NON-EXTENSIVE STATISTICAL MODEL FOR STRANGE AND NON-STRANGE HADRON SPECTRA AT RHIC AND LHC ENERGIES*

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We review the basic assumptions, proofs and phenomenological applications of non-extensive thermodynamics and statistical models to high-energy elementary and heavy-ion collisions. We also speculate about physical processes in terms of classical field theory which may mimic a thermal source of the experimentally observed spectra.

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1. Introduction

When dealing with thermal models and their predictions, it is frequently asked where is the thermal bath and the time long enough to reach equipartition. Despite the phenomenological success of such models a deeper understanding of the physical mechanisms behind this success is much unknown.

In this paper we review, on the one hand, possible generalizations of the classical basis of thermodynamics towards incorporating more phenomena than those described by exponential energy distributions and, on the other hand, we also point out theoretical possibilities along which an apparently thermal behaving spectrum of particles may arise without a physical thermal bath being ever present in any single event of the studied ensemble.

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2. Generalized thermal equilibrium

The classical concept of thermal equilibrium deals with the long term result of equilibration processes between large systems containing many dynamical degrees of freedom. In this limit, the assumption of dynamical and statistical independence of large parts of even larger physical systems is a commonplace, and the main consequences, as the equipartition, the entropy maximum in equilibrium and the transitivity of the concept of the absolute temperature among several bodies in pairwise thermal equilibrium, are widely known. In this section, we present a possible generalization of these conceptual fundamentals.

We connect the additivity of the thermodynamical extensives, in particular that of entropy and energy, with the zeroth theorem declaring the universality of the absolute temperature and indicate a certain way to loosen the requirement of additivity. For the details of the derivation see [1]. We derive a general formula for the thermodynamical temperature in entropy maximum states and demonstrate that the frequently cited entropy formulas by Renyi [2,3] and Tsallis [4,5] lead to equivalent results in equilibrium. In fact, the Renyi entropy is special as being the formal logarithm to the Tsallis entropy, and as such, an additive quantity.

In classical thermodynamics the maximum entropy principle [6,7], the additivity of the main extensives, like entropy and energy, and the zeroth law defining the absolute temperature are interrelated. Any third can be derived by assuming two others. For example, assuming the addition as the relevant composition law, $S_{12} = S_1 + S_2$ and $E_{12} = E_2 + E_2$ by the unification of two systems, and maximizing

$$\Big(S[p_i] - S(E)\Big) - \alpha\Big(\sum_i p_i - 1\Big) - \beta\Big(\sum_i p_i E_i - E\Big) = \max. \tag{1}$$

one concludes that the Lagrange multiplier β can also be derived from the equation of state, $S(E)$, as being

$$\beta = \frac{1}{T} = S'(E)\,, \qquad S_1'(E_1) \;=\; S_2'(E_2)\,. \tag{2}$$

Assuming, on the other hand, more general composition laws than the addition, the composite quantities, S_{12} and E_{12} may have non-trivial derivatives with respect to their arguments. Then the generalization of Eq. (2) holds

$$\frac{\partial S_{12}}{\partial S_1}\,\frac{\partial E_{12}}{\partial E_2}\,S_1'(E_1) \;=\; \frac{\partial S_{12}}{\partial S_2}\,\frac{\partial E_{12}}{\partial E_1}\,S_2'(E_2)\,. \tag{3}$$

This equation leads to a general requirement with respect to the composition functions. Their factors, depending only on one or the other system's variable, are factorisable into a quantity for the system 1 equal to the same quantity for the system 2 type formula. Remaining factors depending on both systems' variables then have to satisfy equal ratios

$$\frac{H_{12}(S_1,S_2)}{H_{21}(S_1,S_2)} = \frac{C_{12}(E_1,E_2)}{C_{21}(E_1,E_2)}\,. \tag{4}$$

This ratio is a constant if it is valid for any arbitrary functional connections, $S_1(E_1)$ and $S_2(E_2)$, *i.e.*, for any equations of state for the systems in equilibrium. It has twofold consequences: *(i)* the absolute temperature is generalized to a form

$$\frac{1}{T} = \frac{\partial \hat{L}(S)}{\partial L(E)}\,, \tag{5}$$

where $\hat{L}(S)$ and $L(E)$ are monotonic functions of the entropy and energy and *(ii)* the composition formulas are equivalent with the relations

$$\begin{aligned} \hat{L}_{12}(S_{12}) &= \hat{L}_1(S_1) + \hat{L}_2(S_2)\,, \\ L_{12}(E_{12}) &= L_1(E_1) + L_2(E_2)\,. \end{aligned} \tag{6}$$

Due to the property of "mapping a general composition law to the addition" these functions are named *formal logarithms*.

The composition formula satisfied by the Tsallis entropy, for example,

$$S_{12} = S_1 + S_2 + (q-1)S_1S_2\,, \tag{7}$$

leads to

$$\hat{L}(S) = \frac{1}{q-1}\ln\left(1+(q-1)S\right)\,. \tag{8}$$

In fact, applying this transformation to the Tsallis entropy formula the Renyi entropy emerges [1,8].

3. Canonical and microcanonical spectra

One is familiar with the exponentially decreasing Boltzmann–Gibbs energy distribution as the general canonical formula. As we have demonstrated in the previous section, the definition of the thermodynamical temperature can be generalized, so can be the canonical distribution. Now we consider the additive, but parameter dependent Renyi entropy to be maximized with the canonical constraint on the average energy per particle. This constraint leads to a perfectly power-law tailed canonical energy distribution, as follows.

The formula for the Renyi entropy,

$$S_{\mathrm{R}} = \frac{1}{1-q} \ln \sum_i p_i^q \,, \tag{9}$$

contains the parameter q. It is an additive measure of entropy for any fixed q; it gives the sum of the corresponding entropy contributions if the joint probabilities factorize. For $q = 1$ it resembles the Boltzmann–Gibbs–Shannon entropy formula

$$S_{\mathrm{R}}\big|_{q=1} = S_{\mathrm{BGS}} = -\sum_i p_i \ln p_i \,. \tag{10}$$

The entropy maximum with Renyi's formula at fixed normalization and average energy contains two Lagrange multipliers, α and β

$$\frac{1}{1-q} \ln \sum_i p_i^q - \alpha \sum_i p_i - \beta \sum_i p_i E_i = \max. \tag{11}$$

Varying this expression with respect to the p_i probabilities delivers

$$\frac{1}{1-q} \frac{q p_i^{q-1}}{\sum_j p_j^q} = \alpha + \beta E_i \,. \tag{12}$$

This equation can easily be resolved for the canonical equilibrium distribution

$$p_i = e^{-S_{\mathrm{R}}} \left(1 + (1-q) \frac{\beta}{q} \left(E_i - E \right) \right)^{\frac{1}{q-1}} \,. \tag{13}$$

This is a cut power-law offering a thermal interpretation to the Pareto–Tsallis–Levy fits in high energy experiments.

According to the discussion presented in the previous section, however, not only the average energy, but also the average of a special function of the energy, its formal logarithm, $L(E)$, may make a sense to be fixed. In this case, the quantities E_i and E are to be replaced by their respective formal logarithms, $L(E_i)$ and $L(E)$ in Eq. (13). Fig. 1 presents the schematic form of Boltzmann–Gibbs (lowest curve), the q-entropic (uppermost curve) and the q-entropic approach with the formal logarithm of the individual energies (middle curve).

Considering a statistical model for jet fragmentation in e^+e^- collisions, an effective dimensionality D can be considered between 1 and 3 due to some transverse diversity of momenta inside the jets. Applying the above formula to $x = E_i/(\sqrt{s}/2)$ energy fraction, the formal logarithm of a non-additive energy composition law with $a = -2/\sqrt{s}$ gives rise to the $-\ln(1-x)$

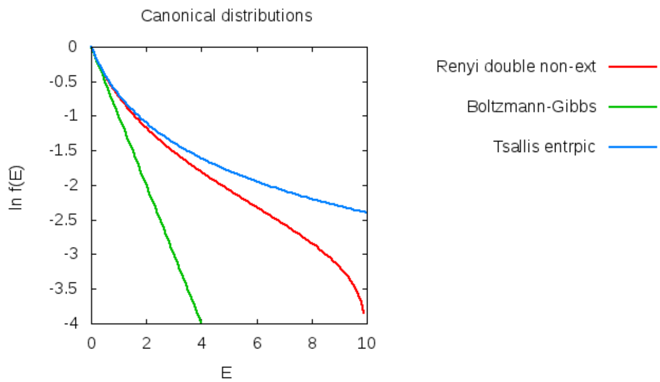


Fig. 1. (Color online) Canonical distributions in classical and generalized thermodynamics (schematic curves).

argument. Supplemented with phase space and multiplicity factors, powers of x and $(1-x)$ may also occur. Also in a microcanonical approach [9, 10, 11, 12] an equilibrium statistical distribution can be derived; it differs from the canonical one only near to the edge of the one-particle energy, close to $\sqrt{s}/2$. We have fitted the corresponding microcanonical formula [12] to various experimental data on pion and kaon yields from high-energy e^+e^- collisions

$$\frac{1}{\sigma}\frac{d\sigma}{dx} = \frac{A\,x^{D-1}(1-x)^{D(N_0-1)-1}}{\left(1-\dfrac{q-1}{T/(\sqrt{s}/2)}\ln(1-x)\right)^{1/(q-1)}} . \tag{14}$$

The results are plotted in Fig. 2, while Fig. 3 displays the fitted parameters T and q.

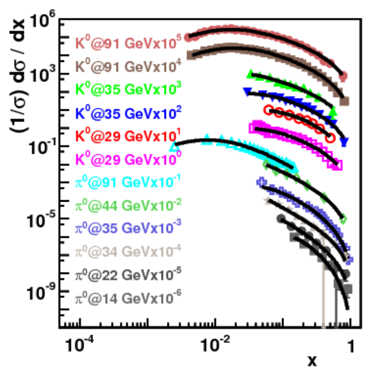
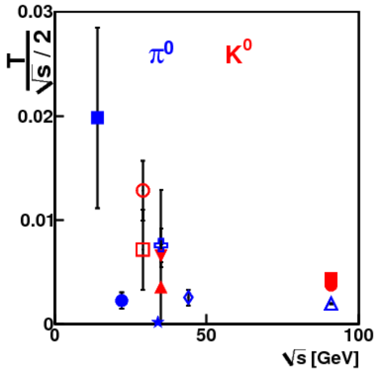


Fig. 2. Fragmentation functions of π^0s and K^0s measured at various collision energies (data, from top to bottom, are published in Refs. [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 22, 22]) and fitted with microcanonical Tsallis–Pareto distributions (Eq. (14) with $D = 3$ and $N_0 = 1 + 1/D$).



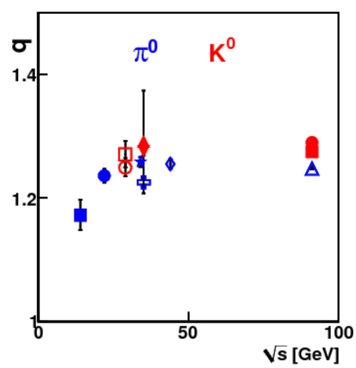


Fig. 3. Fitted values of the T and q parameters in Eq. (14) with $D = 3$ and $N_0 = 1 + 1/D$ to measured fragmentation functions shown in Fig. 2.

4. Superstatistics

The power-law tailed canonical energy distribution is an exponential distribution in terms of the corresponding formal logarithm of the energy, occurring in the zeroth law

$$e^{-\beta L(E)} = e^{-\frac{1}{aT}\ln(1+aE)} = (1+aE)^{-1/aT} . \tag{15}$$

Such an expression can also be viewed as an integral of momentaneous Gibbs factors over an Euler–Gamma distribution of β values with expectation value of $1/T$

$$p_i^{\mathrm{eq}} = \frac{1}{Z}\left(1+\hat{\beta}E_i/c\right)^{-c} = \frac{1}{Z}\int\limits_0^\infty dx\, \frac{c^c}{\Gamma(c)}\, x^{c-1}e^{-cx}\, e^{-x\hat{\beta}E_i} . \tag{16}$$

Here $\hat{\beta} = \beta/(1+(q-1)\alpha)$, $Z = (1+(q-1)\alpha)^c$ and $c = 1/a = 1/(q-1)$ connects to the Renyi or Tsallis entropy formula.

In general, a distribution of the thermodynamical intensives is considered by superstatitics [8,23]. Not only the above connection between the traditional Boltzmann–Gibbs exponential and the power-law tailed canonical Tsallis–Pareto energy distribution maximizing the Renyi entropy is known, but an even more familiar case, the negative binomial multiplicity distribution in high energy hadron production [24]. The negative binomial, containing the ratio of two negative integer factorials is finite,

$$\begin{pmatrix} -k-1 \\ n \end{pmatrix} = (-1)^n \begin{pmatrix} k+n \\ n \end{pmatrix} . \tag{17}$$

It defines a Bernoulli distribution,

$$P_{n,k} = \begin{pmatrix} k+n \\ n \end{pmatrix} f^n (1+f)^{-k-1-n}\,, \tag{18}$$

which by using the identity

$$\int\limits_0^\infty x^N\, e^{-ax}\, dx = \frac{N!}{a^{N+1}} \tag{19}$$

can be also written as an integral

$$P_{n,k} = \frac{f^n}{k!n!} \int\limits_0^\infty x^{k+n}\, e^{-(1+f)x}\, dx\,. \tag{20}$$

This form is, in fact, a convolution of a Poisson distribution in the multiplicity n with the mean value xf (instead of f) with an Euler–Gamma distribution in x

$$P_{n,k} = \int\limits_0^\infty \frac{(xf)^n}{n!} e^{-xf}\, \frac{x^k}{k!}\, e^{-x}\, dx\,. \tag{21}$$

Note that the Euler–Gamma distribution in the scale factor x with parameter k is at the same time a Poisson distribution in the integer k with the mean value x. If the statistical power-law tail is due to event by event multiplicity fluctuations, then the relation

$$q = \frac{k}{k+1} \tag{22}$$

should hold. This relation may be experimentally checked in the future.

5. Apparently "thermal" spectra

Even if thermal and hydrodynamical models assuming local equilibrium with a locally thermalized environment have enormous success in describing experimental spectra stemming from heavy ion reactions and also some success for data from the more elementary proton–proton collisions, the physical mechanism behind this success remains obscure. Here we indicate that, theoretically, even without ever assuming a temperature in the familiar sense, apparently thermal spectra may emerge for single-particle observables. For the sake of simplicity we deal with the semiclassical photon bremsstrahlung. For details see [25].

The invariant gamma spectrum can be obtained by assuming that each photon contributes with the energy $\hbar\omega$ to the classical intensity, calculated by squaring the Fourier-transform amplitude,

$$\frac{dN}{k_{\rm T}dk_{\rm T}d\eta d\psi} = \frac{1}{\hbar k_{\rm T}^2 \cosh^2\eta} \left|\vec{A}\right|^2 \tag{23}$$

with the vector potential Fourier-amplitude using $K^2 = e^2/4\pi c$,

$$\vec{A} = K\vec{e} \int e^{i\phi} \frac{du}{d\xi}\, d\xi \tag{24}$$

while using the photon four-momentum $k = k_{\rm T}(\cosh\eta, \sinh\eta, \cos\psi, \sin\psi)$ and the charge four-velocity $U = (\cosh\xi, \sinh\xi, 0, 0)$ on a straight line. The quantity u is the orthogonally projected velocity component together with a Doppler factor, $u = \sinh\xi/\cosh(\xi-\eta)$. The phase is taken at retarded times, integrating the $d\phi/d\tau = \omega\gamma(1-v\cos\theta)$ relation, with the Lorentz factor $\gamma = 1/\sqrt{1-v^2/c^2}$ and $\cos\theta = \tanh\eta$ due to the $\eta = \ln\cot(\theta/2)$ photon angle-rapidity relation. This leads to $\phi = (\omega c/g)\sinh(\xi-\eta) = \ell k_{\rm T}\sinh(\xi-\eta)$, by introducing $\ell = c^2/g$ with $g = c d\xi/d\tau$ constant proper acceleration. Finally, the amplitude can be expressed by an integral over the charge rapidity

$$\vec{A} = K\vec{e} \int e^{i\ell k_{\rm T}\sinh(\xi-\eta)} \frac{\cosh\eta}{\cosh^2(\xi-\eta)}\, d\xi\,. \tag{25}$$

For an infinite time path this integral can be calculated analytically with the result

$$\frac{dN}{k_{\rm T}dk_{\rm T}d\eta d\psi} = \frac{4\alpha}{\pi}\, \ell^2\, K_1^2(\ell k_{\rm T})\,, \tag{26}$$

with the Bessel-K function and the electromagnetic coupling $\alpha = e^2/4\pi\hbar c$. This formula for low values of $\ell k_{\rm T}$ is conformal being proportional to $1/k_{\rm T}^2$, but for high $\ell k_{\rm T}$ it develops an exponential tail, $\exp(-2\ell k_{\rm T})$. The corresponding slope temperature is π times the Unruh temperature, $T = 1/2\ell = \pi T_{\rm U}$. The rapidity distribution is flat in this case.

Biographical notes on J.R.

Finally, since this paper was given as a talk by Tamás Biró in the session celebrating Jan Rafelski's 60-th birthday, let T.S.B. finishes with a few personal remarks.

Jan (Johann) Rafelski, shortly J.R., has certainly left his personal and unique imprints on our international heavy-ion community in the past decades. Born in 1950, here in Kraków (Cracow), studied, taught and conducted research in Frankfurt in close collaboration with Walter Greiner and his at that time fellow colleague Berndt Müller, visited CERN, worked there with professor Rolf Hagedorn, became professor at the University of Cape Town and later at the University of Arizona, in Tucson, USA.

He dealt with important and exciting physics research problems, among others with strong fields, mild fusion, quark-gluon plasma (QGP) and lately with extreme light (after the extreme difficult). Like his favorite models, his personality occurs with numerous facets, once thermally excited, then statistically balanced, then spiced with chemistry, or a bit fugacitive, another time inducing certain resonances ...

In the beginning of my carrier I had an indirect affair with J.R., not that I had wanted such. As a very beginner, one of my tasks was to calculate certain quark level processes producing strangeness, set by my PhD advisor, Jozsef Zimanyi. I did the job — for the only Feynman graph I had got as a problem. Contemporarily J.R. had the idea to detect QGP due to its enhanced strangeness. The strange quark pair production rate from light quark annihilation, we obtained, seemed to be too low for that. Our result, communicated to Walter Greiner by Zimanyi in Frankfurt, caused this way some extra days for J.R. with heavy labour. But — as everyone knows today — there are other, gluon fusion processes producing strange quark pairs, which were calculated rapidly and supported the proposal of enhanced strangeness as a QGP signal at the end.

After such a sinister start we gradually became into direct personal contact over the years. Jan visited Budapest, the Strangeness '96 conference, and later also Quark Matter in 2005. He even invited me once to Arizona. We keep meeting on conferences, lately more than earlier, mainly due to the involvement of both of us in the series of meetings on Strangeness in Quark Matter, as this conference. So let the final word of this talk be my thank for the opportunity to contribute to the celebration in my own way, and to wish Jan peace, wisdom and containment for the future and happy birthday with this bottle of special Tokaj wine.

REFERENCES

[1] T.S. Biro, P. Van, *Phys. Rev.* **E83**, 061147 (2011).

[2] A. Renyi, *Acta Math. Acad. Sci. Hung.* **10**, 193 (1959).

[3] A. Renyi, *On Measures of Entropy and Information*, in: Proc. 4th Berkeley Symp. Math. Statist. Probability, 1960, vol. I, University of California Press, Berkeley 1961, pp. 547–561.

[4] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).

[5] C. Tsallis, *Introduction to Nonextensive Statistical Mechanics*, Springer, 2009.

[6] C.E. Shannon, *Bell Syst. Tech. J.* **27**, 379 (1948); *Bell Syst. Tech. J.* **27**, 623 (1948).

[7] E.T. Jaynes, *Phys. Rev.* **160**, 171 (1957); *Phys. Rev.* **168**, 620 (1957).

[8] C. Tsallis, *Eur. Phys. J.* **A40**, 257 (2009); C. Beck, *Eur. Phys. J.* **A40**, 267 (2009); G. Kaniadakis, *Eur. Phys. J.* **A40**, 275 (2009); T. Kodama, T. Koide, *Eur. Phys. J.* **A40**, 289 (2009); G. Wilk, Z. Wlodarczyk, *Eur. Phys. J.* **A40**, 299 (2009); W.M. Alberico, A. Lavagno, *Eur. Phys. J.* **A40**, 313 (2009); T.S. Biro, G. Purcsel, K. Urmossy, *Eur. Phys. J.* **A40**, 325 (2009).

[9] V.V. Begun, M. Gazdzicki, M.I. Gorenstein, *Phys. Rev.* **C78**, 024904 (2008).

[10] F. Becattini, L. Ferroni, *Eur. Phys. J.* **C35**, 243 (2004); **C38**, 225 (2004).

[11] F.M. Liu, K. Werner, J. Aichelin, *Phys. Rev.* **C68**, 024905 (2003); F.M. Liu *et al.*, *J. Phys. G* **30**, S589 (2004); F.M. Liu, K. Werner, *Phys. Rev.* **D74**, 034024 (2006).

[12] K. Urmossy, G.G. Barnafoldi, T.S. Biro, *Phys. Lett.* **B701**, 111 (2011).

[13] P. Abreu *et al.* [CERN-LEP-DELPHI], *Zeit. Phys.* **C65**, 587 (1995).

[14] R. Barate *et al.* [CERN-LEP-ALEPH], CERN-PPE-96-186, 1996.

[15] H.-J. Behrend *et al.* [DESY-PETRA-CELLO], *Zeit. Phys.* **C46**, 397 (1990).

[16] W. Braunschweig *et al.* [DESY-PETRA-TASSO], *Zeit. Phys.* **C47**, 167 (1990).

[17] H. Aihara *et al.* [SLAC-PEP-004 (TPC)], *Phys. Rev. Lett.* **53**, 2378 (1984).

[18] H. Schellman *et al.* [SLAC-PEP-005 (MARK-II)], *Phys. Rev.* **D31**, 3013 (1985).

[19] M. Acciarri *et al.* [CERN-LEP-L3], *Phys. Lett.* **B328**, 223 (1994).

[20] W. Braunschweig *et al.* [DESY-PETRA-TASSO], *Zeit. Phys.* **C42**, 189 (1989).

[21] H.-J. Behrend *et al.* [DESY-PETRA-CELLO], *Zeit. Phys.* **C47**, 1 (1990).

[22] H.-J. Behrend *et al.* [DESY-PETRA-CELLO], *Zeit. Phys.* **C20**, 207 (1983).

[23] C. Beck, E.G.D. Cohen, *Physica A* **322**, 267 (2003); *Physica A* **344**, 393 (2004); H. Touchette, C. Beck, *Phys. Rev.* **E71**, 016131 (2005); S. Abe, C. Beck, G.D. Cohen, *Phys. Rev.* **E76**, 031102 (2007).

[24] M. Biyajima *et al.*, *Eur. Phys. J.* **C48**, 597 (2006); G. Wilk, Z. Wlodarczyk, *Eur. Phys. J.* **A40**, 299 (2009).

[25] T.S. Biro, M. Gyulassy, Z. Schram, *Phys. Lett.* **B708**, 276 (2012), [`arXiv:1111.4817v1 [hep-ph]`].