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BOSE GAS IN POWER-LIKE SPHERICALLY SYMMETRIC POTENTIAL IN ARBITRARY SPATIAL DIMENSIONALITY

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Temperature of the Bose–Einstein condensation and the temperature behavior of the chemical potential and other thermodynamical functions of the ideal Bose gas are found for the arbitrary power-like spherical-symmetric potential at an arbitrary space dimension. It is shown that the recently observed Bose–Einstein condensation of photons in the cavity is the phase transition of the third kind.

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1. Introduction

The majority of studies, both theoretical and experimental, on the Bose–Einstein condensation (BEC) was carried out for gases in the harmonic traps and in three space dimensions [1]. In the meantime, a number of works were devoted to the study of the thermodynamic properties of the Bose gas trapped in arbitrary power-like spherical-symmetric potential in three spacial (3D) dimensions [2], and to the harmonic trap at lower dimension $D = 2$ [3]. Recent publication [4] contains the calculation of the temperature and the fraction of the condensed particles in the case of Bose–Einstein condensation of the ideal Bose gas in free space at any space dimension. As far as the physical realization is concerned, the harmonic traps of lower dimensions are effectively modeled by the sharply anisotropic frequencies [3]. A direct realization of BEC in the 2D harmonic and isotropic trap was recently achieved with the photons confined in the spherical cavity [5]. The confinement results both in the effective nonrelativistic dependence of the photon energy on the transverse momentum, the harmonic oscillator potential in the transverse direction,

$$\begin{aligned} E &= \hbar c\sqrt{k_{\|}^2(r)+k_r^2} \approx \hbar c k_{\|}(r) + \frac{\hbar c k_r^2}{2k_{\|}(r)} \\ &\approx \frac{\pi\hbar c s}{D_0} + \frac{\pi\hbar c s r^2}{R D_0^2} + \frac{\hbar c D_0 k_r^2}{2\pi s} \equiv E_0 + \frac{m_{\rm eff}\omega^2 r^2}{2} + \frac{p_r^2}{2m_{\rm eff}}\,, \end{aligned} \tag{1.1}$$

with a tiny effective photon mass $m_{\rm eff} = \pi\hbar c s/D_0 \approx 5\times10^{-33}\,$g, the frequency $\omega = c\sqrt{2/(RD_0)} \approx 4\times10^{11}\,{\rm s}^{-1}$, and $E_0 = \pi\hbar c s/D_0 \approx 3$ eV [5]. Here, $D_0 = 1.4\times10^{-4}\,$cm is the distance along the optical axes between the spherical mirrors, each having the radius of curvature $R = 100$ cm, $s = 7$ is the excitation number of the longitudinal standing wave of the photon, and r, $k_r = p_r/\hbar$ are, respectively, the distance from the optical axes and the conjugate wave number in the transverse direction, c and $\hbar$ are the velocity of light and the Planck constant.

However, the general analysis of the possibility of BEC in a power-law potential in an arbitrary space dimension D, as well as the degree of singularity of the chemical potential and other thermodynamic functions, is absent in the literature. In the present work, we fill the gap and find both the conditions for BEC, as well as the degree of singularity of the thermodynamical functions of the ideal Bose gas, appearing in the case of arbitrary power-like spherical potential and for arbitrary space dimensionality. Despite the fact that the ideal Bose gas approximation adopted in the present work is poor for the gases of alkali metals, it is meaningful for BEC of photons studied in Ref. [5], in addition to the case of the hypothetical weakly interacting axions, proposed as dark-matter candidate in Ref. [6].

The rest of the paper is organized as follows. The density of states is obtained in Sec. 2. Section 3 is devoted to the elucidation of the behavior of the chemical potential near the the BEC transition temperature. The results and conclusions are presented in Sec. 4. The Appendix contains the necessary information concerning the polylogarithm function.

2. Density of states and the critical temperature of BEC

The starting point is the semi-classical one-particle density of states $\nu(\epsilon)$ in the case of spherically symmetric single-particle potential of the form

$$U(r) = U_0\left(\frac{r}{a}\right)^{\gamma}\,, \tag{2.1}$$

characterized by its strength $U_0 > 0$, the scale a, and the exponent γ. In effect, the single-particle density of states can be defined as

$$\nu(\epsilon) = \int \frac{d^D r d^D p}{(2\pi\hbar)^D} \delta\left[\epsilon - \frac{p^2}{2m} - U_0\left(\frac{r}{a}\right)^\gamma\right], \tag{2.2}$$

where $\delta(x)$ is the Dirac delta function. Hereafter, D is the space dimensionality of the gas, the particle spin is assumed to be zero, and p and r are the moduli of the D-dimensional vectors of the momentum and the radius, respectively. The phase-space integral (2.2) is calculated in the multi-dimensional spherical coordinates, where the volume element integrated over angular variables takes the form

$$d^D r = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)} r^{D-1} dr$$

see [7], the formula 4.632(2) differentiated over R. Analogously for $d^D p$, with the use of the table integral [8]

$$\int\limits_0^1 x^{\xi-1}(1-x)^{\eta-1} dx = \frac{\Gamma(\xi)\Gamma(\eta)}{\Gamma(\xi+\eta)},$$

the calculation leads to the explicit expression

$$\nu(\epsilon) = A\epsilon^{\alpha-1}, \tag{2.3}$$

where

$$\begin{aligned} \alpha &= \frac{D}{2} + \frac{D}{\gamma}, \\ A &= \left(\frac{ma^2}{2\hbar^2 U_0^{2/\gamma}}\right)^{D/2} \frac{2\Gamma\left(\frac{D}{\gamma}\right)}{\gamma\Gamma\left(\frac{D}{2}\right)\Gamma(\alpha)}, \end{aligned} \tag{2.4}$$

and $\Gamma(x)$ is the Euler gamma function.

The case of $\gamma = 0$, at first sight, seems to be singular. However, it corresponds to the situation, when the gas is under the influence of the constant potential U_0. As is usual in the case of free particles or particles in the constant potential, one should put the system into the finite volume. The problem is reduced to the case of bosons in D spatial dimensions, with the only difference that they condense to the state with the energy U_0.

The temperature of BEC in this case is the same as found in [4]. The thermodynamical properties of the system do not depend on changing the energy reference point.

In the meantime, there is another way to get the case of free bosons in the space with no external potential, confined inside the sphere with the radius a. This is obtained in the limit $\gamma \to \infty$. Then

$$U(r) = U_0 \left(\frac{r}{a}\right)^{\gamma} \to \left\{ \begin{array}{l} 0,\ \text{if}\ r < a\,, \\ \infty,\ \text{if}\ r > a\,, \end{array} \right. \tag{2.5}$$

and the expressions (2.4) are reduced, respectively, to the following ones

$$\begin{aligned} \alpha &= \frac{D}{2}\,, \\ A &= \left(\frac{m}{2\hbar^2}\right)^{D/2} \frac{a^D}{\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}+1\right)}\,. \end{aligned} \tag{2.6}$$

The density of states in this limit coincides with that obtained earlier [4] for free bosons in a finite volume. As for the dependence of the chemical potential on the temperature, in the cases of $\gamma = 0$ or $\gamma \to \infty$, we will comment on this point after establishing the analogous dependence in the general case of finite $\gamma \neq 0$.

As usual, the inequalities $\mu(T) \leq 0$ and $\left(\frac{\partial\mu}{\partial T}\right)_{U_0} < 0$ result from the condition of having the particle number bosons in the gas, *i.e.*,

$$N = \int\limits_0^\infty \frac{\nu(\epsilon)d\epsilon}{e^{(\epsilon-\mu)/T} - 1}\,, \tag{2.7}$$

where $\mu \equiv \mu(T)$ is the chemical potential. The above equation is valid only if $U_0 > 0$. The Boltzmann constant is set to unity. BEC is possible if the integral defining the temperature of BEC T_0,

$$N = \int\limits_0^\infty \frac{\nu(\epsilon)d\epsilon}{e^{\epsilon/T_0} - 1} = \left(\frac{ma^2}{2\hbar^2 U_0^{2/\gamma}}\right)^{D/2} \frac{2\Gamma\left(\frac{D}{\gamma}\right)\zeta\left(\alpha\right)T_0^{\alpha}}{\gamma\Gamma\left(\frac{D}{2}\right)}\,, \tag{2.8}$$

has a definite value. Hereafter

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

is the Riemann zeta function. This function is defined at $x > 1$. Hence, the restriction on the parameters of the problem γ and D, resulting from the definition of zeta function, is

$$\alpha = \frac{D}{2} + \frac{D}{\gamma} > 1\,. \tag{2.9}$$

One obtains that BEC is possible in the space dimension $D = 1, 2$, and 3 at, respectively, $\gamma < 2$, $\gamma > 0$, and $\gamma > -6$. In particular, the confinement in the harmonic trap, $\gamma = 2$, admits BEC in any space dimension when $D > 1$. The critical temperature T_0 of BEC is found from Eq. (2.8)

$$T_0 = \left[N \left(\frac{2\hbar^2 U_0^{2/\gamma}}{ma^2} \right)^{D/2} \frac{\gamma \Gamma\left(\frac{D}{2}\right)}{2\Gamma\left(\frac{D}{\gamma}\right)\zeta(\alpha)} \right]^{1/\alpha} . \tag{2.10}$$

In particular, one has $D = 3$, $\gamma = 2$ for BEC of alkali atoms in the traps with the typical magnitude of the frequency $\omega \sim 10^2\,\mathrm{s}^{-1}$. Then, $ma^2/(2\hbar^2 U_0) = (\hbar\omega)^{-2}$, and the BEC temperature is

$$T_0 = \hbar\omega \left[\frac{N}{\zeta(3)} \right]^{1/3} \sim 10^{-7}\ \mathrm{K}$$

for $N \sim 10^6$. Hereafter, the necessary expressions for the zeta function at particular arguments are taken from [9]. On the other hand, under the conditions of the experiment [5], the parameters are $D = 2$, $\gamma = 2$, $\omega \sim 10^{11}\,\mathrm{s}^{-1}$, and the BEC temperature is

$$T_0 = \frac{\hbar\omega}{\pi}\sqrt{3N} \sim 10^3\ \mathrm{K}\,,$$

at $N \sim 10^4$ photons in the cavity. The above equation allows for two polarization states of the photon. For the oscillator potential in space with arbitrary dimension D, one finds

$$T_0 = \hbar\omega \left[\frac{N}{\zeta(D)} \right]^{1/D} . \tag{2.11}$$

Since $\lim_{D\to\infty}\zeta(D) = 1$, one can see from (2.11) that, in this limit, $T_0 \to \hbar\omega$ independently of the particle number N.

The chemical potential $\mu(T)$ at $T < T_0$ is zero with all its derivatives. Hence, the number of particles sitting at the excited energy levels, $N_>(T)$, is given by Eq. (2.8), in which one should make the replacements $N \to N_>(T)$

and $T_0 \to T$. Then, the number of particles in BEC, $N_0(T) = N - N_>(T)$, is given by the expression

$$N_0(T) = N\left[1 - \left(\frac{T}{T_0}\right)^{\alpha}\right]. \tag{2.12}$$

This expression is valid for the finite values of the power γ in the external potential. It coincides with the earlier expression $\alpha = 3/2 + 3/\gamma$ [1, 2] at $D = 3$. As is explained above, the case of ideal Bose gas put in a finite volume, upon neglecting the interaction with the external field, corresponds to taking the limit $\gamma \to \infty$, and the power α is reduced to the value $\alpha = 3/2$ in the three-dimensional case.

3. Thermodynamic functions near BEC transition temperature

The temperature behavior of the thermodynamic functions in the vicinity of the critical temperature T_0 is found from the behavior of $\mu(T)$. Since $\mu(T) = 0$ at $T \leq T_0$, one should consider the region $T = T_0 + \Delta T$, where $\Delta T \ll T_0$. $\mu(T)$ is found from Eq. (2.7)

$$\left(\frac{T_0}{T}\right)^{\alpha} = \frac{\mathrm{Li}_{\alpha}\left(e^{\mu/T}\right)}{\mathrm{Li}_{\alpha}(1)}, \tag{3.1}$$

where $\mathrm{Li}_{\alpha}(z)$ is the polylogarithm function. Its definition and some properties necessary in the context of the present work, are gathered in Appendix. With the help of Eq. (A.1) in the Appendix one can find the internal energy in terms of $\mu \equiv \mu(T)$. To this end one should use the following relations

$$\begin{aligned} E &= A\int\limits_0^{\infty}\frac{\epsilon^{\alpha}d\epsilon}{e^{(\epsilon-\mu)/T}-1} = AT^{\alpha+1}\Gamma(\alpha+1)\mathrm{Li}_{\alpha+1}\left(e^{\mu/T}\right), \\ N &= A\int\limits_0^{\infty}\frac{\epsilon^{\alpha-1}d\epsilon}{e^{(\epsilon-\mu)/T}-1} = AT^{\alpha}\Gamma(\alpha)\mathrm{Li}_{\alpha}\left(e^{\mu/T}\right). \end{aligned}$$

Excluding from these equations the multiplicative factor A, one obtains the internal energy in the following form

$$E = NT\alpha\frac{\mathrm{Li}_{\alpha+1}\left(e^{\mu/T}\right)}{\mathrm{Li}_{\alpha}\left(e^{\mu/T}\right)}. \tag{3.2}$$

Although Eqs. (3.1) and (3.2) are exact, they are not very useful for practical purposes. One should consider them in various limiting cases, in order to obtain explicit expressions for μ and other thermodynamic functions. As will be clear, the temperature behavior of $\mu(T)$ is different at different values of α. If $\alpha > 2$ (that is, if $\gamma < 2/3$, $\gamma < 1$, and $\gamma < 6$ at $D = 1$, $D = 2$, and $D = 3$, respectively) then one may use power expansion in fugacity $e^{\mu/T}$

$$\begin{aligned} N &= A\int\limits_0^\infty \frac{\epsilon^{\alpha-1}d\epsilon}{e^{(\epsilon-\mu)/T}-1} = A\sum_{n=0}^{\infty} e^{\mu(n+1)/T}\int\limits_0^\infty d\epsilon \\ &\times\epsilon^{\alpha-1}e^{-\epsilon(n+1)/T} = A\Gamma\left(\alpha\right)T^{\alpha}\left[\zeta\left(\alpha\right)+\sum_{n=1}^{\infty}\frac{e^{\mu n/T}-1}{n^{\alpha}}\right]. \qquad (3.3)\end{aligned}$$

Since $\mu \ll T$, which is valid in the vicinity of T_0, then, to the first order, $e^{\mu n/T} - 1 \approx \mu n/T$, and the summation is obtained in the following closed form

$$\frac{\mu}{T}\sum_{n=1}^{\infty}\frac{1}{n^{\alpha-1}} = \frac{\mu}{T}\zeta\left(\alpha-1\right).$$

The explicit equation for finding the chemical potential valid at $\alpha > 2$ is

$$N = N\left(\frac{T}{T_0}\right)^{\alpha}\left[1+\frac{\mu\zeta\left(\alpha-1\right)}{T\zeta\left(\alpha\right)}\right],$$

whose solution at $T > T_0$ is

$$\mu(T) \approx T_0\left[\left(\frac{T_0}{T}\right)^{\alpha}-1\right]\frac{\zeta\left(\alpha\right)}{\zeta\left(\alpha-1\right)} \approx -(T-T_0)\frac{\alpha\zeta\left(\alpha\right)}{\zeta\left(\alpha-1\right)}.$$

One can see from this expression that $\mu(T)$ is continuous at T_0, but its first derivative has a discontinuity

$$[\mu'] = -\frac{\alpha\zeta\left(\alpha\right)}{\zeta\left(\alpha-1\right)}.$$

Hereafter,

$$[f] \equiv \lim\nolimits_{\epsilon\to 0}\left[f(T_0+\epsilon)-f(T_0-\epsilon)\right]$$

designates the discontinuity of the function $f(T)$ at the BEC transition temperature T_0. Hence, the energy of the Bose gas is continuous but the heat capacity at constant external field defined as

$$C = \left(\frac{\partial E}{\partial T}\right)_{N,U} = A\int\limits_0^\infty \frac{\epsilon^{\alpha}e^{(\epsilon-\mu)/T}}{\left[e^{(\epsilon-\mu)/T}-1\right]^2}\left(\frac{\epsilon-\mu}{T^2}+\frac{\mu'}{T}\right)d\epsilon$$

is discontinuous at $T = T_0$, *i.e.*,

$$[C] \;=\; A\frac{[\mu']}{T_0}\int\limits_0^\infty \frac{\epsilon^\alpha e^{\epsilon/T}d\epsilon}{\left(e^{\epsilon/T}-1\right)^2} = -N\frac{\alpha^2\zeta\left(\alpha\right)}{\zeta\left(\alpha-1\right)}\,. \tag{3.4}$$

The case when the inequality

$$1<\alpha\leq 2 \tag{3.5}$$

is satisfied, demands the separate treatment, because $e^{\mu n/T}$ under the sum in Eq. (3.3) cannot be expanded in powers of μ/T due to the divergence of the summation over n. In due turn, both the cases of the strict inequality $1<\alpha<2$ and the equality $\alpha=2$ in (3.5) should also be treated separately. Depending on the number of space dimensions $D = 1$, 2, and 3, the strict inequality is satisfied for the power γ in the intervals $2/3<\gamma<2$, $\gamma>2$, and $|\gamma|>6$, respectively.

First, let us consider the case of the strict inequality $1<\alpha<2$. At small μ, the dominant contribution to the sum in Eq. (3.3) comes from large n, so the summation can be approximated by the integration with the help of the formula due to Euler and Maclaurin [9]

$$\sum_{n=1}^\infty f(n+a) \;\approx\; \int\limits_a^\infty f(x)dx - \frac{1}{2}f(a) - \frac{1}{12}f'(a)\,. \tag{3.6}$$

To this end one should write

$$\sum_{n=1}^\infty f(n) = \sum_{l=0}^\infty f(l+1) = f(1) + \sum_{l=1}^\infty f(l+1)\,,$$

and apply (3.6) in the particular case $a=0$. Upon neglecting the non-singular terms of the order of μ/T, one finds that the dominant contribution to the equation relating the chemical potential μ to the particle number N is thus represented in the form

$$N \;=\; N\left(\frac{T}{T_0}\right)^\alpha\left[1+\frac{1}{\zeta\left(\alpha\right)}\int\limits_1^\infty\left(e^{x\mu/T}-1\right)\frac{dx}{x^\alpha}\right] \approx N\left(\frac{T}{T_0}\right)^\alpha \times\left[1-\frac{1}{\zeta\left(\alpha\right)\left(\alpha-1\right)}\left(\frac{|\mu|}{T}\right)^{\alpha-1}\Gamma\left(2-\alpha,\frac{|\mu|}{T}\right)\right]\,,$$

where

$$\Gamma(\alpha,x) = \int\limits_x^\infty e^{-z} z^{\alpha-1} dz = \Gamma(\alpha) - \sum_{n=0}^{\infty} \frac{(-1)^n x^{\alpha+n}}{n!(\alpha+n)} \tag{3.7}$$

is the incomplete gamma function [7], and $x = |\mu|/T \ll 1$. Since x is small, one can keep only the term with $n = 0$ in the sum over n in Eq. (3.7). As a result, after keeping the terms of the lower order in the ratio μ/T, one obtains the first-order expression for the nonzero value of $\mu(T)$ for $T > T_0$

$$\mu(T) \approx -T_0 \left[\frac{\alpha\,(\alpha-1)\,\zeta\,(\alpha)\,(T-T_0)}{T_0 \Gamma\,(2-\alpha)}\right]^{\frac{1}{\alpha-1}} . \tag{3.8}$$

This expression shows that the chemical potential is continuous at $T = T_0$. Indeed, $\mu(T)$ vanishes at $T \leq T_0$ with all its derivatives. Since, in the present case, $1/(\alpha-1) > 0$, then, at $T > T_0$, the expression $(T-T_0)^{\frac{1}{\alpha-1}}$ vanishes at $T \to T_0$, too. As for the derivatives of the chemical potential over temperature, the evaluation of the order k derivative can be expressed as follows

$$\mu^{(k)}(T) \equiv \frac{\partial^k \mu}{\partial T^k} = -\frac{1}{T_0^{k-1}} \left[\frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)}\right]^{1/(\alpha-1)} \times \left(\frac{T-T_0}{T_0}\right)^{1/(\alpha-1)-k} \prod_{l=0}^{k-1} \left(\frac{1}{\alpha-1} - l\right) ,$$

which shows that, provided the condition

$$\alpha = 1 + \frac{1}{k} \tag{3.9}$$

is fulfilled, where $k \geq 2$, the k-th order derivative of the chemical potential as a function of temperature is discontinuous at $T = T_0$. The magnitude of the discontinuity is

$$\left[\mu^{(k)}\right] = -\frac{k!}{T_0^{k-1}} \left[\frac{(k+1)\zeta(1+1/k)}{k^2 \Gamma(1-1/k)}\right]^k . \tag{3.10}$$

Since, in the present case, $\alpha < 2$, the case $k = 1$ is beyond the treatment. As will be clear later on, when $\alpha = 2$, the temperature behavior of the chemical potential becomes non-analytical at $T = T_0$.

Let us turn to the case $\alpha = 2$. Note that this equality takes place, if, at the space dimension $D = 1, 2, 3$, the power of the coordinate dependence of the trapping potential $U(r) \propto r^\gamma$ equals, respectively, $\gamma = 2/3, 2, 6$. It is the situation $D = 2$, $\gamma = 2$ which was realized recently in the experiments with the Bose–Einstein condensation of photons in the cavity [5]. The chemical potential at $T > T_0$ can be found from Eq. (3.1), with $\alpha = 2$

$$\left(\frac{T_0}{T}\right)^2 = \frac{\mathrm{Li}_2\left(e^{\mu/T}\right)}{\mathrm{Li}_2(1)} \approx \frac{\mathrm{Li}_2\left(1 - |\mu|/T\right)}{\mathrm{Li}_2(1)} . \tag{3.11}$$

The corresponding special function is now $\mathrm{Li}_2(x)$. Its definition and some properties necessary for the present treatment, are listed in Appendix. Using Eq. (A.3) with $\epsilon = |\mu|/T$, the equation for the determination of $|\mu|$ at $0 < T - T_0 \ll T_0$, reads

$$\frac{|\mu|}{T} \ln \frac{|\mu|}{T} \approx -\frac{\pi^2 (T - T_0)}{3T_0} ,$$

whose solution, with the logarithmic accuracy, can be represented in the form

$$\mu(T) \approx -\frac{\pi^2 (T - T_0)}{3 \ln \frac{3T_0}{\pi^2 (T - T_0)}} . \tag{3.12}$$

One can see that, in the experimentally accessible case $D = 2$, $\gamma = 2$ [5], the temperature dependence $\mu(T)$ is non-analytical at $T = T_0$. The differentiation over temperature shows that the chemical potential and its first derivative are continuous but the second derivative has an infinite discontinuity at the critical temperature. Hence the temperature derivative of the heat capacity is discontinuous.

Let us comment on the thermodynamics of the Bose gas in the limiting cases $\gamma = 0$ or $\gamma \to \infty$. The first one is reduced to the constant external potential $U(r) = U_0 = \mathrm{const}$. In this case, the thermodynamical properties are meaningful, if one restricts the system to the finite D-dimensional volume. Since the effect of the potential $U(r) = \mathrm{const}$ can be removed by the change of the energy reference point, the thermodynamical properties at $\gamma = 0$ are the same as in the case of free bosons. Curiously, but the seemingly opposite case $\gamma \to \infty$ corresponds exactly to the same case of the free Bose gas. The only difference is that the effect of the finite volume is reached naturally, see Eq. (2.5). The behavior of the BEC fraction $N_0(T)$ and the chemical potential $\mu(T)$ near the BEC transition point T_0 can be inferred from the results obtained in the preceding section by making the replacement $\alpha \to D/2$. In particular,

$$N_0(T) = N \left[1 - \left(\frac{T}{T_0}\right)^{D/2}\right]$$

which coincides with the results of [4]. For the ideal Bose gas, BEC is possible when $D > 2$. If $D > 4$, the chemical potential is continuous at $T = T_0$, but its derivative over T has the discontinuity of the magnitude

$$[\mu'] = -\frac{D\zeta(D/2)}{2\zeta(D/2-1)} \,.$$

If $D/2 = 1+1/k$, where $k = 2, 3, \ldots$, that is, $D = 3, 8/3, \ldots$, then the second, third, ... derivative of the chemical potential is discontinuous. In fact, one finds from (3.10), that, in the three-dimensional space, the magnitude of the discontinuity is

$$[\mu''] = -\frac{9\zeta^2(3/2)}{8\pi T_0} \approx -\frac{2.44}{T_0} \,.$$

This coincides with the textbook result. Finally, if $D = 4$, then $\mu(T)$ for $T > T_0$ is given by the expression (3.12), the same as in the case of the oscillatory external potential in the two-dimensional space.

4. Discussion and conclusion

The temperature behavior of the chemical potential as the function of the temperature is important from the point of view of establishing the kind of the phase transition, if such a transition indeed takes place. There is a well-known Ehrenfest classification of the phase transitions according to which the transition is of the first order, if the energy (in particular, the chemical potential) is discontinuous, of the second order, if the derivative of the energy over temperature (the heat capacity) is discontinuous, *etc.* If one treats the Bose–Einstein condensation as the phase transition, then the results of the preceding section help to establish the kind of this transition. One can see that in all cases except the one with $\alpha = 2$, the BEC transition is indeed the phase transition of some particular kind. Specifically, when $\alpha > 2$, the phase transition is of the second kind (see Eq. (3.4)), when $\alpha = 1 + 1/k$, where $k \geq 2$, the transition is of the $(k + 1)$-th kind (see Eq. (3.10)). But in the singular case of $\alpha = 2$, which is realized in $D = 4$ for the bosons in zero external potential (unphysical case), and in the two-dimensional oscillator (physically realized in [5]), the BEC transition is the phase transition of the third kind.

The kind of the phase transition could be established in the caloric experiments, *i.e.* via the measurement of the temperature dependencies of various thermodynamical quantities. Is the crossover character (3.12) of the BEC transition realized or not, was not studied in the experiment [5] with the $D = 2$ Bose gas of photons in the oscillator potential $U(r) \propto r^2$. The fact is that the Bose–Einstein condensation in the experiment [5] took place at the fixed room temperature. BEC was manifested as the macroscopic occupation of the state with the energy $E_0 = \hbar cs/D_0$, see Eq. (1.1), upon reaching some critical pump power of the laser corresponding to $N \sim 10^4$ photons in the cavity. This type of the experiment is not the caloric one.

To summarize, the power γ of the trapping spherically symmetric potential, $U \propto r^\gamma$, for dimension of the space D which admits the Bose–Einstein condensation of the ideal Bose gas, is found. The degree of singularity of the chemical potential and other thermodynamic functions appears to be very different for different space dimensions D and different power γ of the coordinate dependence of the trapping potential encoded in the quantity $\alpha = D/2 + D/\gamma$. In particular, the case $\alpha = 2$ corresponding to $D = 2$ and $\gamma = 2$, accessible through the recent experiment [5] with photons in the cavity, results in the non-analytic behavior of the temperature dependence of the chemical potential and other thermodynamic functions. The results obtained in the paper, could be possibly applied in the case of the hypothetical axionic BEC as the dark matter [6], in some power-like spherical-symmetric potentials in 3D, realized in the self-gravitating distributions of the above hypothetical particles.

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Appendix

Polylogarithm function

The polylogarithm function $\mathrm{Li}_\alpha(z)$ is defined by the following integral and series representations [10]

$$\mathrm{Li}_\alpha(z) = \frac{z}{\Gamma(\alpha)} \int\limits_0^\infty \frac{x^{\alpha-1}dx}{e^x - z} = \sum_{k=1}^\infty \frac{z^k}{k^\alpha}\,. \tag{A.1}$$

This function is defined at $|z| \leq 1$, and $\alpha > 1$. As is clear from this definition, $\mathrm{Li}_\alpha(1) = \zeta(\alpha)$. In particular, another equivalent representations for $\mathrm{Li}_2(z)$ is used in the paper

$$\mathrm{Li}_2(z) = -\int\limits_0^z \ln(1-t)\frac{dt}{t}\,. \tag{A.2}$$

To prove this representation, one should use the series expansion of $\ln(1-t)$ [11]. Then

$$\frac{\ln(1-t)}{t} = -\sum_{k=1}^{\infty} \frac{t^{k-1}}{k} \,,$$

and

$$-\int_0^z \ln(1-t)\frac{dt}{t} = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \mathrm{Li}_2(z) \,.$$

The expansion of $\mathrm{Li}_2(z)$ near $z=1$, used in the body of the paper, is

$$\mathrm{Li}_2(1-\epsilon) \approx \frac{\pi^2}{6} + \epsilon \ln \epsilon \,, \tag{A.3}$$

where $\epsilon \ll 1$. Indeed, one has the following chain of equations

$$\begin{aligned} \mathrm{Li}_2(1-\epsilon) &= -\int_0^{1-\epsilon} \ln(1-t)\frac{dt}{t} = -\left(\int_0^1 - \int_{1-\epsilon}^1\right) \ln(1-t)\frac{dt}{t} \\ &= \mathrm{Li}_2(1) + \int_{1-\epsilon}^1 \ln(1-t)\frac{dt}{t} = \mathrm{Li}_2(1) + \int_0^{\epsilon} \frac{du \ln u}{1-u} \\ &\approx \frac{\pi^2}{6} + \epsilon \ln \epsilon \,. \end{aligned}$$

The approximate equality has the logarithmic accuracy, and the relation $\zeta(2) = \pi^2/6$ is taken into account.

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