

ON THE QUANTUM COULOMB FIELD

BOGDAN DAMSKI

Los Alamos National Laboratory, Theoretical Division
MS B213, Los Alamos, NM, 87545, USA

PIOTR MARECKI

Fakultät für Physik, Universität Duisburg-Essen
Lotharstrasse 1, 47057 Duisburg, Germany*(Received January 12, 2012)*

The quantum theory of the Coulomb field has been developed by Andrzej Staruszkiewicz in the long series of papers. This theory explains the universality and quantization of the electric charge observed in Nature. Moreover, the efforts have been made to determine the value of the elementary charge from its mathematical structure. Nonetheless, no other immediate applications of this theory have been proposed. We make such an attempt by *(i)* considering the classical energy operator and defining its counterpart in the quantum theory of the Coulomb field; *(ii)* determining the eigenstates of the energy operator and assigning energy to the excitations of the theory; and *(iii)* proposing a simple theoretical scheme to estimate the effect of the quantum fluctuations of the Coulomb field on the energy levels of hydrogen-like atoms. We argue that the recent experimental advances in hydrogen and muonic-hydrogen spectroscopy may provide the unique window of opportunity for the verification of the Staruszkiewicz theory.

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1. Introduction

The theory of the quantum Coulomb field (QCF) has been proposed by Staruszkiewicz [1, 2, 3] (see also Ref. [4] for the discussion of this theory from a different angle and Refs. [5, 6, 7] for other approaches to the quantization of long-range fields). It supplements standard quantum electrodynamics (QED) by describing *long-range* quantum fluctuations of the Coulomb field [8]. The principal need for the development of this theory is

that even though the electric charge of every particle type *could* be arbitrary in the QED framework, all electric charges are multiples of a single quantum with a stunning accuracy [9].

The studies of the quantum Coulomb field have been centered on charge quantization and universality as well as on the extensive search for a distinguished value of the fine structure constant (see *e.g.* Ref. [10]). While the former efforts have succeeded, the Staruszkiewicz theory predicts that charged particles carry an integer multiple of the same elementary charge, the latter efforts are still ongoing. Indeed, even though it was found that the mathematically distinguished interval of the allowed values of α is $0 < \alpha < \pi$, the hunt for the theoretical determination of the exact value of the fine structure constant $\alpha = 1/137.036 \dots$ is unfinished. We remark that, to the best of our knowledge, the Staruszkiewicz theory provides the first theoretical framework in which such a hunt is at all possible.

Our goal is to extend the QCF theory to study its experimentally-relevant implications. We discuss the electric and magnetic field operators of the QCF, define a set of distinguished states of the QCF that might be carried by charged particles, and develop a simple theory providing a basis for the quantitative studies of the shifts of the energy levels of atoms resulting from the quantum fluctuations of the Coulomb field. The possibility that there may be additional level shifts absent in standard QED is strongly suggested by the puzzling recent precision spectroscopy experiment on muonic hydrogen [11].

2. Basics of the quantum Coulomb field theory

In this section, we will briefly summarize Staruszkiewicz's QCF theory. The central object of the theory is the phase field $S(x)$. It may be derived on the classical level in two ways.

Such a degree of freedom is always present for charged fields. This is seen by writing the action for the electromagnetic field and its source in the form

$$-\frac{1}{16\pi} \int d^4x F^{\mu\nu} F_{\mu\nu} + \int d^4x \mathcal{L} [eA_\mu + \partial_\mu S, \dots] , \quad (1)$$

where the phase S of the charged matter field appears in the gauge-invariant combination $eA_\mu + \partial_\mu S$, while the dots stand for other degrees of freedom of the charged matter field, *e.g.*, its real amplitude, relative phases between the different components of a spinor, *etc.*

To illustrate this point, one can consider the simplest example of a field theoretical system with the electric charge: the Klein–Gordon theory discussed in this context in Ref. [12]. Its gauge-invariant action reads

$-\frac{1}{16\pi} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \int d^4x |(\partial^\mu + ieA^\mu)\phi|^2 - m^2|\phi|^2$. Writing the Klein-Gordon field ϕ as $R \exp(iS)$ one finds that this action equals

$$-\frac{1}{16\pi} \int d^4x F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \int d^4x [\partial^\mu R \partial_\mu R + R^2 (\partial_\mu S + eA_\mu) (\partial^\mu S + eA^\mu) - m^2 R^2] ,$$

in agreement with Eq. (1). In this particular case, the dots in the argument of $\mathcal{L}$ stand for R (the real amplitude of the charged matter field) and its derivatives.

Coming back to the discussion of Eq. (1), we note that its variation with respect to A_μ defines the current j_μ to be equal to $\frac{1}{4\pi} \partial^\nu F_{\mu\nu}$. The variation of the action with respect to S leads to the charge conservation law: $\partial_\mu j^\mu = 0$. The momentum canonically conjugated with the phase field S is

$$\pi_S = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial S}{\partial t} \right)} = -\frac{j_0}{e} .$$

The canonical quantization $[\hat{S}(x), \hat{\pi}_S(y)]_{x^0=y^0} = i\delta(\mathbf{x} - \mathbf{y})$ results in

$$[\hat{j}_0(x), \hat{S}(y)]_{x^0=y^0} = ie\delta(\mathbf{x} - \mathbf{y}) .$$

Integrating it over the hyperplane $x^0 = y^0$, one obtains

$$[\hat{Q}, \hat{S}(y^0, \mathbf{y})] = ie , \quad (2)$$

where $\hat{Q} = \int d^3x \hat{j}_0$ is the charge operator, $e = 1/\sqrt{137.036\dots}$ stands for the unit of the electric charge, and $\hbar = c = 1$. This interesting relation, however, is rather useless unless further input is provided about the phase field $\hat{S}(x)$. This is done in the following way.

The electric charge can be also defined by looking only at the electromagnetic field. In order to find the minimal framework for such a consideration, one investigates the following scalar functional of the electromagnetic potential $A_\mu(x)$

$$\mathfrak{s}(x) = -ex^\mu A_\mu(x) .$$

It can be shown that it characterizes the electromagnetic fields of the Coulomb type completely. Namely, all electric and magnetic fields falling off as $1/r^2$ can be uniquely expressed as the appropriate derivatives of $\mathfrak{s}(x)$ through the relation $-ex^\nu F_{\mu\nu} = \partial_\mu \mathfrak{s}(x)$ (see *e.g.* Ref. [9]). Moreover, $\mathfrak{s}(x)$ satisfies the d'Alembert equation: $\square \mathfrak{s} = 0$.

In Ref. [1], Staruszkiewicz formulated a complete field-theoretical system by reflecting on the Gauss law. He identified $S(x)$ (the property of the charged matter) with $\mathfrak{s}(x)$ (the property of the electromagnetic field), and used the commutation relation (2) as the basis of the quantum field theory of the Coulomb field. Note that the identification of a degree of freedom of matter with some degree of freedom of the electromagnetic field is necessary if matter is required to carry a true electric charge even in asymptotic future/past (see *e.g.* Ref. [7]).

The quantization of the phase field proceeds in two steps. Firstly, as the consequence of both the d'Alembert equation satisfied by $S(x) = \mathfrak{s}(x)$ and the required fall-off condition for $S(x)$, one obtains the appropriate field operator [1]

$$\hat{S}(x) = \hat{S}_0 - e\hat{Q} \tanh(\psi) + \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} [\hat{c}_{\ell m} f_{\ell m} + \text{h.c.}] ,$$

$$f_{\ell m}(\psi, \theta, \varphi) = \left[{}_2F_1\left(-\frac{\ell+1}{2}, \frac{\ell}{2}; \frac{1}{2}; \tanh^2 \psi\right) \frac{G_{\ell}}{2} - \frac{i}{G_{\ell}} \tanh(\psi) {}_2F_1\left(-\frac{\ell}{2}, \frac{\ell+1}{2}; \frac{3}{2}; \tanh^2 \psi\right) \right] Y_{\ell m}(\theta, \varphi) ,$$

where (r, θ, φ) are the spherical coordinates around the charge, $\psi = \text{arctanh}(t/r)$ and

$$G_{\ell} = \sqrt{\frac{\ell}{\ell+1}} \frac{\Gamma\left(\frac{\ell}{2}\right)}{\Gamma\left(\frac{\ell+1}{2}\right)} .$$

The phase field $\hat{S}$ is quantized *outside* of the light cone $x_{\mu}x^{\mu} = t^2 - \mathbf{x}^2 = 0$, *i.e.*, for $r > |t|$. The non-vanishing commutators are

$$[\hat{c}_{\ell m}, \hat{c}_{\ell' m'}^{\dagger}] = 4\pi e^2 \delta_{\ell\ell'} \delta_{mm'} , \quad [\hat{Q}, \hat{S}_0] = ie .$$

As the second step of quantization, the representation space for the theory is defined to be the Fock space constructed upon the “vacuum” state $|0\rangle$, which is supposed to be annihilated by all $\hat{c}_{\ell m}$ and $\hat{Q}$ (the last relation being highly non-trivial; see also [13]).

The “simplest” state carrying n units of the electric charge, $|n\rangle$, is created from vacuum via $\exp(-in\hat{S}_0)$

$$|n\rangle = \exp\left(-in\hat{S}_0\right) |0\rangle , \quad \hat{Q}|n\rangle = ne|n\rangle , \quad \langle n|m\rangle = \delta_{nm} . \quad (3)$$

We will refer to $|n\rangle$ as the n -charged vacuum state as it is annihilated by all $\hat{c}_{\ell m}$. One should remember that $|n\rangle$ is a short cut for $|n; \{n_{\ell m}\}\rangle$, where $n_{\ell m}$, the occupation of the (ℓ, m) modes, is set to zero. The (ℓ, m) modes are responsible for the global angular distortions of the Coulomb field. Another distinguished charged state of the QCF was found in Ref. [14].

Finally, we would like to draw attention of the reader to the fact that independently of the beautiful physical relevance attached to the field $\hat{S}(x)$, the quantum theory of this field is a perfectly well-defined, highly interesting, and well worked-out quantum field theory in $2 + 1$ dimensional de Sitter space-time (see *e.g.* the recent non-trivial results on the structure of the boost operators and the spectral decomposition of the vacuum state [15, 16]).

3. Fluctuations of the quantum Coulomb field

It is convenient for our calculations to switch to the following notation. For $m = 0$, we define

$$\hat{c}_{\ell 0} = \sqrt{4\pi}ie \hat{b}_{\ell 0}, \quad \tilde{Y}_{\ell 0}(\theta, \varphi) = Y_{\ell 0}(\theta, \varphi),$$

while for $m > 0$, we define

$$\hat{c}_{\ell, \pm m} = \sqrt{2\pi}ie \left(\hat{b}_{\ell m} \pm \hat{b}_{\ell, -m} \right), \quad \tilde{Y}_{\ell, \pm m}(\theta, \varphi) = \frac{Y_{\ell m}(\theta, \varphi) \pm Y_{\ell, -m}(\theta, \varphi)}{\sqrt{2}}.$$

The redefined operators satisfy $[\hat{b}_{\ell m}, \hat{b}_{\ell' m'}^\dagger] = \delta_{\ell\ell'}\delta_{mm'}$ and $[\hat{b}_{\ell m}, \hat{b}_{\ell' m'}] = 0$.

From a given $\hat{S}(x) = \hat{\mathbf{s}}(x)$ the corresponding electromagnetic Coulomb fields can be computed using

$$\hat{A}^\mu(x) = -\frac{1}{e} \frac{x^\mu}{xx} \hat{S}(x).$$

In the standard spherical orthonormal tetrad $(\mathbf{N}, \mathbf{\Theta}, \mathbf{\Phi})$

$$\begin{aligned} \mathbf{N} &= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \\ \mathbf{\Theta} &= (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \\ \mathbf{\Phi} &= (-\sin \varphi, \cos \varphi, 0), \end{aligned}$$

we find

$$\begin{aligned} \hat{\mathbf{E}} &= -\frac{1}{er} \left[\mathbf{N} \frac{\partial}{\partial t} + \frac{t}{r^2 - t^2} \left(\frac{\mathbf{\Phi}}{\sin \theta} \frac{\partial}{\partial \varphi} + \mathbf{\Theta} \frac{\partial}{\partial \theta} \right) \right] \hat{S}, \\ \hat{\mathbf{H}} &= \frac{1}{e} \frac{1}{r^2 - t^2} \left(\frac{\mathbf{\Theta}}{\sin \theta} \frac{\partial}{\partial \varphi} - \mathbf{\Phi} \frac{\partial}{\partial \theta} \right) \hat{S}. \end{aligned}$$

Within the theory of Staruszkiewicz, the Coulomb electromagnetic fields are given only in the space-time region $|t| < r$ (the theory is originally formulated in spatial infinity outside of the light cone, where this condition does not cause problems; we extrapolate the fluctuations from the spatial infinity to near the charge center assuming homogeneity of the Coulomb field). This reflects the lack of a complete theory of charged matter accompanied by such fields (for attempts in this direction see Ref. [7]). In this paper, we circumvent that difficulty by dealing exclusively with effects on the surface $t = 0$, for which the theory gives complete information about the electromagnetic fields (apart from the singular point $r = 0$)

$$\hat{\mathbf{E}}(0, \mathbf{x}) = \left[\hat{Q} - \sqrt{4\pi} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{\tilde{Y}_{\ell m}}{G_{\ell}} \hat{b}_{\ell m} + \text{h.c.} \right) \right] \frac{\mathbf{N}}{r^2}, \quad (4)$$

$$\hat{\mathbf{H}}(0, \mathbf{x}) = \frac{\sqrt{\pi}}{r^2} \left[\frac{\boldsymbol{\Theta}}{\sin \theta} \frac{\partial}{\partial \varphi} - \boldsymbol{\Phi} \frac{\partial}{\partial \theta} \right] \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \left(i \tilde{Y}_{\ell m} G_{\ell} \hat{b}_{\ell m} + \text{h.c.} \right). \quad (5)$$

Several remarks are in order now.

First, these expressions show that there is a fluctuating electric and magnetic field around a charge: an expected feature of the quantum Coulomb field. The amount of fluctuation depends on the quantum state of its Coulomb field. Physically sound states describing the QCF of a charge at rest are those reproducing the classical result

$$\langle \hat{\mathbf{E}}(0, \mathbf{x}) \rangle = ne \frac{\mathbf{N}}{r^2}, \quad \langle \hat{\mathbf{H}}(0, \mathbf{x}) \rangle = 0. \quad (6)$$

This condition, however, is satisfied by the infinite number of states. For example, by any single Fock state $|n, \{n_{\ell m}\}\rangle$, where $n_{\ell m}$ is an arbitrary non-negative integer. It is therefore fundamentally important to ask which quantum state of the Coulomb field is carried by a charged particle, say an electron or a proton?

Second, the amount of the fluctuation of the quantum Coulomb field should be experimentally measurable through the studies of the level shifts of atoms (especially hydrogen-like for which the techniques of precision spectroscopy work best [17, 18]; we will come back to this point later). This is a simple conclusion well founded on the success of QED to explain the Lamb shift. The Lamb shift, originally the splitting of the $2S_{1/2}$ and $2P_{1/2}$ levels in hydrogen, results from the coupling between the orbiting charge and the fluctuating vacuum field [19, 20, 21].

Third, one easily sees from (4) that the relative fluctuation of the QCF with respect to its classical counterpart is (i) independent of the distance from the charge center as both decay as $1/r^2$ and (ii) decays as the inverse

of the charge carried by the QCF (fluctuations are charge independent while the classical field goes linearly with charge). The (i) effect is in stark contrast to what one finds in QED, where the relative modification of the classical Coulomb field by virtual electron–positron pairs is short-ranged [21, 22].

4. Energy associated with the quantum Coulomb field

Having realized that there are many quantum states of the Coulomb field that are physically allowed, we propose a simple physically-motivated approach allowing for finding a set of distinguished states of the QCF. Namely, we propose to “label” different configurations of the QCF by an energy associated with them. To this end, we define electromagnetic energy operator on the surface $t = 0$

$$\hat{\mathcal{E}}_C = \frac{1}{8\pi} \int d^3x : \hat{\mathbf{E}}(0, \mathbf{x}) \hat{\mathbf{E}}(0, \mathbf{x}) + \hat{\mathbf{H}}(0, \mathbf{x}) \hat{\mathbf{H}}(0, \mathbf{x}) :, \quad (7)$$

where $::$ stands for normal ordering [23].

In classical electrodynamics the expression (7) corresponds to the energy of field configurations. In the QCF theory states reproducing on average the classical Coulomb field do exist, and we feel it justified to claim that their energy should be computed from the quantum version of the classical energy functional. However, since $\hat{\mathbf{E}}(0, \mathbf{x})$ and $\hat{\mathbf{H}}(0, \mathbf{x})$ scale as $1/r^2$, this operator has problems due to the divergence of the integral at $r = 0$. Moreover, it does not have the interpretation of the generator of the time-translation symmetry (the Hamiltonian). We will discuss the former problem below and mention here that it is our assumption to assign the energy to the excitations of the QCF through the operator (7).

Putting operators (4) and (5) into the above expression, we obtain

$$\hat{\mathcal{E}}_C = \frac{\varepsilon_c}{e^2} \hat{Q}^2 + \frac{\varepsilon_c}{e^2} \sum_{\ell=1}^{\infty} \gamma_{\ell} \hat{\mathcal{E}}_{\ell}, \quad (8)$$

$$\begin{aligned} \hat{\mathcal{E}}_{\ell} = & \hat{b}_{\ell 0}^{\dagger} \hat{b}_{\ell 0} + \frac{\beta_{\ell} (-1)^{\ell}}{2} \left(\hat{b}_{\ell 0}^2 + \hat{b}_{\ell 0}^{\dagger 2} \right) \\ & + \sum_{m=1}^{\ell} \left[\hat{b}_{\ell m}^{\dagger} \hat{b}_{\ell m} + \hat{b}_{\ell, -m}^{\dagger} \hat{b}_{\ell, -m} + (-)^{m+\ell} \frac{\beta_{\ell}}{2} \left(\hat{b}_{\ell m}^2 - \hat{b}_{\ell, -m}^2 + \text{h.c.} \right) \right]. \end{aligned} \quad (9)$$

We introduced above

$$\beta_{\ell} = \frac{4 - G_{\ell}^4 \ell(\ell + 1)}{4 + G_{\ell}^4 \ell(\ell + 1)}, \quad \gamma_{\ell} = \frac{2}{G_{\ell}^2} + \frac{G_{\ell}^2 \ell(\ell + 1)}{2},$$

and

$$\varepsilon_c = \frac{e^2}{8\pi} \int \frac{d^3x}{r^4}. \quad (10)$$

The latter corresponds to the classical electromagnetic energy of a unit charge at rest. We do not propose any new solution to the obvious lack of convergence of this integral at $r = 0$. On general grounds, however, we expect that in any such solution, the parameter ε_c should be of the order of the electron's rest mass m_e . This is in line with the century long efforts to associate the mass of an electron with the energy of its field [24] (see also Ref. [25] for the relevant hadronic example). We shall also argue below that the actual spectroscopic consequences of the quantum theory of Coulomb field will be in a good position to determine the numerical value of ε_c .

To look for the spectrum of the electromagnetic energy operator, we diagonalize (8). This is done by applying the Bogolubov transformation

$$\hat{B}_{\ell m} = \cosh(u_{\ell m}) \hat{b}_{\ell m} - \sinh(u_{\ell m}) \hat{b}_{\ell m}^\dagger, \quad (11)$$

where assuming that $m > 0$

$$u_{\ell 0} = \frac{(-)^{\ell+1} \operatorname{arctanh}(\beta_\ell)}{2}, \quad u_{\ell, \pm m} = \frac{\mp (-)^{m+\ell} \operatorname{arctanh}(\beta_\ell)}{2}. \quad (12)$$

The transformation (11) can be easily expressed as the explicit function of ℓ employing

$$\operatorname{arctanh}(\beta_\ell) = \ln \frac{2}{G_\ell^2 \sqrt{\ell(\ell+1)}}.$$

Finally, we note that $[\hat{B}_{\ell m}, \hat{B}_{\ell' m'}^\dagger] = \delta_{\ell\ell'} \delta_{mm'}$ and $[\hat{B}_{\ell m}, \hat{B}_{\ell' m'}] = 0$ hold for the redefined operators.

Putting Eqs. (11) and (12) into Eqs. (8) and (9), we find

$$\begin{aligned} \hat{\mathcal{E}}_C = & \hat{Q}^2 \frac{\varepsilon_c}{e^2} + \frac{\varepsilon_c}{2e^2} \sum_{\ell=1}^{\infty} \gamma_\ell \left(\sqrt{1 - \beta_\ell^2} - 1 \right) (2\ell + 1) \\ & + \frac{2\varepsilon_c}{e^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\ell(\ell+1)} \hat{B}_{\ell m}^\dagger \hat{B}_{\ell m}, \end{aligned} \quad (13)$$

where we have employed the identity $\gamma_\ell \sqrt{1 - \beta_\ell^2} = 2\sqrt{\ell(\ell+1)}$ to simplify the result.

The first term in (13) recovers the classical electromagnetic energy of a charge at rest. The second one is the contribution of the zero point modes (see also [23])

$$\frac{\varepsilon_c}{2e^2} \sum_{\ell=1}^{\infty} \gamma_{\ell} \left(\sqrt{1 - \beta_{\ell}^2} - 1 \right) (2\ell + 1) \approx -5 \times 10^{-2} \frac{\varepsilon_c}{e^2}. \quad (14)$$

The third term in Eq. (13) provides the spectrum of the electromagnetic energy operator, which is degenerate for all m in every ℓ sector. We define the ground state of $\hat{\mathcal{E}}_C$, say $|G\rangle$, as the state that is annihilated by all $\hat{B}_{\ell m}$ operators. For generality, we assume that it carries n quanta of the electric charge: $\hat{Q}|G\rangle = ne|G\rangle$. The excited states are created in the n -charged sector by acting (an arbitrary number of times) $\hat{B}_{\ell m}^{\dagger}$ on $|G\rangle$. It is worth to mention that the excitation gap to the ℓ sector equals

$$\frac{2\varepsilon_c \sqrt{\ell(\ell+1)}}{e^2}, \quad (15)$$

which is a huge number assuming that indeed $\varepsilon_c = \mathcal{O}(m_e)$. Given the physical interpretation of the electromagnetic energy operator and an expected huge gap in the excitation spectrum, we propose the ground state $|G\rangle$ as the natural candidate for representing the quantum state of the Coulomb field of a stable charged particle (*e.g.* an electron or a proton [26]). Note that both the charged vacuum state $|n\rangle$ and the ground state $|G\rangle$ are spherically symmetric. The expectation value of the energy operator is lowest in the state $|G\rangle$ (the state $|n\rangle$ is not an eigenstate of this operator). In what follows, we describe some of its basic properties.

First, we note that the expectation value of the electric and magnetic field operators, calculated in the ground state $|G\rangle$, reproduces the classical result (6). States supposed to correspond to moving particles can be constructed because the generators of the boosts are explicitly known (see Refs. [27, 28] and the Appendix). For example, the quantum state of the Coulomb field of the charge moving in the $+z$ direction with velocity $v = \tanh\lambda$ is

$$|G, \lambda\rangle = \exp\left(-i\lambda\hat{M}^{03}\right)|G\rangle,$$

where $\hat{M}^{03}$ is the boost operator (A.6).

We expect that the averages,

$$\langle G, \lambda | \hat{\mathbf{E}}(0, \mathbf{x}) | G, \lambda \rangle, \quad \langle G, \lambda | \hat{\mathbf{H}}(0, \mathbf{x}) | G, \lambda \rangle,$$

computed as the functions of $\mathbf{x}$ on the surface $t = 0$, are equal to the classical fields of a moving charge crossing this surface at $\mathbf{x} = 0$ [29]

$$\mathbf{E}_{\text{cl}} = N \frac{ne}{r^2} \frac{1 - v^2}{(1 - v^2 \sin^2 \theta)^{3/2}}, \quad \mathbf{H}_{\text{cl}} = \mathbf{v} \times \mathbf{E}_{\text{cl}}.$$

The theory has enough structure to prove that it is the case, but as a check we have verified the result by a direct calculation to the order λ^4 . We also note that the action of the boost operator on the ground state $|G\rangle$ additionally populates the (ℓ, m) modes in the $|G, \lambda\rangle$ state. This allows for the addition of the non spherically-symmetric component to the quantum Coulomb field, which is indispensable for the reproduction of the above classical result.

Second, it turns out that $|G\rangle$ is a squeezed vacuum state $|n\rangle$ (see Ref. [30] for the discussion of the squeezed states in the quantum optics context)

$$|G\rangle = \exp(\hat{D}) |n\rangle, \quad \hat{D} = \frac{1}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} u_{\ell m} (\hat{b}_{\ell m}^{\dagger 2} - \hat{b}_{\ell m}^2). \quad (16)$$

This is easily seen from the following identity $\exp(-\hat{D}) \hat{B}_{\ell m} \exp(\hat{D}) = \hat{b}_{\ell m}$. Moreover, the overlap between the ground state and the charged vacuum state is

$$\langle n|G\rangle = \prod_{\ell=1}^{\infty} \left[\cosh \left(\frac{\text{arctanh}(\beta_{\ell})}{2} \right) \right]^{-\ell - \frac{1}{2}} \approx 0.997.$$

The two spherically-symmetric states are, therefore, quite similar. Interestingly, their overlap is independent of the fine structure constant $\alpha = e^2$. Thus, it is of purely geometric nature. Furthermore, we can study the population of the (ℓ, m) modes,

$$\langle G | \hat{b}_{\ell m}^{\dagger} \hat{b}_{\ell m} | G \rangle = \sinh^2 \left(\frac{\text{arctanh}(\beta_{\ell})}{2} \right) = \frac{\gamma_{\ell}}{4\sqrt{\ell(\ell+1)}} - \frac{1}{2}. \quad (17)$$

It equals about 3×10^{-3} for $\ell = 1$, decays monotonically with ℓ , and for $\ell \gg 1$ it approaches $1/64\ell^4$: only low (ℓ, m) modes are noticeably populated.

Third, while the expectation value of the field operators (4) and (5) reproduces the classical result, their fluctuations in the ground state have absolutely no classical counterpart. This is so because the long-range fields are now described by a quantum field theory, and not just by the classical functions. Indeed,

$$\begin{aligned} \left\langle \hat{\mathbf{E}}(0, r, \theta, \varphi) \hat{\mathbf{E}}(0, r, \theta', \varphi') \right\rangle &= \frac{n^2 e^2}{r^4} \\ &+ \frac{1}{r^4} \sum_{\ell=1}^{\infty} \frac{2\ell+1}{G_{\ell}^2} \eta_{\ell} P_{\ell}(\cos \theta \cos \theta' + \cos(\varphi - \varphi') \sin \theta \sin \theta'), \end{aligned}$$

where $\eta_\ell = 1$ when the average is calculated in the n -charged vacuum state $|n\rangle$ and

$$\eta_\ell = \frac{G_\ell^2 \sqrt{\ell(\ell+1)}}{2} > 1$$

when it is calculated in the ground state $|G\rangle$. (P_ℓ stands for the Legendre polynomial.) This shows explicitly that the charged states $|n\rangle$ and $|G\rangle$ (*i*) differ by the pattern of the fluctuations of the electric field associated with them; (*ii*) are spherically symmetric.

5. Discussion of the results

To start, this work proposes that there can be a *fluctuating long-range* electromagnetic field around every charge. This field originates from the quantum fluctuations of the Coulomb field. We propose that it should be considered in addition to the fluctuating vacuum field studied in the standard QED framework. Since such a field has not been observed yet, the confirmation of its presence will fundamentally update our understanding of charged particles. It is thus critical to find out the experimentally accessible consequences of this *conjecture*.

General experience with quantum systems interacting with the fluctuating electromagnetic fields is that (at the very least) their energy levels are subjected to shifts [21]. We propose to employ the spectroscopic measurements of the energy levels of the bound systems such as hydrogen, muonic hydrogen, muonium, *etc.* in the hunt for the experimental evidence of the quantum nature of the Coulomb field.

To quantify the expected level shifts, one can adopt the perturbative formalism analogous to the one employed for the calculation of the Lamb shift [19,20,21,31]. In this approach the quantum state of the Coulomb field of a nucleus is modeled by our ground state $|G\rangle$ (16). The field operators used for writing down the coupling between the orbiting charge and the fluctuating Coulomb field are given by Eqs. (4) and (5). The excited states of the QCF and their energies (both required in the perturbative expansion) are provided by the eigenstates and eigenvalues of our electromagnetic energy operator (13). Since the eigenvalues depend on ε_c , we propose to treat it as the (only) free parameter and argue that its value can be experimentally determined from the spectroscopic measurements. Naturally, the larger ε_c the smaller the level shift due to the QCF should be.

The magnitude of the level shifts resulting from the presence of the QCF is bounded by the discrepancies between the experimental measurements and the QED calculations. For example, the frequency of the 1S–2S transition in hydrogen is experimentally known with the fantastic accuracy of about 40 Hz corresponding to the relative accuracy of about 1 part in 10^{14} [17,18]. Even

more amazingly, this result is expected to be significantly improved in the foreseeable future [18]. The QED calculations match it with the relatively thick error bar of several tens of kHz [22]. The most of this theoretical uncertainty comes from the imprecise knowledge about the “distribution of charge” in a proton. Thus, the QCF corrections to this transition are bounded from above by a few tens of kHz.

The above-outlined possibility to consider additional level shifts in hydrogen-like atoms is especially appealing in the light of the current discrepancies between the standard QED predictions and the spectacular recent measurements of the Lamb shift in muonic hydrogen [11]. It is also quite natural given the fact that precision spectroscopy has been serving for a long time as the Rosetta Stone for deciphering the laws of quantum physics [18]. The formalism proposed in this manuscript should lay the ground for the studies of the spectroscopic consequences of the quantum Coulomb field. The work along these lines is already ongoing [31].

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Appendix

We use spherical harmonics defined as

$$Y_{\ell m}(\theta, \varphi) = (-)^{(m+|m|)/2} i^\ell \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}} P_\ell^{|m|}(\cos\theta) \exp(im\varphi), \quad (\text{A.1})$$

following the convention introduced by Staruszkiewicz in Ref. [3]. For the convenience of the reader we list the first three ℓ sectors:

$$\begin{aligned} Y_{00} &= \frac{1}{\sqrt{4\pi}}, & Y_{10} &= i\sqrt{\frac{3}{4\pi}} \cos\theta, \\ Y_{1,\pm 1} &= \mp i\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\varphi}, & Y_{20} &= \sqrt{\frac{5}{16\pi}} (1 - 3\cos^2\theta), \\ Y_{2,\pm 1} &= \pm\sqrt{\frac{15}{32\pi}} \sin(2\theta) e^{\pm i\varphi}, & Y_{2,\pm 2} &= -\sqrt{\frac{15}{32\pi}} \sin^2\theta e^{\pm 2i\varphi}. \end{aligned}$$

In particular, this implies that

$$\begin{aligned} Y_{\ell m}^*(\theta, \varphi) &= (-)^{m+\ell} Y_{\ell, -m}(\theta, \varphi), \\ \hat{L}_{\pm} Y_{\ell m} &= \sqrt{\ell(\ell+1) - m(m \pm 1)} Y_{\ell, m \pm 1}, \end{aligned} \quad (\text{A.2})$$

where

$$\hat{L}_{\pm} = e^{\pm i\varphi} \left(\pm \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right).$$

In our manuscript, we use the redefined spherical harmonics $\tilde{Y}_{\ell m}$. Considering $m > 0$, one has

$$\tilde{Y}_{\ell m}^*(\theta, \varphi) = (-)^{m+\ell} \tilde{Y}_{\ell m}(\theta, \varphi), \quad \tilde{Y}_{\ell, -m}^*(\theta, \varphi) = -(-)^{m+\ell} \tilde{Y}_{\ell, -m}(\theta, \varphi). \quad (\text{A.3})$$

Noting that

$$\sum_{m=-\ell}^{\ell} \tilde{Y}_{\ell m}(\theta, \varphi) \tilde{Y}_{\ell m}^*(\theta', \varphi') = \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi) Y_{\ell m}^*(\theta', \varphi'),$$

we obtain

$$\sum_{m=-\ell}^{\ell} \tilde{Y}_{\ell m}(\theta, \varphi) \tilde{Y}_{\ell m}^*(\theta', \varphi') = \frac{2\ell+1}{4\pi} P_{\ell}(\cos \theta \cos \theta' + \cos(\varphi - \varphi') \sin \theta \sin \theta'), \quad (\text{A.4})$$

where P_{ℓ} is the Legendre polynomial. Eqs. (A.2), (A.3) and (A.4) are useful in the derivation of both the electromagnetic energy operator and the expression for the two-point correlation function.

The boost operator $\hat{M}^{03}$, that we use in the paper, was calculated in Ref. [27] (see also Ref. [28]). We list it below for reader's convenience

$$\begin{aligned} \hat{M}^{03} &= \frac{1}{\sqrt{6\pi e}} \hat{Q} \left(\hat{c}_{10} + \hat{c}_{10}^{\dagger} \right) \\ &+ \frac{i}{4\pi e^2} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{\ell^2-1}{4\ell^2-1}} \sqrt{\ell^2-m^2} \left(\hat{c}_{\ell m}^{\dagger} \hat{c}_{\ell-1, m} - \text{h.c.} \right), \end{aligned} \quad (\text{A.5})$$

which in our notation corresponds to

$$\begin{aligned} \hat{M}^{03} &= i \sqrt{\frac{2}{3\pi}} \hat{Q} \left(\hat{b}_{10} - \hat{b}_{10}^{\dagger} \right) + \\ &i \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sqrt{\frac{\ell^2-1}{4\ell^2-1}} \sqrt{\ell^2-m^2} \left(\hat{b}_{\ell m}^{\dagger} \hat{b}_{\ell-1, m} - \text{h.c.} \right). \end{aligned} \quad (\text{A.6})$$

We mention in passing that the derivation of the boost operators (A.5) and (A.6) assumes that spherical harmonics are given by Eq. (A.1).

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