CYCLIC SOLUTIONS IN CHAOS AND THE SHARKOWSKII THEOREM*

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The fixed points of the logistic map at full chaos are the roots of a special class of polynomials. These polynomials are solvable by a method of multiple angles. The solutions are expressible in cyclic form. By using the theorem due to Sharkovskii we show that the fixed point spectrum has a finite measure. We argue that chaos in 1d, defined through a finite spectral measure, is superior to any phenomenological definitions of chaos such as the Lyapunov exponent.

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1. Introduction

The Lyapunov exponent lambda is what one uses to determine whether the trajectories in iterative space are periodic or chaotic. It is defined as: $\lambda = \lim_{(n\to\infty)} \log|df^n(x^*)/dx|$, where f is a map, x a real number and x^* a fixed point of $f^n(x)$, where $f^n = f(f^{n-1})$, $n = 1, 2, \dots$ with $f^1 \equiv f$. If $\lambda \leq 0$, the trajectories are periodic. If $\lambda > 0$, they are chaotic. This is how one usually knows when there is chaos in 1d maps [1,2]. In the logistic map (the harmonic oscillator of chaos), well past the stable bifurcation there exists a stable 3-cycle, the onset of which is analytically proved in [3]. The parametric value and the fixed points for the super stable 3-cycle are also analytically proved [4]. The Lyapunov exponent in the 3-cycle window would measure nonpositive, that is, the trajectories are periodic, not chaotic.

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The above conventional observation flies in the face of a deep math theorem due to Sharkovskii [5] that the existence of a 3-cycle means chaos. This theorem applies to 1d chaotic maps like the logistic map. This theorem was published in 1964. But it seems to have remained largely unknown for many years. More than ten years later Li and Yorke published a theorem entitled "Period-3 implies chaos" [6]. Period-3 is another name for 3-cycle. Apparently unaware of the earlier theorem due to Sharkovskii, Dyson wrote only recently that the theorem by Li and Yorke is the only rigorous theorem on chaos and "one of the important gems" in mathematics [7]. It is now known that the theorem by Li and Yorke is a corollary of Sharkovskii's theorem [8]. In any event it is clear that there is chaos in the 3-cycle window. The Lyapunov exponent simply fails to detect chaos therein. The Lyapunov exponent is a phenomenologically defined quantity, practical and sensible, but not rooted in anything more fundamental. Thus this difficulty over the 3-cycle window should come as no surprise.

Given this situation, it seems desirable to find a definition of chaos from "first principles". One might say that chaos is a manifestation of some basic property or properties of a fundamental equation or theorem of chaos. If unearthed, they could serve as the definition of chaos. Such a definition could not go awry as the Lyapunov exponent. It could even provide deeper insights into chaos.

Our idea is that by considering Sharkovskii's as the fundamental theorem of chaos and by applying it on the logistic map, we might arrive at that definition by uncovering the basic properties. The route to finding them has to begin at 3-cycle. Why a 3-cycle gate? Since 3-cycle implies chaos, a path that connects to other cycles must begin at 3-cycle gate.

For the logistic map the 3-cycle window begins from $a = b_3 = 1 + \sqrt{8}$ [3] to $a = 4$. A conventional definition of the 3-cycle window would range only, where the cycle is stable. But neither Sharkovskii's theorem nor Li–Yorke's theorem makes reference to stability, so that our 3-cycle window is wider beginning from $a = b_3$ and extending to $a = 4$. When $a = 4$, it is termed fully developed chaos or simply full chaos. In this window we want to find the fixed points of a 3-cycle for some values of a in the interval $b_3 \leq a \leq 4$. At that value of a, we want to find the fixed points of a 5-cycle and so on. The possible special values of a are b_3 [3], $\bar{b}_3$ [4], and $a = 4$ [4], for which their fixed points are already known analytically. Obtaining the fixed points of a 5-cycle at the first two special values b_3 and $\bar{b}_3$, would prove too difficult. We shall thus choose $a = 4$ (full chaos) [9]. We shall thus choose a 3-cycle gate at full chaos in an attempt at reaching other cycles. At full chaos no cycles are stable.

2. 3-cycles in logistic map

Since this map is already well known [1,2], we will describe only its basic properties that are needed. The logistic map is defined for two real variables x and x', both in the interval (0,1), by $x' = f(x)$, where $f(x) = ax(1-x)$, for the control parameter $0 < a \leq 4$. An n-cycle is defined by

$$f^n(x) - x = 0\,, \qquad n = 1, 2, \ldots\,, \tag{1}$$

where $f^n(x) = f(f^{n-1}(x))$ with $f^1 \equiv f$. Thus the roots of $f^n(x) - x$ are the fixed points of $f^n(x)$. If $n = 3$, the fixed points of $f^3(x)$ include the fixed points of $f(x)$. For the unique fixed points of $f^3(x)$ we must exclude the fixed points of $f(x)$. To obtain the unique ones only, we solve the characteristic equation $Q_3(x) = 0$, where

$$Q_3 = \left(f^3 - x\right)/((f - x)\,. \tag{2}$$

It is straightforward to show that if $a = 4$

$$Q_3 = 21 - 105t + 189t^2 - 157t^3 + 65t^4 - 13t^5 + t^6\,, \tag{3}$$

where $t = 4x$. Thus $Q_3 = 6(P)$, where $N(p)$ means a polynomial of degree N.

If (3) were a general sextic equation, no algebraic solutions, only numerical solutions, would be possible. Studying special classes of polynomials that occur in physical problems has attracted much interest. See *e.g.* [10,11,12].

According to Abel's theorem, the highest general polynomial solvable by radicals is 4. But Q_3 may not be a general sextic equation since it arises from a 3-cycle. Since the 3-cycle has the structure of the regular triangle, Q_3 must have the symmetry of the triangle. It means that Q_3 is composed of the cubic equation and is solvable algebraically. Indeed, we find that it is a product of two cubic equations

$$Q_3 = \left(7 - 14t + 7t^2 - t^3\right) \times \left(3 - 9t + 6t^2 - t^3\right)\,. \tag{4}$$

There are two independent 3-cycles, not one. There must be two sets of 3 unrelated fixed points. We shall term them *isocycles*. Isocycles are cycles of the same order but with different fixed points. Neither Sharkovskii's theorem nor Li–Yorke's theorem says anything about the isocycles of order 3, just the cycle of order 3. At this stage we do not know the significance of isocycles with respect to these theorems. For the general interest in 3-cycles, also see [13].

The solutions for the two cubic equations may be given as (recalling that $t = 4x$)

$$x = \sin^2 \pi y/2\,, \tag{5}$$

where for the first cubic equation

$$y/2 = 1/7\,, 2/7 \quad \text{and} \quad 3/7\,, \tag{6}$$

while for the second cubic equation

$$y/2 = 1/9\,, 2/9 \quad \text{and} \quad 4/9\,. \tag{7}$$

Observe that in (7), $3/9$ is missing. This value is the solution for the 1-cycle, removed by the construction of the characteristic equation Q_3. There are thus altogether 6 real roots to Q_3. A solution is said to be *cyclic* if it has the form of (5) and $y/2$ or y its *cyclic values.*

By Sharkovskii's theorem, we now have established that when $a = 4$, there are all other cycles of order $n = 4, 5, 6, 7, \ldots$ since a pair of 3-cycles have been shown to exist. The theorem, however, does not say that the other cycles are cyclic or have isocycles. Let us call cycles *similar* if they have these two properties.

To look for what underlies chaos, we have begun on the route of full chaos. Going on this route past the gate of the 3-cycle, we now want to know whether other cycles are similar. How far on this route or how many posts of higher cycles do we need to go? If it were necessary to go past an infinite number of them, it would not be possible. Sharkovskii's ordering might allow us a short cut.

3. Sharkovskii ordering

We digress briefly here to introduce what is known as Sharkovskii's ordering of the natural numbers, which forms the heart of Sharkovskii's theorem. According to Sharkovskii, the existence of a 3-cycle implies the existence of a 5-cycle and so in a remarkable sequence shown below:

$$
\begin{array}{ll}
3 \to 5 \to 7 \to 9 \to \ldots \to (\text{all odd numbers}) \to & [b \text{ subordering}] \\
2 \times 3 \to 2 \times 5 \to 2 \times 7 \to \ldots \to 2 \ \times (\text{all odd numbers}) \to & [c \text{ subordering}] \\
4 \times 3 \to 4 \times 5 \to 4 \times 7 \to \ldots \to 2^2 \times (\text{all odd numbers}) \to & [d \text{ subordering}] \\
8 \times 3 \to 8 \times 5 \to 8 \times 7 \to \ldots \to 2^3 \times (\text{all odd numbers}) \to & [e \text{ subordering}] \\
\ldots \to \ldots \to & \\
2^\infty \to \ldots \to 2^3 \to 2^2 \to 2 \to 1\,. & [a* \ \text{subordering}]
\end{array}
$$

Given the above ordering, we first want to see whether 5-cycles are similar. We shall do this by solving the 5-cycle problem. If 6-cycles are found similar, it would imply that all cycles in b-subordering are similar. If 4-cycles are found similar, it would imply that all cycles in subordering from c to $a*$

excluding cycles of order 2 and 1 are similar. The cycles of order 1 and 2 are cyclic but they do not have isocycles. Sharkovskii's ordering allows us to consider cycles of order 5, 6 and 4 only instead of infinitely many. This consideration makes it possible to continue our analysis. But what must be done is still by no means simple. To solve the cycles of order 5, 6 and 4 one must tackle polynomials of high degrees.

4. Cycle solutions

4.1. $f^5(x) = x$

The unique fixed points of $f^5(x)$ are the roots of $Q_5 = (f^5 - x)/(f - x) = 30(p)$. Again, replacing x by $t = 4x$, we obtain the characteristic equation

$$\begin{aligned} Q_5 &= \left(11 - 55t + 77t^2 - 44t^3 + 11t^4 - t^5\right) \times A \times B \\ &\equiv q_5 \times A \times B\,, \end{aligned} \tag{8}$$

where $A = 15(p)$ and $B = 10(p)$, too long to be displayed here. We consider q_5 a quintic equation on the r.h.s. of (8). If q_5 were a general quintic equation, we could not solve it. As in the 3-cycle problem, the 5-cycle must reflect the symmetry of the regular pentagon. We consider the multiple angles of $\sin 11\alpha$, expressing in terms of $2\sin\alpha$

$$\begin{aligned} \sin 11\alpha/\sin\alpha &= 11 - 55(2\sin\alpha)^2 + 77(2\sin\alpha)^4 \\ &\quad -44(2\sin\alpha)^6 + 11(2\sin\alpha)^8 - (2\sin\alpha)^{10}\,. \end{aligned} \tag{9}$$

If one compares the coefficients of q_5 with the coefficients of those in the r.h.s., they are term by term identically the same. Therefore, q_5 is congruent to $\sin 11\alpha/\sin\alpha$ if t is identified with $(2\sin\alpha)^2$ or x by $(\sin\alpha)^2$. Thus the roots of q_5 are the zeros of $\sin 11\alpha/\sin\alpha$. The 5 roots of q_5 may be expressed by (5) with the cyclic values $y/2$

$$1/11 \quad 2/11 \quad 4/11 \quad 3/11 \quad 5/11\,. \tag{10}$$

This method of solution will be called the method of multiple angles. Observe that the above cyclic values are all rational numbers as those of the 3-cycles, see (6). There are no common cyclic values in the three sets (6), (7) and (10). They are unique as they are solutions of the three unique polynomials.

The above set (10) satisfies the 5-cycle definition $f(x_i) = x_{i+1}$, $i = 1, 2, 3, 4, 5$ with $x_6 \equiv x_1$ in sequence as given. Evidently q_5 is cyclic. But are there isocycles? If so, they would have to be contained in $A = 15(p)$ and $B = 10(p)$. Both A and B are not further reducible to products of polynomials of lower order. But they can be solved by the method of multiple angles: If $x = \sin^2\alpha$

$$A = \sin 31\alpha / \sin\alpha \,. \tag{11}$$

Thus, the 15 roots of A have the following cyclic values $y/2$

$$\begin{array}{lllll} 1/31 & 2/31 & 4/31 & 8/31 & 15/31 \end{array} \tag{12a}$$
$$\begin{array}{lllll} 3/31 & 6/31 & 12/31 & 7/31 & 14/31 \end{array} \tag{12b}$$
$$\begin{array}{lllll} 5/31 & 10/31 & 11/31 & 9/31 & 13/31\,. \end{array} \tag{12c}$$

There are 15 distinct roots for $A = 15(p)$. But they do not represent one 15-cycle or the cycle of order 15. Instead there are 3 isocycles of order 5. The three sets or groups satisfy the 5-cycle definition separately.

By the method of multiple angles,

$$B = \sin\alpha\,\sin 33\alpha / \sin 3\alpha\,\sin 11\alpha \tag{13}$$

if $x = \sin^2\alpha$. Thus, the 10 roots of B have the following cyclic values $y/2$

$$\begin{array}{lllll} 1/33 & 2/33 & 4/33 & 8/33 & 16/33 \end{array} \tag{14a}$$
$$\begin{array}{lllll} 3/33 & 10/33 & 13/33 & 7/33 & 14/33\,. \end{array} \tag{14b}$$

The 10 distinct roots do not represent one cycle of order 10. Instead there are two cycles of order 5. Both sets satisfy the 5-cycle definition separately reflecting the Sharkovskii groups. The 5-cycles are cyclic and there are 6 isocycles of order 5. That the 5-cycles are similar must also be implied by the 3-cycles. The cyclic values for the 3- and 5-cycles are all rational numbers and they are all unique.

4.2. $f^6(x) = x$

The unique fixed points of $f^6(x)$ are the roots of the characteristic equation $Q_6 = (f^6 - x)(f - x)/(f^2 - x)(f^3 - x) = 54(p)$. The 6-cycle has the symmetry of the regular hexagon. Thus, as with Q_3 which consists of a pair of the cubic equation, Q_6 should contain a pair of the hexatic equation. Indeed we obtain, using $t = 4x$ as before,

$$\begin{aligned} Q_6 &= \left(13 - 91t + 182t^2 - 156t^3 + 65t^4 - 13t^5 + t^6\right) \\ &\quad \times \left(1 - 16t + 60t^2 - 78t^3 + 44t^4 - 11t^5 + t^6\right) \times C \times D \\ &\equiv q_6 \times q_6' \times C \times D \,, \end{aligned} \tag{15}$$

where $C = 18(p)$ and $D = 24(p)$, not displayed. By the method of multiple angles and if $x = \sin^2\alpha$

$$\begin{aligned} q_6 &= \sin 13\alpha/\sin\alpha\,, && (16)\\ q_6' &= \sin\alpha\,\sin 21\alpha/\sin 3\alpha\,\sin 7\alpha\,, && (17)\\ C &= \sin\alpha\,\sin 63\alpha/\sin 3\alpha\,\sin 21\alpha\,, && (18)\\ D &= \sin\alpha\,\sin 65\alpha/\sin 5\alpha\,\sin 13\alpha\,. && (19)\end{aligned}$$

Thus the 54 roots of Q_6 have the following cyclic values $y/2$

1/13	2/13	4/13	5/13	3/13	6/13	(20a)
1/21	2/21	4/21	8/21	5/21	10/21	(20b)
1/63	2/63	4/63	8/63	16/63	31/63	(20c)
5/63	10/63	20/63	23/63	17/63	29/63	(20d)
11/63	22/63	19/63	25/63	13/63	26/63	(20e)
1/65	2/65	4/65	8/65	16/65	32/65	(20f)
3/65	6/65	12/65	24/65	17/65	31/65	(20g)
7/65	14/65	28/65	9/65	18/65	29/65	(20h)
11/65	22/65	21/65	23/65	19/65	27/65 .	(20i)

The two polynomials $C = 18(p)$ and $D = 24(p)$ yield 3 and 4 isocycles of order 6. There are altogether 9 isocycles of order 6. The 6-cycles are similar. The cyclic values are all rational numbers and are unique.

4.3. $f^4(x) = x$

The unique fixed points of $f^4(x)$ are the roots of $Q_4 = (f^4 - x)/(f^2 - x) = 12(p)$. The 4-cycle has the symmetry of the square, implying that Q_4 must contain the quartic equation. We obtain using $t = 4x$ again

$$\begin{aligned} Q_4 &= \left(1 - 8t + 14t^2 - 7t^3 + t^4\right)\\ &\quad\times\left(17 - 204t + 714t^2 - 1122t^3 + 935t^4 - 442t^5 + 119t^6 - 17t^7 + t^8\right)\\ &\equiv q_4 \times q_8\,. \end{aligned} \tag{21}$$

By the method of multiple angles and if $x = \sin^2\alpha$

$$\begin{aligned} q_4 &= \sin\alpha\,\sin 15\alpha/\sin 3\alpha\,\sin 5\alpha\,, && (22)\\ q_8 &= \sin 17\alpha/\sin\alpha\,. && (23)\end{aligned}$$

The 12 roots of Q_4 have the following cyclic values $y/2$

$$1/15 \quad 2/15 \quad 4/15 \quad 7/15 \tag{24a}$$
$$1/17 \quad 2/17 \quad 4/17 \quad 8/17 \tag{24b}$$
$$3/17 \quad 6/17 \quad 5/17 \quad 7/17\,. \tag{24c}$$

All the cyclic values are rational numbers and they are unique.

4.4. 2-cycle and 1-cycle

For completeness we include here the two trivial cases. The fixed points of $f^2(x)$ are the roots of the characteristic equation $Q_2 = (f^2 - x)/(f - x)$. It follows that $Q_2 = 2(p)$:

$$Q_2 = 5 - 20x + 16x^2 = \sin 5\alpha / \sin\alpha\,. \tag{25}$$

Thus, the two roots of Q_2 have the following cyclic values $y/2$

$$1/5 \quad 2/5\,. \tag{26}$$

Finally, the fixed point of $f(x)$ is given by $Q_1 = 3 - 4x = \sin 3\alpha/\sin\alpha$. The one root has the cyclic value

$$y/2 = 1/3\,. \tag{27}$$

There are no isocycles in cycles of order 2 and 1. The cyclic values are, however, rational numbers, distinct from all others previously obtained.

5. Cycles and isocycles

All the cyclic values obtained for cycles and isocycles of the order of $n = 3, 5, 6, 4, 2$ and 1 leads us to the following two conditions:

a. All cyclic values are rational numbers in $0 < y < 1$.

b. All cyclic values are unique.

We now assert on the strength of Sharkovskii's theorem that all other cycles must satisfy the above two conditions. Since there are infinitely many cycles, the cyclic values assume all possible rational points in the interval (0,1) with no gaps. Thus,

$$x = \sin^2 \pi y/2\,, \qquad y = (0,1)\,. \tag{28}$$

To see the significance, we consider a couple of rational values for y: If $y = 1/3$, $x = 3/4$, another rational number in the interval (0,1). If $y = 1/5$,

$x = (5 - \sqrt{5})/8$, now an irrational number but in the same interval (0,1). Thus the rational points of set y in the interval (0,1) generate rational and irrational points of set x in the interval (0,1). If the cardinality of set y were finite, the cardinality of set x would be the same, one to one. In (26), the cardinality of set y is not finite, but transfinite. Thus we can conclude that

$$x = x(\mathcal{R}) + x(\mathcal{I})\,, \tag{29}$$

where $\mathcal{R}$ and $\mathcal{I}$ mean infinite sets of rational points and irrational points in the interval (0,1), respectively. If $\mu(x)$ means the Lebesgue measure of x in the interval (0,1),

$$\mu(x) = \mu(x(\mathcal{I}) = 1\,. \tag{30}$$

Above we have used the additivity property of measure and also $\mu(x(\mathcal{R})=0$. Thus the cyclic solution (28) implies that the fixed point spectrum has measure 1. The distribution of the irrational points in the interval (0,1) for this spectrum may be calculated as follows: $\rho(x) = \Delta\mu(x(\mathcal{I}))/\Delta x$. Thus,

$$1 = \int_0^1 \rho(x)dx\,. \tag{31}$$

In a slight digression we show here how $\rho(x)$ may be simply deduced by noting that (31) corresponds to a calculation of the number of particles in statistical mechanics. If N is the total number of ideal spinless Fermi particles in a 1d box of length L, where N and L are both very large, then

$$N = \sum_k \langle \boldsymbol{n}_k \rangle\,, \tag{32}$$

where $\langle \boldsymbol{n}_k \rangle$ is the Fermi function and k the wave vector. Since L is very large, the sum may be converted to an integral. At the ground state $\langle \boldsymbol{n}_k \rangle$ is a step function: 1 if $|k| \leq k_{\mathrm{F}}$ and 0 if otherwise, where k_{F} is the Fermi wave vector. Thus

$$1 = \frac{L}{\pi N} \int_0^{k_{\mathrm{F}}} dk\,. \tag{33}$$

If $\varepsilon = \varepsilon(k)$, known as the dispersion relation of k,

$$1 = \int_0^{\varepsilon_{\mathrm{F}}} g(\varepsilon)d\varepsilon\,, \tag{34}$$

where $\varepsilon_{\rm F} = \varepsilon(k_{\rm F})$ the Fermi energy and $g(\varepsilon) = L/\pi N \; dk/d\varepsilon$ the density of states. If ε is scaled by $\varepsilon_{\rm F}$, so that the upper limit becomes 1, (34) is identical to (31) in structure.

If the cyclic solution (28) is viewed as the dispersion relation of y, we see the correspondence: $\varepsilon \leftrightarrow x$ and $k \leftrightarrow y$, thus $g(\varepsilon/\varepsilon_{\rm F}) \leftrightarrow \rho(x)$. It follows that $\rho(x) = M dy(x)/dx$, where M is the normalization constant. We obtain

$$\rho(x) = \frac{1}{\pi\sqrt{x(1-x)}}\,, \qquad 0 < x < 1\,. \tag{35}$$

Observe that the distribution of the irrational points is greater towards the edges than towards the center. Everywhere $\rho(x) > 0$ implying that there are no gaps [14, 15].

6. Concluding remarks

The logistic map is a rich source of polynomials of a special interest. At full chaos, different cycles generate special classes of polynomials which we find solvable by the method of multiple angles. In other domains of a, where a 3-cycle exists, *i.e.* $b_3 < a < 4$, where $b_3 = 1 + \sqrt{8}$ [3], solutions for the polynomials are cyclic modified. As a result, it is more difficult to apply Sharkovskii's theorem than at full chaos. If one could carry out an analysis much like that at full chaos, there should also be found fixed-point spectra of finite measure. It should result in the same kind of definition for chaos that we have been able to obtain at full chaos. Our definition of chaos by finite spectral measure may be said to be of first principles since it is derived by application of a fundamental theorem of chaos on a model of chaos which is the logistic map.

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[15] It is known that (5) is a transformation equation that transforms the logistic map to the tent map. See [14]. To our knowledge no one has shown that (5) gives the fixed points of all cycles if and only if $y = y(\mathcal{R})$ and $\mu(y) = 0$. The tent map does not restrict y to $\mathcal{R}$.