

provided that

$$L^2 = \lambda^2\left(\frac{v^2}{c^2}-1\right) = \frac{\lambda^2}{2}\left[1-\frac{v^2}{c^2}+\sqrt{\left(1-\frac{v^2}{c^2}\right)^2+\frac{1}{3}\left(\frac{l}{\lambda}\right)^2}\,\right]$$

which for the velocity v gives

$$\frac{v^2}{c^2} = 1 + \frac{1}{\sqrt{24}}\,\frac{l}{\lambda}\,. \tag{62}$$

It is also interesting to consider an opposite limit $m_1 \gg m_2$ of (59). To simplify the treatment, we neglect the anharmonicity this time while relaxing the (58) condition to the following one (introducing a dimensionless parameter ϵ)

$$\epsilon^2\,\frac{k_1}{k_2} = \frac{m_1}{m_2} = \frac{V_1}{V_2} = \frac{V_0}{V_1}\,. \tag{63}$$

Then, in the limit $m_1 \gg m_2$, we obtain a system of coupled sine-Gordon equations

$$\begin{aligned}\frac{1}{c^2}\,\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial x^2} &= \frac{1}{\lambda^2}\,\frac{m_1}{m_2}\,\sin\left(\Psi-\Phi\right),\\ \frac{1}{(\epsilon c)^2}\,\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} &= \frac{1}{\lambda^2}\left(\frac{m_1}{\epsilon m_2}\right)^2\sin\left(\Phi-\Psi\right).\end{aligned} \tag{64}$$

Note the physical meaning of the parameter ϵ

$$\epsilon = \frac{c_2}{c_1}\,, \tag{65}$$

where c_1 and c_2 are sound velocities associated to the individual Frenkel–Kontorova chains.

It will be convenient to introduce dimensionless variables instead of t and x through

$$\tilde{x} = \frac{x}{\lambda}\frac{m_1}{m_2}\,, \qquad \tilde{t} = \frac{ct}{\lambda}\frac{m_1}{m_2}\,. \tag{66}$$

Then, we get from (64)

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial \tilde{t}^2} - \frac{\partial^2 \Phi}{\partial \tilde{x}^2} &= -\delta^2 \sin\left(\Phi - \Psi\right), \\ \frac{\partial^2 \Psi}{\partial \tilde{t}^2} - \epsilon^2\,\frac{\partial^2 \Psi}{\partial \tilde{x}^2} &= \sin\left(\Phi - \Psi\right), \end{aligned} \tag{67}$$

where

$$\delta^2 = \frac{m_2}{m_1}\,. \tag{68}$$

The system (67) is a generalization of the Frenkel–Kontorova model of crystal dislocations [108]. Indeed, in the limit $\delta \to 0$ ($m_1 \gg m_2$) and for $\Phi = 0$, we obtain for Ψ the sine-Gordon equation for the long-wavelength description of the displacements of the particles of mass m_2 while treating the much heavier particles of mass m_1 as motionless. Interestingly, the system (67) with $\epsilon = 1$ was suggested to describe soliton excitations in deoxyribonucleic acid (DNA) double helices [109]. In the general case $\epsilon \neq 1$, $\delta \neq 0$, solutions of (67) were considered in [108, 110, 111]. When $\epsilon \neq 1$ (that is when c_1 and c_2 sound velocities are different), the coupled system (67) is not Lorentz invariant. As a result, traveling wave solutions do appear with supersonic velocities, as we now demonstrate, following [108].

Let us seek the solution of (67) in the form

$$\Phi\left(\tilde{x}, \tilde{t}\,\right) = A\,p(\xi)\,, \qquad \Psi\left(\tilde{x}, \tilde{t}\,\right) = B\,p(\xi)\,, \tag{69}$$

where $\xi = \tilde{x} - \nu\tilde{t}$, and ν, A, B are some constants such that

$$A - B = 1\,. \tag{70}$$

Substituting (69) into (67), we get

$$\begin{aligned} A\left(\nu^2 - 1\right)\frac{d^2 p}{d\xi^2} &= -\delta^2 \sin p\,, \\ B\left(\nu^2 - \epsilon^2\right)\frac{d^2 p}{d\xi^2} &= \sin p\,. \end{aligned} \tag{71}$$

Therefore,

$$\frac{A\left(\nu^2 - 1\right)}{B\left(\nu^2 - \epsilon^2\right)} = -\delta^2\,, \tag{72}$$

and this relation together with (70), determines A and B constants as

$$A=\frac{\delta^2\left(\nu^2-\epsilon^2\right)}{\delta^2\left(\nu^2-\epsilon^2\right)+\nu^2-1}\,,\qquad B=\frac{1-\nu^2}{\delta^2\left(\nu^2-\epsilon^2\right)+\nu^2-1}\,,\tag{73}$$

while for $p(\xi)$ we get the equation

$$\frac{d^2p}{d\xi^2}+\mu\,\sin p=0\,,\tag{74}$$

where

$$\mu=\frac{\delta^2\left(\nu^2-\epsilon^2\right)+\nu^2-1}{\left(\nu^2-\epsilon^2\right)\left(\nu^2-1\right)}=\frac{\delta^2}{\nu^2-1}+\frac{1}{\nu^2-\epsilon^2}\,.\tag{75}$$

We will assume $\epsilon<1$ and first consider the case $\mu>0$ which is only possible if

$$\epsilon^2<\nu^2<\frac{1+\delta^2\,\epsilon^2}{1+\delta^2}\qquad\text{or}\qquad 1<\nu^2<\infty\,.\tag{76}$$

Assuming

$$\frac{dp}{d\xi}(\xi=0)=0\qquad\text{and}\qquad p(\xi=0)=p_0$$

we have from (74) the following first integral

$$\frac{1}{2}\left(\frac{dp}{d\xi}\right)^2=\mu(\cos p-\cos p_0)=2\mu\left(\sin^2\frac{p_0}{2}-\sin^2\frac{p}{2}\right)\,.$$

Therefore,

$$\int\limits_{p_0}^{p}\frac{d(p/2)}{\sqrt{\sin^2\frac{p_0}{2}-\sin^2\frac{p}{2}}}=\sigma\sqrt{\mu}\,\xi\tag{77}$$

with $\sigma=\pm1$. Let us make the following substitution in the integral

$$\sin\frac{p}{2}=k\,\sin\phi\,,\qquad k=\sin\frac{p_0}{2}\,.$$

Then, we get

$$\int\limits_{\pi/2}^{\phi}\frac{d\phi}{\sqrt{1-k^2\,\sin^2\phi}}=\sigma\sqrt{\mu}\,\xi$$

and, therefore,

$$\mathrm{sn}^{-1}\left(\frac{1}{k}\,\sin\frac{p}{2},k\right)-K(k)=\sigma\sqrt{\mu}\,\left(\tilde{x}-\nu\tilde{t}\right)\,,\tag{78}$$

where $\mathrm{sn}(u,k)$ is one of the Jacobian elliptic functions (for elementary theory of elliptic functions see, for example, [112]) and $K(k)$ is the complete elliptic integral of the first kind. It is clear from (78) that $K(k)$ can be absorbed by the redifinition of the spatial origin and we ignore it in the following.

Therefore, the case $\mu > 0$ corresponds to the fast traveling waves [108] which can propagate with velocity $v = \nu c \equiv \nu c_1$ in the range

$$c_2^2 < v^2 < \frac{m_1 c_1^2 + m_2 c_2^2}{m_1 + m_2} \qquad \text{or} \qquad c_1^2 < v^2 < \infty\,, \tag{79}$$

and have the form (we have switched back from dimensionless quantities)

$$\begin{aligned}
\Phi(x,t) &= \frac{2\,\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}\ \arcsin\left[k\,\mathrm{sn}\left(\frac{\sigma}{\tilde{L}}(x-vt),\,k\right)\right],\\
\Psi(x,t) &= \frac{2\left(1-\frac{v^2}{c^2}\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}\ \arcsin\left[k\,\mathrm{sn}\left(\frac{\sigma}{\tilde{L}}(x-vt),\,k\right)\right],
\end{aligned} \tag{80}$$

where

$$\tilde{L} = \frac{\lambda\delta^2}{\sqrt{\mu}} = \lambda\delta^2\sqrt{\frac{\left(\frac{v^2}{c^2}-\epsilon^2\right)\left(\frac{v^2}{c^2}-1\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}}\,. \tag{81}$$

Now we consider the case $\mu < 0$, that is (remember that we have assumed $\epsilon < 1$)

$$0 \le \nu^2 < \epsilon^2 \qquad \text{or} \qquad \frac{1+\delta^2\,\epsilon^2}{1+\delta^2} < \epsilon^2 < 1\,. \tag{82}$$

If we take $p = \tilde{p} - \pi$, then the equation for $\tilde{p}$ will be

$$\frac{d^2\tilde{p}}{d\xi^2} + (-\mu)\,\sin\tilde{p} = 0\,. \tag{83}$$

That is for $\tilde{p}$ we have the previously discussed case $\tilde{\mu} = -\mu > 0$ and, therefore, we already know the solution of (83). Using the relation $\mathrm{dn}^2(u,k) = 1 - k^2\mathrm{sn}^2(u,k)$ among Jacobi elliptic functions, we finally get a slow traveling wave solution which can propagate with velocities in the range

$$0 \le v^2 < c_2^2 \qquad \text{or} \qquad \frac{m_1 c_1^2 + m_2 c_2^2}{m_1 + m_2} < v^2 < c_1^2\,, \tag{84}$$

and have the form

$$\begin{aligned}\Phi(x,t) &= \frac{2\,\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}\,\arcsin\left[\mathrm{dn}\left(\frac{\sigma}{L}(x-vt),\,k\right)\right],\\ \Psi(x,t) &= \frac{2\left(1-\frac{v^2}{c^2}\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}\,\arcsin\left[\mathrm{dn}\left(\frac{\sigma}{L}(x-vt),\,k\right)\right],\end{aligned} \tag{85}$$

where

$$L=\frac{\lambda\delta^2}{\sqrt{-\mu}}=\lambda\delta^2\sqrt{\frac{\left(\epsilon^2-\frac{v^2}{c^2}\right)\left(\frac{v^2}{c^2}-1\right)}{\delta^2\left(\frac{v^2}{c^2}-\epsilon^2\right)+\frac{v^2}{c^2}-1}}\,. \tag{86}$$

Note that the slow traveling waves may move with velocities which can exceed c_2, the limiting velocity for the second Frenkel–Kontorova chain. This waves are a generalization of the Frenkel–Kontorova solitons (13), while the fast traveling waves, (80), are a generalization of the tachyonic anti-dislocations (15). Indeed, let us consider the limit

$$k\to 1\,,\qquad \epsilon\to 1\,,\qquad \delta\to 0\,,\qquad \lambda\,\delta\to\lambda_0\,. \tag{87}$$

Using

$$\arcsin x = 2\,\arctan\frac{x}{1+\sqrt{1-x^2}}$$

and the relation $\mathrm{dn}(u,1)=\mathrm{sech}\,u$ [112], we get for the slow traveling waves, in the limit (87), $\Phi=0$ and

$$\Psi=-4\arctan\,\exp\left[-\frac{\sigma}{L}(x-vt)\right]=-2\pi+\arctan\,\exp\left[\frac{\sigma}{L}(x-vt)\right],$$

where, according to (86), L takes the form (12), with λ_0 instead of λ, in this limit.

As for the fast traveling waves, we can use the relation $\mathrm{sn}(u,1)=\tanh u$ [112] to get from (80), in the limit (87),

$$\sin\frac{\Psi}{2}=\tanh\left(-\frac{\sigma}{\tilde{L}}(x-vt)\right),$$

and therefore,

$$\exp\left(-\frac{\sigma}{\tilde{L}}(x-vt)\right)=\sqrt{\frac{1-\sin\frac{\Psi}{2}}{1-\sin\frac{\Psi}{2}}}=\tan\left(\frac{\pi}{4}-\frac{\Psi}{4}\right),$$

from which it follows that Ψ is an anti-dislocation (15).

7. Concluding remarks

Summing up, what is the elvisebrion hypothesis about? It suggests a possible existence of a hidden sector which is either not Lorentz invariant or it is Lorentz invariant but with a different limiting speed. If the two sectors (hidden and visible) are connected very weakly, we can encounter a situation in which the Lorentz invariance is a very good approximation in the visible sector but nevertheless the whole setup is no longer Lorentz invariant and, therefore, a possibility of hidden-sector induced superluminal phenomena does appear.

On the eve of Higgs discovery, one can speculate about the possibility that the Higgs field may provide a portal into hidden sectors [113]. Usually, the Higgs portal is introduced through the renormalizable coupling $\mathcal{L}_{\rm int} = g\Phi^+\Phi\Psi^+\Psi$ of the Higgs field Φ to a scalar Ψ from a hidden sector. If the hidden sector is not Lorentz invariant, such a coupling may induce a contribution to the Higgs mass term which is space dependent and, therefore, breaks Lorentz invariance in the Higgs sector. This violation of Lorentz invariance is then propagated at one loop level to all other Standard Model particles. Such a scenario was considered in [114].

However, if the hidden sector is considered to be not Lorentz invariant, it is not necessary to assume that its couplings to the visible sector are Lorentz invariant either. To illustrate the elvisebrion idea, we can consider, for example, the following minimal model given by the Lagrangian $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_{\rm int}$, where

$$\mathcal{L}_1 = \frac{1}{c_1^2}\frac{\partial\Phi^+}{\partial t}\frac{\partial\Phi}{\partial t} - \nabla\Phi^+\cdot\nabla\Phi + \mu^2\Phi^+\Phi - \frac{1}{2}\lambda\left(\Phi^+\Phi\right)^2 \tag{88}$$

is the Lagrangian for the Standard Model Higgs doublet Φ,

$$\mathcal{L}_2 = \frac{1}{2}\left[\frac{1}{c_2^2}\frac{\partial\Psi}{\partial t}\frac{\partial\Psi}{\partial t} - \nabla\Psi\cdot\nabla\Psi\right] \tag{89}$$

is the Lagrangian for the hidden sector presented here (rather unrealistically) by only one massless real scalar field Ψ with limiting velocity c_2 which we assume to exceed the light velocity c_1, and

$$\mathcal{L}_{\rm int} = g\,\Phi^+\Phi\,\frac{\partial\Psi}{\partial x} \tag{90}$$

is the interaction Lagrangian. Note that $\mathcal{L}_{\rm int}$ breaks not only the Lorentz symmetry, but also the spatial isotropy as it assumes the existence of a preferred direction taken here to be the x-direction.

Classical equations of motion that follow from such a Lagrangian are

$$\frac{1}{c_1^2}\frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi = \mu^2\Phi - \lambda\left(\Phi^+\Phi\right)\Phi + g\,\Phi\frac{\partial\Psi}{\partial x}\,,$$

$$\frac{1}{c_2^2}\frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi = -g\,\frac{\partial}{\partial x}(\Phi^+\Phi)\,. \tag{91}$$

As we see, we have a system of coupled nonlinear Klein–Gordon equations with different speeds. It will be convenient to introduce dimensionless variables

$$\tilde{t} = c_1\,\mu\,t\,, \qquad \tilde{x} = \frac{c_1}{c_2}\,\mu\,x\,, \qquad \Phi = \frac{\mu}{\sqrt{\lambda}}\,\phi\,, \qquad \Psi = \frac{c_2}{c_1}\,\frac{\mu}{\sqrt{\lambda}}\,\psi\,. \tag{92}$$

Then the system (91) takes the form

$$\begin{aligned} \frac{\partial^2\phi}{\partial\tilde{t}^2} - \epsilon^2\,\tilde{\Delta}\phi &= \phi - \left(\phi^+\phi\right)\phi + \frac{g}{\sqrt{\lambda}}\,\phi\,\frac{\partial\psi}{\partial\tilde{x}}\,, \\ \frac{\partial^2\psi}{\partial\tilde{t}^2} - \tilde{\Delta}\psi &= -\frac{g}{\sqrt{\lambda}}\,\frac{\partial}{\partial\tilde{x}}\left(\phi^+\phi\right)\,, \end{aligned} \tag{93}$$

where $\epsilon = c_1/c_2$. When $g/\sqrt{\lambda} = 1$, the system (93) describes the three-dimensional dynamics of a twisted elastic rod near its first bifurcation threshold [115, 116] and has many interesting solutions [115]. The similar solutions do exist even in the case of small g. For example, it can be checked that we have the following traveling pulse solution

$$\phi = \begin{pmatrix} 0 \\ A\,\mathrm{sech}\left(\frac{\tilde{x}-\nu\tilde{t}}{l}\right) \end{pmatrix}\,, \qquad \psi = B\tanh\left(\frac{\tilde{x}-\nu\tilde{t}}{l}\right)\,, \tag{94}$$

where

$$A^2 = \frac{2\left(\nu^2-1\right)}{\nu^2 - 1 + g^2/\lambda}\,, \qquad B = \frac{-2gl/\sqrt{\lambda}}{\nu^2-1+g^2/\lambda}\,, \qquad l = \sqrt{\nu^2-\epsilon^2}\,. \tag{95}$$

If we assume $\epsilon < 1$, then the solution (94) does exist provided $\nu > 1$. In dimensionfull quantities, the pulse (94) moves with the superluminal velocity $v = \nu\,c_2 > c_2$ and has the width

$$L = \frac{lc_2}{\mu\,c_1} = \frac{1}{\mu}\sqrt{\frac{v^2}{c_1^2} - 1}\,.$$

The solution (94) survives in the limit $g = 0$, $\epsilon = 1$ and describes a tachyonic pulse of the Higgs field. It is a common consensus that such tachyonic pulses cannot convey information at superluminal speeds. Does the situation change when the pulse contains a small admixture of the hidden scalar Ψ whose limiting velocity c_2 exceeds the light velocity c_1? We suspect that the answer is yes but do not know for sure. This is just one question from many that the elvisebrion hypothesis raises. For example, how is the gravity modified when the hidden sector violates Lorentz invariance? Will some kind of "mirror gravity" [117, 118] emerge? At this point we do not know answers to these questions.

An unofficial history of tachyons begins with a brief 1959 paper by Sudarshan sent to *Physical Review* [119]. The paper was, however, rejected with a referee report saying that everything was wrong in the paper. Sudarshan requested a second referee and a new report claimed that everything was right in the paper but all the results were well known. The culmination of the story was the report of the third referee saying "I have read the manuscript, and the two referee reports. I agree with both of them" [119]. As a result, the paper was not published and the official history of tachyons begins with another paper [59]. We hope that referees will be more friendly to the elvisebrion hypothesis and it will not generate confusing and contradictory reports. But what is our own confidence in the elvisebrion hypothesis?

Martin Rees once said that he is sufficiently confident about the Multiverse to bet his dog's life on it. He was supported by Andrei Linde who was ready to bet his own life, and by Steven Weinberg who had just enough confidence in the Multiverse hypothesis to bet the lives of both Andrei Linde and Martin Rees's dog [120]. We cannot bet the lives of our pets on the elvisebrion hypothesis, but have enough confidence in it to bet the lives of both Wigner's friend and Schrödinger's cat!

Will special relativity, as a fundamental theory, survive for the next hundred years? We are not as certain about this as were several years ago. Nowadays the Lorentz symmetry is frequently questioned by scientists from various points of views [121], but there is still no single reliable experimental fact indicating the breakdown of special relativity.

True superluminal particles, elvisebrions, if found, will indicate that special relativity does not encompass the whole world of material beings, but it may still be an extremely good approximation in our sector of the world for energies that are not very high (compared, probably, to the Planck energy, $E_{\rm P} \sim 10^{19}$ GeV). Therefore, the impressive experimental support for special relativity cannot be used as an argument against a possible existence of superluminal particles. History of Nature's exploration teaches us that "her bag of performable tricks" is full of wonders.

It is certain, however, that special relativity will remain a precious diamond of the twentieth century physics. Future developments can only place it in the proper framework of more general and powerful theory, emphasizing its sparkling beauty.

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Note added: After the main part of this work was completed, we became aware that similar ideas had been formulated by Gonzalez-Mestres (see [122, 123] and references therein). He considers a hypothetical situation when the excitations of the "substrate" are also governed by special relativity (effective or fundamental) with the invariant speed $c_{\rm s}$ which is much larger than the light velocity, the invariant speed of the effective relativity realized in our sector of the world. Interestingly, there exists a ready condensed matter analogy, albeit two-dimensional, of such a situation. In graphene, the low energy electronic states are described by the Dirac equation for zero-mass particles [124] and an effective relativity emerges with the Fermi velocity, $v_{\rm F} \sim 10^6$ m/s, in the role of invariant velocity. The Fermi velocity in graphene is much smaller than the velocity of light. Therefore, cosmic ray particles which traverse a graphene sheet will appear as elvisebrions (Gonzalez-Mestres calls them superbradyons) in the world of graphene electrons, and their velocity can easily exceed $v_{\rm F}$.

After this article was completed, we became aware of the very interesting paper [125] by Geroch which presents other arguments explaining why elvisebrions could exist without any conflict with the well-established and overwhelming experimental evidence of relativity.

Let us also mention an interesting contribution by Unzicker [126]. He reconsiders the compatibility of the concept of the aether with special relativity and concludes that "not the concept of the aether as such is wrong, but the idea of particles consisting of external material passing through the aether. Rather the aether is a concept that yields special relativity in a quite natural way, provided that topological defects are seen as particles" [126]. Although he does not considers elvisebrions, from such a picture (relativistic particles as topological defects of the aether) there is just one step to assume a possibility of coexistence both of topological defects and of external particles passing through the aether (elvisebrions).

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