THE MIKHAUILOV–NOVIKOV–WANG HIERARCHY AND ITS HAMILTONIAN STRUCTURES*

Xiaoying Shan

Pinghu Campus, Jiaxing University, Pinghu, Zhejiang 314200, China

Junyi Zhu

Department of Mathematics, Zhengzhou University, Henan 450001, China
jyzhu@zzu.edu.cn

(Received July 3, 2012; revised version received September 10, 2012)

Using the Lie algebra approach, we construct a Mikhauilov–Novikov–Wang hierarchy associated with the 3×3 matrix spectral problem. It is shown that the hierarchy of nonlinear evolution equations is integrable in the Lax sense and possesses Hamiltonian structures.

DOI:10.5506/APhysPolB.43.1953
PACS numbers: 02.30.Jr, 47.10.Df, 02.30.Ik

1. Introduction

The main problems in the theory of integrable systems are: *(i)* For a given nonlinear equation, how to determine whether it is integrable or not, and if it is, how to find its Lax representation; *(ii)* To find as many as possible integrable systems such that they become significant equations. In general, it is very difficult to solve the two problems. On the other hand, a nonlinear evolution equation is integrable, that is, it can be expressed as the compatibility condition of two linear spectral problems or possesses a Lax pair, which plays a crucial role in the inverse scattering transformation [1, 2] and Darboux transformation [3, 4] and others [5, 6]. In Ref. [7], a Lie algebraic method was developed to search for new integrable nonlinear evolution equations and their Hamiltonian structures on the basis of trace identity.

* This work was supported by the National Natural Science Foundation of China (project No. 10871182 and No. 11001250), Innovation Scientists and Technicians Troop Construction Projects of Henan Province (084200410019), and SRFDP (200804590008).

In Ref. [8], Mikhailov, Novikov, and Wang proposed an integrable nonlinear evolution equation of the form

$$
\begin{aligned}
u_t &= -u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} - 20u^2 u_x + 9v_x\,, \\
v_t &= 3vu_{xxx} - 4v_x u^2 + v_x u_{xx} - 24vuu_x
\end{aligned}
\tag{1}
$$

by using the symmetry approach. It is shown that this equation is bi-Hamiltonian and possesses a recursion operator [8]. Equation (1) is called the Mikhailov–Novikov–Wang (MNW) equation. It is easy to see that Eq. (1) can be reduced to the well-known Kaup–Kupershmidt equation [9, 10]

$$
u_t = -u_{xxxxx} + 10uu_{xxx} + 25u_x u_{xx} - 20u^2 u_x \tag{2}
$$

by setting $v = 0$. The author of Ref. [11] introduced a 3×3 matrix spectral problem, from which a zero curvature representation for (1) was given.

The aim of the present paper is to construct the MNW hierarchy of nonlinear evolution equations associated with the 3×3 matrix spectral problem and establish their Hamiltonian structures. The compatibility condition between the given matrix spectral problem $y_x = U(s, \lambda)y$ and its auxiliary problem $y_{t_m} = V^{(m)}y$ yields the zero-curvature equation $U_{t_m} - (V^{(m)})_x + [U, V^{(m)}] = 0$, where each entry of $V^{(m)}$ is a Laurent expansion in λ. Then, the zero-curvature equation is equivalent to a set of equations about the coefficients, which can be obtained in terms of the potential function s by solving the stationary zero-curvature equation. As a result, the hierarchy of soliton equation is derived.

In Section 2, we introduce the Lie algebraic method and solve the stationary zero-curvature equation associated with the 3×3 matrix spectral problem, from which we derive two cases of the MNW hierarchy for different choice of the coefficients in the Laurent expansion. The MNW equation (1), as the first nontrivial equation, appears in the first case of the MNW hierarchy. In Section 3, we find a recursion operator and a symplectic operator. Resorting to the trace identity, we give Hamiltonian structures for the two cases of the hierarchy of nonlinear evolution equations and prove that they are Liouville integrable.

2. The hierarchy of nonlinear evolution equations

Let us introduce a 3×3 matrix spectral problem

$$
y_x = U(s, \lambda)y\,, \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ u & 0 & 1 \\ \lambda + \lambda^{-1}v & u & 0 \end{pmatrix}, \tag{3}
$$

where $s = (u, v)^T, \lambda$ is a constant spectral parameter. We construct a Lie algebra G over $\mathbb{C}$, $G = \mathrm{span}\{\sigma_i\}_{i=1}^{8}$ with the base

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\sigma_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \sigma_8 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

equipped with the commutation relations $[\sigma_j, \sigma_k] = \sum_{l=1}^{8} C_{jk}^{l} \sigma_l$, or explicitly

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8
σ_1	0	$-\sigma_1$	$3\sigma_8$	0	σ_2	$-\sigma_3$	$-\sigma_6$	$2\sigma_4$
σ_2		0	0	$2\sigma_4$	$-\sigma_5$	$-\sigma_6$	$-2\sigma_7$	σ_8
σ_3			0	0	$3\sigma_6$	$3\sigma_5$	0	$-3\sigma_1$
σ_4				0	$-\sigma_8$	σ_1	σ_2	0
σ_5					0	$-2\sigma_7$	0	σ_3
σ_6						0	0	$-\sigma_2$
σ_7							0	$-\sigma_5$
σ_9								0 .

One loop algebra generated by G is expressed as $\widetilde{G} = \mathrm{span}\{\sigma_i(n)\}_{i=1}^{8}$, $\sigma_i(n) = \sigma_i \lambda^n$, $i = 1,2,\ldots,8$, along with the commutative operations

$$[\sigma_j(n), \sigma_k(m)] = \sum_{l=1}^{8} C_{jk}^{l} \sigma_l(n+m)\,, \tag{4}$$

where the structural constants C_{jk}^{l} can be found in the above table.

Then U in (3) takes the form

$$U = R + u\sigma_5(0) + v\sigma_7(-1)\,,$$

where $R = \sigma_1(0) + \sigma_7(1)$. It is easy to see that $\widetilde{G} = \ker \mathrm{ad}R + \mathrm{Im}\ \mathrm{ad}R$ and $\ker \mathrm{ad}R$ is commutative. The gradation for $\widetilde{G}$ is defined by

$$\begin{aligned} \deg(\sigma_4(m)) &= 3m + 2\,, & \deg(\sigma_1(m)) = \deg(\sigma_8(m)) = 3m + 1\,, \\ \deg(\sigma_2(m)) &= \deg(\sigma_3(m)) = 3m\,, & \deg(\sigma_7(m)) = 3m - 2\,, \\ \deg(\sigma_5(m)) &= \deg(\sigma_6(m)) = 3m - 1\,. \end{aligned} \tag{5}$$

Therefore, we have

$$\begin{aligned} \deg(R) &= \deg(\sigma_1(0)+\sigma_7(1)) = 1\,, \\ \deg(\sigma_5(0)) &= -1\,, \qquad\qquad \deg(\sigma_7(-1)) = -5\,, \end{aligned}$$

In order to solve the associated stationary zero-curvature equation

$$V_x = [U, V]\,, \tag{6}$$

we define

$$\mathrm{rank}(\sigma_j(m)) = \deg(\sigma_j(m))$$

and

$$\mathrm{rank}(\lambda) = 3\,, \qquad \mathrm{rank}(u) = 2\,, \qquad \mathrm{rank}(v) = 6\,, \qquad \mathrm{rank}(\partial) = 1$$

which imply that the equation (6) has homogeneous rank. Let the solution of the initial problem of (6) be

$$\begin{aligned} V &= \sum_{m\geq 0} (a_m\sigma_1(-m) + b_m\sigma_2(-m) + c_m\sigma_3(-m) + d_m\sigma_4(-m) + e_m\sigma_5(-m) \\ &\quad + f_m\sigma_6(-m) + g_m\sigma_7(-m) + h_m\sigma_8(-m))\,, \end{aligned} \tag{7}$$

where the initial values are

$$a_0 = b_0 = c_0 = d_0 = f_0 = h_0 = 0\,. \tag{8a}$$

By substituting (7) into (6), we obtain

$$\begin{aligned} a_{m,x} &= -b_m\,, \qquad\qquad b_{m,x} = e_m - ua_m - d_{m+1} - vd_{m-1}\,, \\ c_{m,x} &= uh_m - f_m\,, \qquad\qquad d_{m,x} = 2h_m\,, \\ e_{m,x} &= ub_m - h_{m+1} - vh_{m-1}\,, \\ f_{m,x} &= -(3uc_m + g_m) + a_{m+1} + va_{m-1}\,, \\ g_{m,x} &= -2uf_m + 2b_{m+1} + 2vb_{m-1}\,, \\ h_{m,x} &= 3c_m + ud_m\,, \qquad\qquad (m \geq 1) \end{aligned} \tag{8b}$$

with

$$\begin{aligned} b_{0,x} &= e_0 - ua_0 - d_1\,, \qquad\qquad e_{0x} = ub_0 - h_1\,, \\ f_{0,x} &= -(3uc_0 + g_0) + a_1\,, \qquad g_{0,x} = -2uf_0 + 2b_1\,. \end{aligned} \tag{8c}$$

Resorting to (8), we get $b_1 = h_1 = 0$, $e_{0,x} = g_{0,x} = 0$. Because of $\mathrm{rank}(\sigma_5(0)) \neq \mathrm{rank}(\sigma_7(0))$, for e_0 and g_0, one is nonzero constant, the other is zero. Therefore, there are two cases:

$$\textit{(i)}\ a_0 = b_0 = c_0 = d_0 = f_0 = h_0 = g_0 = 0\,, \quad e_0 = \alpha \quad (\alpha = \text{const},\ \alpha \neq 0)\,;$$
$$\textit{(ii)}\ a_0 = b_0 = c_0 = d_0 = f_0 = h_0 = e_0 = 0\,, \quad g_0 = \beta \quad (\beta = \text{const},\ \beta \neq 0)\,.$$

Case *(i)*. From the recurrence equation (8) and the homogeneous rank convention, we calculate successively that

$$\begin{aligned}
a_1 &= b_1 = h_1 = e_1 = 0\,, \qquad c_1 = -\frac{\alpha}{3}u\,, \qquad d_1 = \alpha\,,\\
f_1 &= \frac{\alpha}{3}u_x\,, \qquad g_1 = \frac{\alpha}{9}\left(5u^2 - 2u_{xx}\right)\,,\\
a_2 &= \frac{\alpha}{9}\left(u_{xx} - 4u^2\right)\,, \qquad b_2 = \frac{\alpha}{9}(8uu_x - u_{xxx})\,,\\
e_2 &= \frac{\alpha}{27}\left(-u_{xxxx} + 7uu_{xx} + 9u_x^2 + \frac{4}{3}u^3 + 9v\right)\,,\\
h_2 &= d_2 = c_2 = f_2 = g_2 = 0\,, \qquad a_3 = b_3 = e_3 = 0\,,\\
d_3 &= \frac{2\alpha}{27}\left(u_{xxxx} - 10uu_{xx} - \frac{15}{2}u_x^2 + \frac{20}{3}u^3 - 9v\right)\,,\\
h_3 &= \frac{\alpha}{27}\left(u_{xxxxx} - 25u_xu_{xx} + 20u^2u_x - 10uu_{xxx} - 9v_x\right)\,.
\end{aligned} \tag{9}$$

By using mathematical induction, we get

$$a_{2m+1} = b_{2m+1} = e_{2m+1} = 0\,, \qquad c_{2m} = d_{2m} = f_{2m} = g_{2m} = h_{2m} = 0\,.$$

It is easy to see that

$$\begin{aligned}
\text{rank}(a_{2m}) &= 6m-2\,, \quad \text{rank}(b_{2m}) = 6m-1\,, \quad \text{rank}(e_{2m}) = 6m\,,\\
\text{rank}(h_{2m+1}) &= 6m+1\,, \quad \text{rank}(d_{2m+1}) = 6m\,, \quad \text{rank}(c_{2m+1}) = 6m+2\,,\\
\text{rank}(f_{2m+1}) &= 6m+3\,, \quad \text{rank}(g_{2m+1}) = 6m+4\,,\\
\text{rank}(U) &= 1\,, \quad \text{rank}(V) = -1\,.
\end{aligned}$$

By virtue of (8), we obtain

$$\begin{aligned}
&(\lambda^n V)_{+x} - [U, (\lambda^n V)_+]\\
&= -d_{n+1}\sigma_2(0) + vd_n\sigma_2(-1) - h_{n+1}\sigma_5(0) + vh_n\sigma_5(-1)\\
&\quad + a_{n+1}\sigma_6(0) - va_n\sigma_6(-1) + 2b_{n+1}\sigma_7(0) - 2vb_n\sigma_7(-1)\,,
\end{aligned}$$

where

$$(\lambda^n V)_+ = \sum_{m=0}^{n}[a_m\sigma_1(n-m) + \cdots + h_m\sigma_8(n-m)]\,.$$

Let $n = 2m$, $V^{(m)} = (\lambda^{2m}V)_+ + \Delta_m$, $\Delta_m = va_{2m}\sigma_7(-1) - d_{2m+1}\sigma_5(0)$. Then we have

$$V_x^{(m)} - \left[U, V^{(m)}\right] = -(d_{2m+1,x} + h_{2m+1})\sigma_5(0) + [(va_{2m})_x - 2vb_{2m}]\sigma_7(-1)\,. \tag{10}$$

With the aid of (10) and the zero-curvature equation $U_t - V_x^{(m)} + [U, V^{(m)}] = 0$, we obtain a hierarchy of nonlinear evolution equations

$$\begin{aligned} u_t &= -(d_{2m+1,x} + h_{2m+1}) \,, \\ v_t &= (va_{2m})_x - 2vb_{2m} \,, \qquad m \geq 0 \end{aligned} \tag{11}$$

which is reduced to (1) as $m = 1$, $\alpha = 9$.

Case (*ii*). On the basis of the recurrence equation (8) and the convention on homogeneous rank, the similar data can be calculated successively

$$\begin{aligned}
a_1 &= \beta \,, \qquad\qquad b_1 = c_1 = d_1 = h_1 = f_1 = 0 \,, \\
e_1 &= \frac{\beta}{3} u \,, \qquad g_1 = 0 \,, \qquad a_2 = b_2 = e_2 = 0 \,, \\
c_2 &= \frac{1}{9}\beta\left(-u_{xx} + 2u^2\right) \,, \qquad d_2 = -\frac{2}{3}\beta u \,, \qquad f_2 = \frac{1}{9}\beta(u_{xxx} - 7uu_x) \,, \\
g_2 &= \frac{2}{27}\beta\left(-u_{xxxx} + 9uu_{xx} - \frac{11}{3}u^3 + \frac{15}{2}u_x^2 + 9v\right) \,, \\
h_2 &= -\frac{1}{3}\beta u_x \,, \qquad\qquad d_3 = h_3 = c_3 = f_3 = g_3 = 0 \,, \\
a_3 &= \frac{1}{27}\beta\left(u_{xxxx} - 12uu_{xx} - 6u_x^2 + \frac{32}{3}u^3 - 9v\right) \,, \\
b_3 &= \frac{1}{27}\beta\left(-u_{5x} + 12uu_{xxx} + 24u_x u_{xx} - 32u^2 u_x + 9v_x\right) \,, \qquad a_4 = b_4 = 0 \,, \\
d_4 &= -\frac{2}{81}\beta\Big\{ -u_{6x} + 14uu_{xxxx} + 35u_x u_{xx} + \frac{49}{2}u_{xx}^2 + \frac{56}{3}u^4 \\
&\quad -70uu_x^2 - 56u^2 u_{xx}\Big\} + \frac{4}{9}\beta uv - \frac{2\beta}{9}v_{xx} \,, \\
h_4 &= -\frac{2}{81}\beta\Big\{ -u_{7x} + 49u_x u_{xxxx} + 14uu_{5x} + 84u_{xx}u_{xxx} - 70u_x^3 \\
&\quad -252uu_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x\Big\} + \frac{4}{9}\beta(uv)_x - \frac{2}{9}\beta v_{xxx} \,.
\end{aligned} \tag{12}$$

By using mathematical induction, we get

$$\begin{aligned} a_{2m} &= b_{2m} = e_{2m} = 0 \,, \\ c_{2m+1} &= d_{2m+1} = f_{2m+1} = g_{2m+1} = h_{2m+1} = 0 \,. \end{aligned}$$

Noting that

$$\begin{aligned}
\mathrm{rank}(a_{2m+1}) &= 6m\,, \quad \mathrm{rank}(b_{2m+1}) = 6m+1\,, \quad \mathrm{rank}(e_{2m+1}) = 6m+2\,,\\
\mathrm{rank}(h_{2m}) &= 6m-3\,, \quad \mathrm{rank}(d_{2m}) = 6m-4\,, \quad \mathrm{rank}(c_{2m}) = 6m-2\,,\\
\mathrm{rank}(f_{2m}) &= 6m-1\,, \quad \mathrm{rank}(g_{2m}) = 6m\,,\\
\mathrm{rank}(U) &= 1\,, \quad \mathrm{rank}(V) = -2\,,
\end{aligned}$$

we arrive at

$$\begin{aligned}
&(\lambda^n V)_{+x} - [U, (\lambda^n V)_+]\\
&= -d_{n+1}\sigma_2(0) + v d_n \sigma_2(-1) - h_{n+1}\sigma_5(0) + v h_n \sigma_5(-1)\\
&\quad + a_{n+1}\sigma_6(0) - v a_n \sigma_6(-1) + 2b_{n+1}\sigma_7(0) - 2v b_n \sigma_7(-1)\,.
\end{aligned}$$

Let $n = 2m+1$, $V^{(m)} = (\lambda^{2m+1}V)_+ + v a_{2m+1}\sigma_7(-1) - d_{2m+2}\sigma_5(0)$. We have

$$V_x^{(m)} - \left[U, V^{(m)}\right] = -(d_{2m+2,x} + h_{2m+2})\sigma_5(0) + [(v a_{2m+1})_x - 2v b_{2m+1}]\sigma_7(-1)\,.$$

Then the zero-curvature equation implies a hierarchy of nonlinear evolution equations

$$\begin{aligned}
u_t &= -(d_{2m+2,x} + h_{2m+2})\,,\\
v_t &= (v a_{2m+1})_x - 2v b_{2m+1}\,, \qquad m \geq 0\,.
\end{aligned} \tag{13}$$

So for the case *(ii)*, by taking $m = 1$, $\beta = 27$ in (13), and using the results of (12), we obtain the following new nonlinear evolution equation

$$\begin{aligned}
u_t &= 18v_{xxx} - 36(vu)_x - u_{7x} + 49u_x u_{xxxx} + 14u u_{5x} + 84u_{xx}u_{xxx} - 70u_x^3\\
&\quad -252 u u_x u_{xx} - 56u^2 u_{xxx} + \frac{224}{3}u^3 u_x\,,\\
v_t &= -36 v v_x + v_x u_{xxxx} + 3v u_{5x} - 12 v_x u u_{xx} - 72 v u_x u_{xx} - 36 v u u_{xxx}\\
&\quad -6 v_x u_x^2 + \frac{32}{3}u^3 v_x + 96 v u^2 u_x\,.
\end{aligned}$$

3. The Hamiltonian structures and integrability

In this section, we shall construct the Hamiltonian structures of the resulting nonlinear evolution equations and prove their integrability. To this end we write

$$V = a\sigma_1 + b\sigma_2 + c\sigma_3 + d\sigma_4 + e\sigma_5 + f\sigma_6 + g\sigma_7 + h\sigma_8\,,$$

with

$$a = \sum_{m\geq 0} a_m\lambda^{-m}\,, \quad b = \sum_{m\geq 0} b_m\lambda^{-m}\,, \quad c = \sum_{m\geq 0} c_m\lambda^{-m}\,, \quad d = \sum_{m\geq 0} d_m\lambda^{-m}\,,$$
$$e = \sum_{m\geq 0} e_m\lambda^{-m}\,, \quad f = \sum_{m\geq 0} f_m\lambda^{-m}\,, \quad g = \sum_{m\geq 0} g_m\lambda^{-m}\,, \quad h = \sum_{m\geq 0} h_m\lambda^{-m}\,.$$

It is easy to see that

$$\frac{\partial U}{\partial \lambda} = \sigma_7(0) - v\sigma_7(-2)\,, \qquad \frac{\partial U}{\partial u} = \sigma_5(0)\,, \qquad \frac{\partial U}{\partial v} = \sigma_7(-1)\,.$$

Since $\langle \sigma_1, \sigma_5\rangle = 2$, $\langle \sigma_4, \sigma_7\rangle = 1$, and $\langle \sigma_j, \sigma_5\rangle = 0 = \langle \sigma_l, \sigma_7\rangle$, $(j, l = 1, \cdots, 8;$ $j \neq 1, l \neq 4)$, where $\langle \sigma_i, \sigma_j\rangle = \mathrm{tr}(\sigma_i\sigma_j)$, we get

$$\left\langle V, \frac{\partial U}{\partial \lambda}\right\rangle = d\left(1 - \frac{v}{\lambda^2}\right)\,, \qquad \left\langle V, \frac{\partial U}{\partial u}\right\rangle = 2a\,, \qquad \left\langle V, \frac{\partial U}{\partial v}\right\rangle = \frac{d}{\lambda}\,.$$

Noticing the trace identity [6]

$$\left(\frac{\delta}{\delta u}, \frac{\delta}{\delta v}\right) \mathrm{tr}\left(V\frac{\partial U}{\partial \lambda}\right) = \left[\lambda^{-\varepsilon}\left(\frac{\partial}{\partial \lambda}\right)\lambda^{\varepsilon}\right]\left(\mathrm{tr}\left(V\frac{\partial U}{\partial u}\right), \mathrm{tr}\left(V\frac{\partial U}{\partial v}\right)\right)\,,$$

where ε is a constant to be fixed, we obtain by equating the coefficients of λ^{-2n-2} on both sides that

$$\begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} (d_{n+2} - v d_n) = (-n - 1 + \varepsilon)\begin{pmatrix} 2a_{n+1} \\ d_n \end{pmatrix}\,, \qquad n = 2m+1\,. \tag{14}$$

Now we search for the Hamiltonian structures of nonlinear evolution equations (11). To fix the constant ε, we simply set $n = 1$ in (14) and arrive at $\varepsilon = \frac{1}{3}$. Therefore, we establish the following equation

$$\begin{pmatrix} 2a_{2m+2} \\ d_{2m+1} \end{pmatrix} = \begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} H_m\,, \qquad H_m = \tfrac{3}{5}(v d_{2m+1} - d_{2m+3})\,.$$

By using (8), we have

$$\begin{aligned} a_{2mx} &= -b_{2m}\,, \qquad d_{2m+1,x} = 2h_{2m+1}\,, \\ c_{2m+1} &= \tfrac{1}{3}\left(h_{2m+1,x} - u d_{2m+1}\right) = \tfrac{1}{3}\left(\tfrac{1}{2}\partial^2 - u\right) d_{2m+1}\,, \\ f_{2m+1} &= u h_{2m+1} - c_{2m+1,x} = \tfrac{1}{2}u\partial d_{2m+1} - \tfrac{1}{3}\left(\tfrac{1}{2}\partial^3 - \partial u\right) d_{2m+1} \\ &= \left(-\tfrac{1}{6}\partial^3 + \tfrac{1}{2}u\partial + \tfrac{1}{3}\partial u\right) d_{2m+1}\,, \\ b_{2m} &= \tfrac{1}{2}\left(\partial g_{2m+1} + 2u f_{2m+1} - 2b_{2m+2}\right)\,, \\ v a_{2m} &= \partial f_{2m+1} + 3u c_{2m+1} + g_{2m+1} - a_{2m+2}\,. \end{aligned} \tag{15}$$

Then (11) can be rewritten as

$$\begin{aligned} u_t &= -(d_{2m+1,x} + h_{2m+1}) = -\tfrac{3}{2}\partial d_{2m+1}\,, \\ v_t &= (va_{2m})_x - 2vb_{2m} = -3\partial a_{2m+2} + 3\partial(uc_{2m+1}) + \partial^2 f_{2m+1} - 2uf_{2m+1} \\ &= -\tfrac{3}{2}\partial\,(2a_{2m+2}) + \big(-\tfrac{1}{6}\partial^5 + \tfrac{1}{3}u\partial^3 + \tfrac{1}{3}\partial^3 u \\ &\qquad + \tfrac{1}{2}\partial^2 u\partial + \tfrac{1}{2}\partial u\partial^2 - \tfrac{2}{3}u\partial u - u^2\partial - \partial u^2\big)\, d_{2m+1} \end{aligned}$$

which possesses the following Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J \begin{pmatrix} 2a_{2m+2} \\ d_{2m+1} \end{pmatrix} = J \begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix} H_m\,, \tag{16}$$

where

$$J = \begin{pmatrix} 0 & -\frac{3}{2}\partial \\ -\frac{3}{2}\partial & F \end{pmatrix},$$

with

$$F = -\tfrac{1}{6}\partial^5 + \tfrac{1}{3}u\partial^3 + \tfrac{1}{3}\partial^3 u + \tfrac{1}{2}\partial^2 u\partial + \tfrac{1}{2}\partial u\partial^2 - \tfrac{2}{3}u\partial u - u^2\partial - \partial u^2\,.$$

By (8) and (15), we arrive at

$$\begin{aligned} d_{2m+1} &= e_{2m} - ua_{2m} - vd_{2m-1} - b_{2mx} = e_{2m} - vd_{2m-1} + \left(\partial^2 - u\right) a_{2m}\,, \\ e_{2m,x} &= ub_{2m} - h_{2m+1} - vh_{2m-1} = -u\partial a_{2m} - \tfrac{1}{2}\partial d_{2m+1} - \tfrac{1}{2}v\partial d_{2m-1}\,, \\ e_{2m} &= -\partial^{-1}u\partial a_{2m} - \tfrac{1}{2}d_{2m+1} - \tfrac{1}{2}\partial^{-1}v\partial d_{2m-1}\,, \\ 2a_{2m+2} &= \tfrac{2}{3}\partial^{-1}[Fd_{2m+1} - (\partial v + 2v\partial)a_{2m}]\,. \end{aligned} \tag{17}$$

Therefore, it can be verified that

$$\begin{pmatrix} 2a_{2m+2} \\ d_{2m+1} \end{pmatrix} = L \begin{pmatrix} 2a_{2m} \\ d_{2m-1} \end{pmatrix}\,, \tag{18}$$

with

$$L = \frac{1}{3}\begin{pmatrix} \frac{2}{3}\partial^{-1}F\left(\partial^2 - \partial^{-1}u\partial - u\right) - 2\partial^{-1}v\partial - v & -\frac{2}{3}\partial^{-1}F\left(2v + \partial^{-1}v\partial\right) \\ \partial^2 - \partial^{-1}u\partial - u & -2v - \partial^{-1}v\partial \end{pmatrix}.$$

It is easy to see that both J and JL are skew-symmetric operators, and for $\{H_m\}$ it holds that

$$L^m f(s) = \frac{\delta H_m}{\delta s}\,,$$

where

$$f(s) = \begin{pmatrix} \frac{2\alpha}{9}\left(u_{xx} - 4u^2\right) \\ \alpha \end{pmatrix}.$$

A direct calculation shows that J is symplectic, *i.e.*,

$$\left(J^{'}(s)[Jf]g,h\right)+\left(J^{'}(s)[Jg]h,f\right)+\left(J^{'}(s)[Jh]f,g\right)=0\,.$$

Then, the bracket $\{f,g\}=(\frac{\delta f}{\delta s},J\frac{\delta g}{\delta s})$ is a well-defined Poisson bracket, therefore $\{H_m\}$ are conserved densities for (11) and in involution in pair. Hence (11) is Liouville integrable.

Now, we turn to the search for Hamiltonian structure of (13). It is easy to see that we choose $n=2m$ to be right. To fix the constant ε, we simply set $n=0$ in (14) and arrive at $\varepsilon=\frac{2}{3}$. Therefore, we establish the following equation

$$\begin{pmatrix} 2a_{2m+3} \\ d_{2m+2} \end{pmatrix}=\begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix}\widetilde{H}_m\,, \qquad \widetilde{H}_m=3(vd_{2m+2}-d_{2m+4})\,.$$

Using (8), we have

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=J\begin{pmatrix} 2a_{2m+3} \\ d_{2m+2} \end{pmatrix},$$

where J is the Hamiltonian operator in (16). Consequently, (13) takes the Hamiltonian form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix}=J\begin{pmatrix} 2a_{2m+3} \\ d_{2m+2} \end{pmatrix}=J\begin{pmatrix} \delta/\delta u \\ \delta/\delta v \end{pmatrix}\widetilde{H}_m\,.$$

Based on (8), we derive

$$\begin{pmatrix} 2a_{2m+3} \\ d_{2m+2} \end{pmatrix}=L\begin{pmatrix} 2a_{2m+1} \\ d_{2m} \end{pmatrix}\,,$$

where L is the operator in (18). For $\{\widetilde{H}_m\}$ it holds that

$$L^m\widetilde{f}(s)=\frac{\delta\widetilde{H}_m}{\delta s}\,,$$

where

$$\widetilde{f}(s)=\begin{pmatrix} \frac{2}{27}\beta\left(u_{xxxx}-12uu_{xx}-6u_x^2+\frac{32}{3}u^3-9v\right) \\ -\frac{2}{3}\beta u \end{pmatrix}.$$

Then, $\{\widetilde{H}_m\}$ are conserved densities for (13) and involution in pair. Hence (13) is Liouville integrable [12].

REFERENCES

[1] M.J. Ablowitz, H. Segur, *Solitons and the Inverse Scattering Transformation*, SIAM, Philadephia 1994.

[2] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii, V.E. Zakharov, *Theory of Solitons, The Inverse Scattering Methods*, New York, Consultants Bureau, 1984.

[3] V.B. Matveev, M.A. Salle, *Darboux Transformation and Solitons*, Springer, Berlin 1991.

[4] C. Rogers, W.K. Schief, *Bäcklund and Darboux Transformations, Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, Cambridge 2002.

[5] X.G. Geng, B. Xue, *Nonlinearity* **22**, 1847 (2009).

[6] X.G. Geng, B. Xue, *Adv. Math.* **226**, 827 (2011).

[7] G.Z. Tu, *J. Phys. A: Math. Gen.* **22**, 2375 (1989).

[8] A.V. Mikhauilov, V.S. Novikov, J.P. Wang, *Stud. Appl. Math.* **118**, 419 (2007).

[9] D.J. Kaup, *Stud. Appl. Math.* **62**, 189 (1980).

[10] B.A. Kupershmidt, *Phys. Lett.* **A102**, 213 (1984).

[11] A. Sergyeyev, *J. Math. Sci.* **151**, 3227 (2008).

[12] J.Y. Zhu, X.G. Geng, *Acta Phys. Pol. B* **39**, 1783 (2008).