ANOMALOUS DYNAMICS OF BLACK–SCHOLES MODEL TIME-CHANGED BY INVERSE SUBORDINATORS*

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In this paper we consider a generalization of one of the earliest models of an asset price, namely the Black–Scholes model, which captures the subdiffusive nature of an asset price dynamics. We introduce the geometric Brownian motion time-changed by infinitely divisible inverse subordinators, to reflect underlying anomalous diffusion mechanism. In the proposed model the waiting times (periods when the asset price stays motionless) are modeled by general class of infinitely divisible distributions. We find the corresponding Fractional Fokker–Planck equation governing the probability density function of the introduced process. We prove that considered model is arbitrage-free, construct corresponding martingale measure and show that the model is incomplete. We also find formulas for values of European call and put option prices in subdiffusive Black–Scholes model and show how one can approximate them based on Monte Carlo methods. We present some Monte Carlo simulations for the particular case of tempered α-stable distribution of waiting times. We compare obtained results with the classical and subdiffusive α-stable Black–Scholes prices.

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1. Introduction

The economic and finance theories as we know them today began at the end of the nineteenth century by works of Walras and Pareto and their mathematical description of equilibrium theory (see [1] and references therein).

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The second ground-breaking contribution came soon after. In 1900 Bachelier [2] in his "Théorie de la Spéculation" developed mathematical tools to describe uncertain stock price process based on theory of Brownian motion. He initiated at that time the study of, what we call them today, diffusion processes. Despite the novelty and elegance, the theory of Bachelier was forgotten for over fifty years. It was rediscovered by Samuelson [3] and used to construct more adequate model for description of stock price movement based on the geometric Brownian motion (GBM). This process is lacking the main drawback of Bachelier model, namely it cannot take negative values, while the Bachelier model based on ordinary Brownian motion can. The extensive research on GBM brought in 1973 consistent formulas for the fair prices of European options [4, 5]. Fischer Black, Myron Scholes and Robert Merton determined them based on continuous hedging. Their ideas turned out to be so innovative that Merton and Scholes were awarded the Nobel Prize for Economics in 1997.

Nowadays, after the recent findings, we observe that many of economical processes are far away from the classical models. The most popular Black–Scholes (BS) model of stock prices is, as empirical studies show, incapable to capture many of characteristic properties of present markets. It is worth to mention here such properties as: long-range correlations, heavy-tailed and skewed marginal distributions, lack of scale invariance, periods of constant values, *etc.* In order to overcome these difficulties, dozens of models based on ideas and methods known from field of statistical physics have been proposed [6].

Analysis of various real-life data shows that especially in emerging markets [7], where the number of participants and transactions is rather low, we encounter characteristic periods of time in which economic processes stay motionless. Searching for similar examples, where observed trajectories posses constant periods, we are guided to the physical experiments in systems exhibiting subdiffusion. In such systems motion of small particles is interrupted by the trapping events. During that random period of time investigated particle is immobilized and stays motionless. Subdiffusion is nowadays a very well established phenomenon with many real-life examples: charge carrier transport in amorphous semiconductors, diffusion in percolative and porous systems, transport on fractal geometries, to name only few (see [8] and references therein).

The mathematical description of subdiffusion is the celebrated Fractional Fokker–Planck equation (FFPE) derived from continuous-time random walk scheme with heavy-tailed waiting times [8]. In last years this equation has become a standard mathematical tool in analysis of various complex systems [9, 10, 11, 12]. Equivalent description of subdiffusion is in terms of Langevin equation, where the standard diffusion process is time-changed by the so-called inverse subordinator responsible for trapping events [13, 14, 15, 16]. Following such approach, in this paper we introduce a subdiffusive GBM as a model of stock prices. This process is defined as standard GBM time-changed by infinitely divisible (ID) inverse subordinator.

We stress out that presented here methodology can be applied to different diffusion processes. The case of GBM with α-stable waiting times can be found in [17, 18]. In [19] one can find general discussion of the ID arithmetic Brownian motion as a model of stock prices. In this paper we analyze subdiffusive GBM in full generality.

This article is structured as follows. In Sec. 2 we give precise definition of subdiffusive geometric Brownian Motion and derive the corresponding Fractional Fokker–Planck equation governing the dynamics of probability density function (PDF). In Sec. 3 we will prove that the model is arbitrage free and incomplete. We derive also formulas for fair prices of European options in our model. In Sec. 4 we will present a particular example of waiting times, namely tempered α-stable ones. We discuss the methods of simulation of the trajectories and present some numerical results. Section 5 concludes the paper.

2. Subdiffusive geometric Brownian motion

The price of an asset $Z^{\mathrm{BS}}(t)$ in the classical BS model is represented by GBM

$$Z^{\mathrm{BS}}(t) = Z_0 \exp\{\sigma B(t) + \mu t\}\,, \qquad Z_0 > 0\,. \tag{1}$$

Here $B(t)$ is the standard Brownian motion, $\sigma > 0$ is the volatility and $\mu \in \mathbb{R}$ is the drift parameter. To remove the influence of interest rate $r > 0$ to the price of an asset, we will consider discounted price process. In this discounted world the price of an asset follows

$$Z(t) = e^{-rt} Z^{\mathrm{BS}}(t) = Z_0 \exp\{\sigma B(t) + \mu t - rt\}\,, \qquad Z_0 > 0\,. \tag{2}$$

Applying Itô calculus, Eq. (2) can be equivalently represented in the form of stochastic differential equation (SDE)

$$dZ_t = Z_t \left[\sigma dB(t) + \left(\mu - r + \tfrac{1}{2}\sigma^2\right) dt\right]\,. \tag{3}$$

Such process was used by Black, Scholes and Merton to derive the values of European call and put option prices.

A European call option is a contract between buyer and seller. It gives the buyer right, but not the obligation, to buy from a seller a predetermined amount of an underlying asset $Z(t)$ at a predetermined strike price K within a particular maturity time T. Such right costs the buyer a fee (price). The value of a call option at time T is equal to $\max\{Z(T) - K, 0\}$. A put option is an opposite of a call option and it gives the holder right to sell shares. The value of a put option at T equals $\max\{K - Z(T), 0\}$. By the so-called put–call parity [20] the difference of the value of the call option C and the price of a put option P, both with the identical strike price and a maturity time, is given by

$$C - P = Z(0) - Ke^{-rT}\,. \tag{4}$$

The fair price $C_{\mathrm{BS}}(Z_0, K, T, \sigma)$ of the European call option in the BS model is given by [20]

$$C_{\mathrm{BS}}(Z_0, K, T, \sigma) = Z_0\mathcal{N}(d_+) - e^{-rT}K\mathcal{N}(d_-) \tag{5}$$

with

$$d_\pm = \frac{\log\frac{Z_0}{K} \pm \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\,.$$

Here $\mathcal{N}$ is the cumulative distribution function of standard normal distribution. From put–call parity we can easily determine the put option price

$$P_{\mathrm{BS}}(Z_0, K, T, \sigma) = C_{\mathrm{BS}}(Z_0, K, T, \sigma) + e^{-rT}K - Z_0\,. \tag{6}$$

BS model possesses many advantages, for example closed analytical form for prices of European options, but has also some drawbacks. Growing number of emerging markets, provides us with empirical data of processes with periods of constant prices [7]. These constant periods are similar in nature to the observed trapping events of subdiffusive particles. Following physical description of subdiffusion [8, 16, 18, 19] we propose the generalization of BS model. We introduce the following process

$$Z_\Psi(t) = Z(S_\Psi(t))\,, \tag{7}$$

$t \in [0, T]$ as the model of asset prices. We call $Z_\Psi(t)$ the subdiffusive GBM. It is the standard GBM in operational time $S_\Psi(t)$. $S_\Psi(t)$ is the so-called inverse ID [16] subordinator which is independent of the Brownian motion $B(t)$. We define $S_\Psi(t)$ as

$$S_\Psi(t) = \inf\{\tau : T_\Psi(\tau) > t\}\,, \tag{8}$$

where $T_\Psi(t)$ is the strictly increasing Lévy process [21] with the Laplace transform

$$\langle \exp(-uT_\Psi(t))\rangle = \exp(-t\Psi(u))\,. \tag{9}$$

The so-called Laplace exponent Ψ is given by $\Psi(u) = \int_0^\infty (1 - e^{-ux})\nu(dx)$, where ν is the Lévy measure satisfying $\int_0^\infty (1 \wedge x)\nu(dx) < \infty$. We assume that $\nu(0, \infty) = \infty$, which assures that the trajectories of inverse subordinator

$S_\Psi(t)$ are continuous. We observe that for every jump of $T_\Psi(t)$ there is a corresponding flat period of $S_\Psi(t)$, which is distributed according to ID law (see Fig. 1). Flat periods represent waiting times in which the test particle gets immobilized in the trap. With this property the overall motion is slowed down and $Z_\Psi(t)$ captures the empirical property of constant price periods. The typical trajectory of the process $Z_\Psi(t)$ with tempered α-stable waiting times is shown in Fig. 1. One can see that the main difference between classical and subdiffusive models are the constant periods in trajectory of $Z_\Psi(t)$.

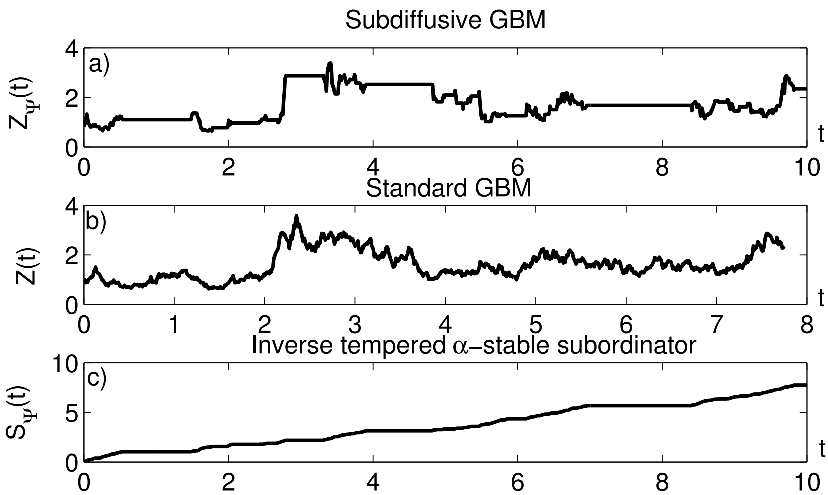

Fig. 1. Sample realizations of: subdiffusive GBM $Z_\Psi(t)$ with tempered stable waiting times (panel (a)), classical GBM $Z(t)$ (panel (b)), inverse ID subordinator $S_\Psi(t)$ (panel (c)). The constant intervals in the trajectory of the process $Z_\Psi(t)$ represent the periods of stagnation in subdiffusive scenario. Parameters are, $r = 0.4$, $\sigma = 0.6$, $Z_0 = 1$, $\lambda = 0.01$, $\alpha = 0.8$.

The main advantage of the proposed model is that the distribution of waiting times can be chosen from rich family of ID distributions. This is an essential extension of the model analyzed in [17, 18]. Important examples of ID laws are: stable, tempered stable, Pareto, gamma, exponential, Mittag–Leffler and Linnik. Choosing different exponent $\Psi(u)$ one obtains different ID distributions of waiting times.

We can describe the probability density function (PDF) of the time-changed process $Z_\Psi(t)$ by the fractional Fokker–Planck equation. Equations of this type are widely used in statistical physics in the context of subdiffusive dynamics [8, 16]. The equation governing the PDF of the process $Z_\Psi(t)$ is derived in the next theorem.

Theorem 2.1. *Let $Z_\Psi(t)$ be the subdiffusive GBM, defined in (7). Then the PDF of $Z_\Psi(t)$ is the solution of the following fractional Fokker–Planck-type equation*

$$\frac{\partial w(x,t)}{\partial t} = \Phi(t)\left[-\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x w(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 w(x,t)\right], \quad (10)$$

$w(x,0) = \delta_{Z_0}(x)$. *Here Φ_t stands for the integro-differential operator defined as*

$$\Phi_t f(t) = \frac{d}{dt}\int_0^t M(t-y)f(y)dy\,, \quad (11)$$

with the memory kernel $M(t)$ defined via its Laplace transform

$$\hat{M}(u) = \int_0^\infty e^{-ut}M(t)dt = \frac{1}{\Psi(u)}\,.$$

For the derivation of formula (10), see Appendix A. One can approximate solutions of (10) by the finite element method [22]. Alternatively, one can use Monte Carlo techniques to estimate the PDF of $Z_\Psi(t)$. For some methods of simulation of $Z_\Psi(t)$, see Sec. 4 and [16].

3. Lack of arbitrage, incompleteness and option pricing formula

Let us now consider a market model, whose evolution up to time T is contained in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the sample probability space, $\mathcal{F}$ is the set containing all information of the evolution of prices, and $\mathbb{P}$ is the "objective" probability measure. $Z_\Psi(t)$ will represent the asset price in this market, and $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $Z_\Psi(t)$ *i.e.* history of the process up to time t. The crucial assumption for pricing rules in a given market is that it does not admit arbitrage opportunities. Arbitrage (also called "free lunch") is an opportunity for investor to profit for simultaneous purchase and sale of an asset. The difference between purchase and sale prices allows investor to make a profit without taking any risk. Formally, an arbitrage opportunity [23, 24] is a self financing investing strategy ϕ, which can lead to a positive terminal gain without any intermediate loss

$$\mathbb{P}(\forall t \in [0,T], V_t(\phi) \geq 0) = 1\,, \qquad \mathbb{P}(V_T(\phi) > V_0(\phi)) \neq 0\,,$$

where $V_t(\phi)$ denotes the value of an investor portfolio at time t. To assure that market model is arbitrage-free, it is enough to prove the existence of the equivalent martingale measure $\mathbb{Q}$. This martingale measure, in contrast to physical measure $\mathbb{P}$, assures that financial assets have the same expected rate of return. The measure given in the next theorem makes the process $Z_\Psi(t)$ a martingale.

Theorem 3.1. *Let $\mathbb{Q}$ be the probability measure defined by the equation*

$$\mathbb{Q}(A) = \int\limits_A \exp\left\{-\gamma B(S_\Psi(T)) - \frac{\gamma^2}{2} S_\Psi(T) d\mathbb{P}\right\}, \tag{12}$$

where $\gamma = \frac{\mu - r + \sigma^2/2}{\sigma}$ and $A \in \mathcal{F}$. Then the process $Z_\Psi(t)$ a martingale is a martingale with respect to $\mathbb{Q}$.

One can easily see that $\mathbb{Q}$ is a probability measure. In Appendix B we prove that the process $(Z_\Psi(t))_{t\in[0,T]}$ is a martingale with respect to $\mathbb{Q}$. As a consequence from Fundamental Theorem of Asset Pricing [23] we obtain that a market model with assets prices modeled by $Z_\Psi(t)$ is a arbitrage-free.

Apart from lack of arbitrage there is another idea that originates from the classical BS model. This is the market completeness. We say that market is complete if every $\mathcal{F}_T$-measurable contingent claim admits a replicating self-financing strategy [23]. In complete markets there is only one fair price of the option. From the Second Fundamental Theorem of asset pricing [23] a market defined by the asset $(Z_\Psi(t))_{t\in[0,T]}$ is complete if and only if there is a unique market martingale measure equivalent to $\mathbb{P}$. The next result postulates that market model described by subdiffusive GBM is incomplete.

Theorem 3.2. *Martingale measure defined in Eq. (12) is not unique, thus the market described by subdiffusive GBM is incomplete.*

In Appendix C we give proof of the incompleteness of financial market based on subdiffusive GBM $Z_\Psi(t)$.

In the next step we derive the option pricing formulas. Here we will concentrate on the European options, however other derivatives can be priced in a similar manner. From Appendix C we know that the subdiffusive model is incomplete, as a consequence different martingale measures lead us to different prices of derivative. In what follows, we will concentrate on the measure $\mathbb{Q}$ defined in (12) since it is a natural extension of martingale measure from the classical BS model. Moreover, the relative entropy (*i.e.* the distance between measures [25]) of $\mathbb{Q}$ is smaller than $\mathbb{Q}_\epsilon$ defined in (C.1). This means that the distance between measures $\mathbb{Q}$ and $\mathbb{P}$ is the smallest. The relative entropy of $\mathbb{Q}$ is equal to

$$D = -\int\limits_\Omega \log \frac{d\mathbb{Q}}{d\mathbb{P}} d\mathbb{P} = \frac{\gamma^2}{2}\left\langle S_\Psi(T)\right\rangle .$$

On the other hand, we have for $\mathbb{Q}_\epsilon$

$$\begin{aligned} D_\epsilon &= -\int\limits_\Omega \log \frac{d\mathbb{Q}_\epsilon}{d\mathbb{P}} d\mathbb{P} = \log \left\langle \exp\left\{-\gamma B(S_\Psi(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right) S_\Psi(T)\right\}\right\rangle \\ &\quad + \left(\epsilon + \frac{\gamma^2}{2}\right) \langle S_\Psi(T)\rangle = \log \langle \exp\{-\epsilon S_\Psi(T)\}\rangle + \left(\epsilon + \frac{\gamma^2}{2} \langle S_\Psi(T)\rangle\right) \\ &\quad \geq \frac{\gamma^2}{2} \langle S_\Psi(T)\rangle = D\,. \end{aligned}$$

Thus, we confirmed that the measure $\mathbb{Q}$ minimizes the relative entropy.

Using martingale measure $\mathbb{Q}$ we can derive fair price of European call option with expiry date T and strike price K. This is a subject of the next theorem

Theorem 3.3. *Assume that the asset price follows GBM $Z_\Psi(t)$ and that the martingale measure $\mathbb{Q}$ is given by (12). Then the BS formula for the European call option price $C_\Psi(X_0, K, T, \sigma, r)$ with interest rate $r > 0$ is given by*

$$\begin{aligned} C_\Psi(X_0, K, T, \sigma, r) &= \langle C(X_0, K, S_\Psi(T), \sigma, r)\rangle \\ &= \int\limits_0^\infty C(X_0, K, x, \sigma, r) g_\Psi(x, T) dx\,. \end{aligned} \tag{13}$$

Here, $g_\Psi(x, T)$ is the PDF of $S_\Psi(T)$ and $C(X_0, K, T, \sigma)$ is given by Eq. (5).

For the proof of the above theorem see Appendix D. The price of a put option can be easily determined from the put–call parity (4)

$$P_\Psi(X_0, K, T, \sigma, r) = C_\Psi(X_0, K, T, \sigma, r) + e^{-rT} K - X_0\,. \tag{14}$$

In order to find the price of a call option in a subdiffusive market model, one can follow two ways. The first way is to approximate the integral in (13). However this can be done only in cases when the PDF $g_\Psi(x, T)$ is known analytically, for instance $\alpha = 1/2$. The second one is by Monte Carlo method. One simulates trajectories of the inverse subordinator $S_\Psi(t)$ on the finite interval $[0, T]$ and calculates the expected value in (13). To do this, one can adopt the following efficient approximation scheme [16] of the process $S_\Psi(t)$

$$S_{\Psi\Delta}(t) = (\min\{n \in \mathbb{N} : T_\Psi(\Delta n) > y\} - 1)\,\Delta\,. \tag{15}$$

Here $\Delta > 0$ is the accuracy parameter and $T_\Psi(\tau)$ is the subordinator defined in (9). Since $T_\Psi(\tau)$ is a Lévy process, for its simulation one can use the general method presented in [27].

4. Example: tempered α-stable waiting times

In this section, we present the numerical analysis of the subdiffusive GBM with tempered stable waiting times.

The class of tempered α-stable times was first introduced by Rosiński [28] and Cartea, del-Castillo-Negrete [29]. The tempered α-stable distributions are invariant under linear transformations and have finite moments of all orders. On the other hand, they resemble stable laws in many aspects (see [28] for details). Tempered α-stable laws are particularly attractive in the modeling of two-regime dynamics. Namely, when we observe the characteristic transition from the initial subdiffusive character of motion in short times to the standard diffusion in long times [30, 31]. Such a transition was confirmed in many experiments, one can mention here the case of random motion of photospheric bright points [32], motion of molecules inside living cells [33, 34] and recently discovered motion of lipid granules in living fission yeast cells [35].

For the case of tempered α-stable waiting times, the Laplace exponent in (9) is given by

$$\Psi(u) = (u+\lambda)^{\alpha} - \lambda^{\alpha}\,, \qquad 0 < \alpha < 1\,, \qquad \lambda > 0\,.$$

For $\lambda \searrow 0$, we recover the Laplace transform of ordinary α-stable distribution.

In order to apply approximation scheme (15), we generate $T_{\Psi(\Delta}n)$, $n = 1, 2, \ldots$, using the standard method of summing up increments [36]

$$\begin{aligned} T_{\Psi}(0) &= 0\,, \\ T_{\Psi(\Delta}n) &= T_{\Psi(\Delta}(n-1)) + Z_n\,, \end{aligned}$$

where Z_n are independent, identically distributed tempered α-stable random variables with the Laplace transform

$$\mathbb{E}\left(e^{-uZ_n}\right) = e^{-\Delta t((u+\lambda)^{\alpha}-\lambda^{\alpha})}\,.$$

The algorithm for generating tempered stable random variable Z with the above Laplace transform, is the following [37]:

I. Generate exponential random variable E with mean λ^{-1};

II. Generate totally skewed α-stable random variable S using the formula [38]

$$S = \Delta t^{1/\alpha} \frac{\sin(\alpha(U+\frac{\pi}{2}))}{\cos(U)^{1/\alpha}} \left(\frac{\cos(U-\alpha(U+\frac{\pi}{2}))}{W}\right)^{(1-\alpha)/\alpha}\,. \tag{16}$$

Here, U is uniformly distributed on $[-\pi/2, \pi/2]$, and W has exponential distribution with mean 1;

III. If $E > S$ put $Z = S$, otherwise go to step I.

Finally, to obtain the trajectory of $Z_\Psi(t) = Z(S_\Psi(t))$, one needs to simulate the process $Z(t)$ and put it together with previously obtained $S_{\Psi\Delta}(t)$. Since $Z(t)$ is a standard diffusion process, its simulation methods are well established [36].

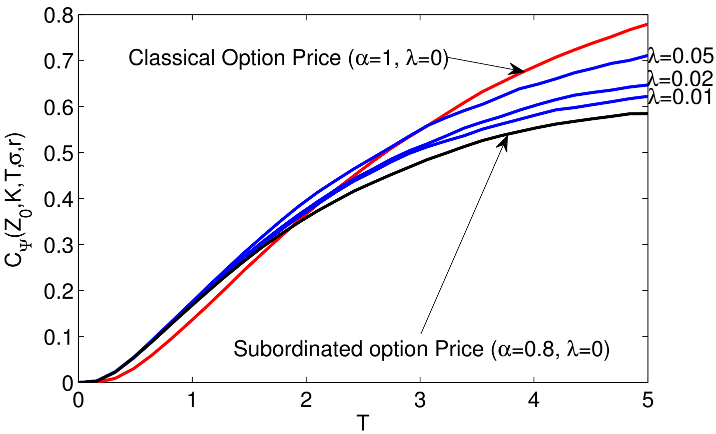

Fig. 2. (Color online) The comparison of the Black–Scholes prices of the European call option $C_\Psi(Z_0, K, T, \sigma, r)$ according to the parameters α and λ. For $\lambda \searrow 0$ option prices with tempered subordinator (blue lines, $\alpha = 0.8$, $\lambda > 0$) tend to the option prices with stable subordinator (black line, $\alpha = 0.8$, $\lambda = 0$). In contrary, when λ is increasing, prices tend to classical Black–Scholes one (red line, $\alpha = 1$, $\lambda = 0$). Here $Z_0 = 1$, $K = 2$, $\sigma = 0.6$, $r = 0.4$, and $T \in [0, 5]$.

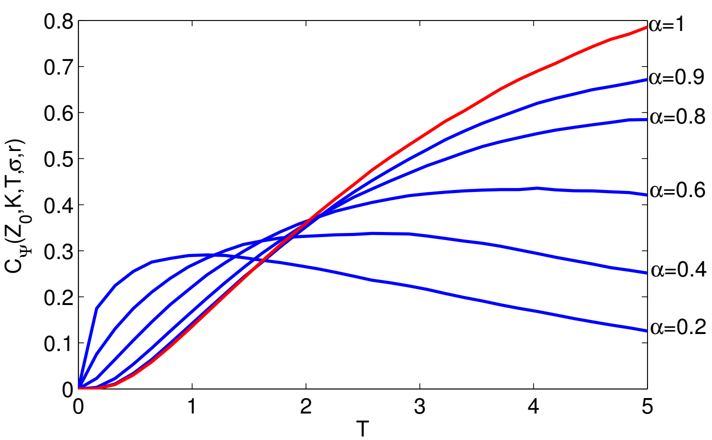

Fig. 3. (Color online) Black–Scholes prices with α-stable waiting times of European call options $C_\Psi(Z_0, K, T, \sigma, r)$ according to α parameter. Here $Z_0 = 1$, $K = 2$, $\sigma = 0.6$, $r = 0.4$, and $T \in [0, 5]$.

In Fig. 2 we have compared the classical Black–Scholes European call option prices with the subordinated ones. One can see that for $\lambda \searrow 0$ option prices with tempered subordinator tend to option prices with stable subordinator. In contrary for increasing λ, option prices are approaching the classical BS prices. In Fig. 3 we present European call option prices in the case, where the classical BS model is subordinated by pure α-stable distribution ($\lambda = 0$) with different αs. When $\alpha \nearrow 1$, the subdiffusive prices tend to the classical one. Fig. 4 depicts the surface of European call option prices in the subdiffusive BS model for different exercise date T and strike price K. The waiting times follow the tempered stable law.

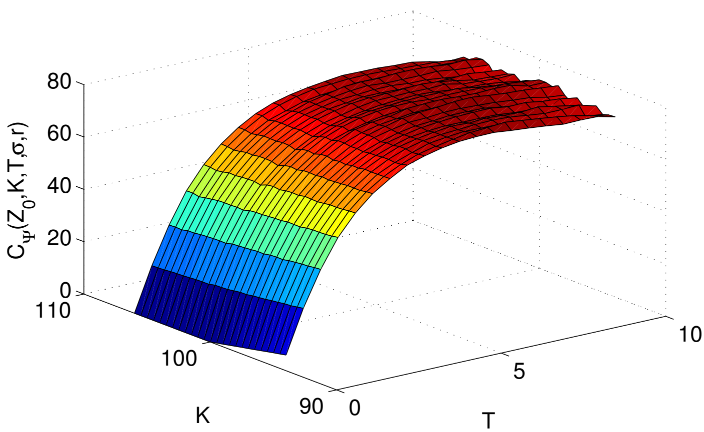

Fig. 4. (Color online) Black–Scholes prices of European call options $C_\Psi(Z_0, K, T, \sigma, r)$ according to exercise date T, strike price K. In the subdiffusive model the waiting times follow the tempered stable law $\Psi(u) = (u + \lambda)^\alpha - \lambda^\alpha$. Here $Z_0 = 1$, $K = 2$, $\sigma = 0.6$, $r = 0.4$, $T \in [0, 10]$ and $\lambda = 0.01$, $\alpha = 0.9$.

5. Conslusions

In this paper, we have introduced an extension of the classical Black–Scholes model which is capable to capture the periods of stagnation of asset prices. Our model is defined as standard GBM, subordinated by the ID inverse subordinator. This assures flexibility of the model since the periods in which the asset price does not change can be chosen from the broad family of nonnegative ID distributions. We have shown that the proposed model is arbitrage free and incomplete. We derived formulas for fair prices of European call an put options and mentioned how to approximate them numerically. To calibrate the model to the real data, one can follow a standard procedure for diffusion processes. First, one needs to remove the waiting times (constant periods) from the trajectory of subdiffusive GBM. This way one obtains the trajectory of the standard GBM, which can be analyzed in the usual way. The removed waiting times form the sample of the underlying ID distribution, which can be recognized with the help of known statistical methods, see for example [39].

We believe that the introduced here model will provide a more adequate description of the assets price processes especially in emerging markets.

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Appendix A

Proof of Theorem 2.1. Let us denote by $p(x,t)$ the PDF of $Z_\Psi(t)$. Using total probability formula we get for the Laplace transform

$$\hat{p}(x,k) = \int_0^\infty e^{-kt} p(x,t)dt = \int_0^\infty f(x,\tau)\hat{g}(\tau,k)d\tau\,. \tag{A.1}$$

Here the $f(x,\tau)$ and $g(\tau,t)$ are PDFs of $Z(\tau)$ and $S_\Psi(t)$, respectively, and $\hat{g}(\tau,k) = \int_0^\infty e^{-kt} g(\tau,t)dt$. The process $Z(t)$ is a standard GBM, thus its PDF $f(x,\tau)$ obeys the ordinary Fokker–Planck Equation

$$\frac{\partial f(x,\tau)}{\partial \tau} = \left[-\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x f(x,\tau) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 f(x,\tau)\right]. \tag{A.2}$$

Taking Laplace transform of the above we get

$$k\hat{f}(x,k) - f(x,0) = -\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x\hat{f}(x,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 \hat{f}(x,k)\,, \tag{A.3}$$

where $\hat{f}(x,k) = \int_0^\infty e^{-k\tau} f(x,\tau)$. We denote the PDF of $T_\Psi(\tau)$. By property $P(T_\Psi(\tau) \geq t) = P(S_\Psi(t) \leq \tau)$ we obtain

$$g(\tau,t) = -\frac{\partial}{\partial \tau}\int_{-\infty}^{t} h(t',\tau)dt'\,.$$

Consequently, by some standard calculations we obtain

$$\hat{g}(\tau,k) = \frac{\Psi(k)}{k} e^{-\tau\Psi(k)}\,.$$

From Eq. (A.1) we have

$$\hat{p}(x,k) = \frac{\Psi(k)}{k}\hat{f}(x,\Psi(k))\,. \tag{A.4}$$

The last formula after the change of variables in Eq. (A.3) $k \to \Psi(k)$ yields

$$\Psi(k)\hat{f}(x,\Psi(k)) - f(x,0) = -\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}x\hat{f}(x,\Psi(k)) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2\hat{f}(x,\Psi(k))\,.$$

Now by the observation from Eq. (A.4) and the fact that $f(x,0) = p(x,0)$ we infer that $\hat{p}(x,k)$ satisfies the equation

$$k\hat{p}(x,k) - p(x,0) = \frac{k}{\Psi(k)}\left[-\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}x\hat{p}(x,k) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2\hat{p}(x,k)\right].$$

Finally inverting the above Laplace transform we obtain

$$\frac{\partial w(x,t)}{\partial t} = \Phi(t)\left[-\left(\mu - r + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}xw(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2 w(x,t)\right],$$

and this completes the proof.

Appendix B

Proof of Theorem 3.1. Let $\mathbb{Q}$ be the probability measure defined in Eq. (12). Let us introduce the filtration $(\mathcal{G}_t)_{t\in[o,T]}$ as

$$\mathcal{G}_t = \mathcal{H}_{S_\Psi(t)}\,, \tag{B.1}$$

where

$$\mathcal{H}_\tau = \bigcap_{u>\tau}\{\sigma(B(y) : 0 \le y \le u) \vee \sigma(S_\Psi(y) : y \ge 0)\}\,. \tag{B.2}$$

One can observe that $\mathcal{F}_t \subseteq \mathcal{G}_t$. Moreover from [18] we get that the processes $B(S_\Psi(t))$ and $\exp\{-\gamma B(S_\Psi(t)) - \frac{\gamma^2}{2}S_\Psi(t)\}$ are $\mathcal{G}_t$-martingales with respect to $\mathbb{P}$. Thus $\mathbb{Q}$ is a probability measure. Clearly, it is equivalent to $\mathbb{P}$. Now we will show that $Z_\Psi(t)$ is a $(\mathcal{G}_t)$-martingale with respect to a new probability measure $\mathbb{Q}$. Let us define

$$K_\Psi(t) = B(S_\Psi(t)) + \gamma S_\Psi(t)\,,$$

and

$$H(t) = \mathbb{E}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|\mathcal{G}_t\right) = \mathbb{E}\left(\exp\left\{-\gamma B(S_\Psi(T)) - \frac{\gamma^2}{2}S_\Psi(T)\right\}\bigg|\mathcal{G}_t\right).$$

From martingale property we get that

$$H(t) = \exp\left\{-\gamma B(S_\Psi(t)) - \frac{\gamma^2}{2} S_\Psi(t)\right\}$$

or, equivalently, in the differential form

$$dH(t) = -\gamma H(t) dB(S_\Psi(t)) \,, \qquad H(0) = 1 \,. \tag{B.3}$$

We easily see from Eq. (7) and the above, that we can write $Z_\Psi(t)$ in terms of $H(t)$ in the following form

$$Z_\Psi(t) = \exp\left\{\sigma K_\Psi(t) - \frac{\sigma^2}{2} S_\Psi(t)\right\} \,. \tag{B.4}$$

Denote by $\langle \cdot, \cdot \rangle$ the quadratic variation. Then, the quadratic variation of $K_\Psi(t)$ satisfies $\langle K_\Psi(t), K_\Psi(t)\rangle = S_\Psi(t)$. Using Girsanov–Meyer theorem [26] and Eq. (B.3) we get that the process

$$\begin{aligned} & B(S_\Psi(t)) - \int\limits_0^t \frac{1}{H(s)} d\left\langle H(s), B(S_\Psi(t))\right\rangle \\ & = B(S_\Psi(t)) + \gamma \int\limits_0^t \frac{1}{H(s)} H(s) d\left\langle B(S_\Psi(t)), B(S_\Psi(t))\right\rangle \\ & = B(S_\Psi(t)) + \gamma S_\Psi(t) = K_\Psi(t) \end{aligned}$$

is a local martingale with respect to $\mathbb{Q}$. By Eq. (B.4) we infer that also the process $Z_\Psi(t)$ is a local martingale with respect to $\mathbb{Q}$. Finally, since

$$\mathbb{E}^{\mathbb{Q}}(Z_\Psi(t)) = \mathbb{E}\left(\exp\left\{\sigma K_\Psi(t) - \frac{\sigma^2}{2} S_\Psi(t) - \gamma B(S_\Psi(t)) - \frac{\gamma^2}{2} S_\Psi(t)\right\}\right) = 1 \,,$$

$Z_\Psi(t)$ is a $(\mathcal{G}_t, \mathbb{Q})$-martingale. This ends the proof.

Appendix C

Proof of Theorem 3.2. For every $\epsilon > 0$ let us define the probability measure

$$\mathbb{Q}_\epsilon(A) = C \int\limits_A \exp\left\{-\gamma B(S_\Psi(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right) S_\Psi(T) d\mathbb{P}\right\} d\mathbb{P} \,, \tag{C.1}$$

where $C = \left(\left\langle \exp\{-\gamma B(S_\Psi(T)) - (\epsilon + \frac{\gamma^2}{2})S_\Psi(T)\}\right\rangle\right)^{-1}$ is the normalizing constant, $\gamma = \frac{\mu - r + \sigma^2/2}{\sigma}$ and $A \in \mathcal{F}$. We will show that $Z_\Psi(t)$ is a $(\mathcal{G}_t)$-martingale with respect to $\mathbb{Q}_\epsilon$, where $\mathcal{G}_t = \mathcal{H}_{S_\Psi(t)}$ is defined in Eq. (12). One clearly see that $\mathbb{Q}_\epsilon$ is equivalent to $\mathbb{P}$. Put

$$Y(t) = \exp\left\{-\gamma B(t) - \frac{\gamma^2}{2}t\right\}, \qquad Z(t) = \exp\{\sigma B(t) + (\mu - r)t\}.$$

Then we have

$$Y(t)Z(t) = \exp\left\{(\sigma - \gamma)B(t) - \frac{(\sigma-\gamma)^2}{2}t\right\}.$$

Thus $Y(t)Z(t)$ is a $(\mathcal{H}_t)$-martingale with respect to $\mathbb{P}$, where $(\mathcal{H}_t)$ is defined in Eq. (B.2). Let us define

$$Z^{S_\Psi(T)}(t) = Z(t \wedge S_\Psi(T)).$$

One can observe that the stopped process $Y(t \wedge S_\Psi(T))Z^{S_\Psi(T)}(t)$ is also a $(\mathcal{H}_t, \mathbb{P})$-martingale. Since the filtration $\mathcal{H}_t$ is right continuous, the bounded random variable $e^{-\epsilon S_\Psi(T)}$ is a $\mathcal{H}_0$-measurable. It follows that the process

$$\left(e^{-\epsilon S_\Psi(T)}Y(t \wedge S_\Psi(T))Z^{S_\Psi(T)}(t)\right)_{t\geq 0}$$

is also a $(\mathcal{H}_t, \mathbb{P})$-martingale. Moreover, for $A \in \mathcal{H}_t$ we have

$$\begin{aligned}
\mathbb{Q}_\epsilon(A) &= \mathbb{E}\left(\mathbf{1}_A \exp\left\{\gamma B(S_\Psi(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right)S_\Psi(T)\right\}\right) \\
&= \mathbb{E}\left(\mathbf{1}_A e^{-\epsilon S_\Psi(T)}\mathbb{E}\left(\exp\left\{\gamma B(S_\Psi(T)) - \frac{\gamma^2}{2}S_\Psi(T)\right\}\middle|\mathcal{H}_t\right)\right) \\
&= \mathbb{E}\left(\mathbf{1}_A e^{-\epsilon S_\Psi(T)}Y(t \wedge S_\Psi(T))\right).
\end{aligned}$$

Thus we obtain that the process $Z^{S_\Psi(T)}(t)$ is a $(\mathcal{H}_t, \mathbb{Q}_\epsilon)$-martingale. Next, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\epsilon}\left(\sup_{t\geq 0} Z^{S_\Psi(T)}(t)\right) = \mathbb{E}^{\mathbb{Q}_\epsilon}\left(\sup_{t\leq S_\Psi(T)} Z(t)\right) \\
&= \mathbb{E}\left(\exp\left\{-\gamma B(S_\Psi(T)) - \left(\epsilon + \frac{\gamma^2}{2}\right)S_\Psi(T)\right\}\sup_{t\leq S_\Psi(T)} Z(t)\right) \\
&\leq \mathbb{E}\left(\exp\{-\gamma B(S_\Psi(T))\}e^{|\mu - r|S_\Psi(T)}\sup_{t\leq T} e^{\sigma B(S_\Psi(t))}\right). \qquad \text{(C.2)}
\end{aligned}$$

For any $p>0$ and $n\in\mathbb{N}$ we have

$$\begin{aligned}\langle S_\Psi^n(t)\rangle &= \int\limits_0^\infty x^{n-1}\mathbb{P}(S_\Psi(T)>x)dx=\int\limits_0^\infty x^{n-1}\mathbb{P}(T_\Psi(x)<t)dx\\ &= \int\limits_0^\infty x^{n-1}\mathbb{P}\left(e^{-pT_\Psi(x)}>e^{-pt}\right)dx\le e^{pt}\int\limits_0^\infty x^{n-1}e^{-x\Psi(p)}=\frac{e^{pt}\Gamma(n)}{\Psi^n(p)}\,.\end{aligned}$$

Thus for any $\lambda>0$ we have

$$\left\langle e^{\lambda S_\Psi(t)}\right\rangle=\sum_{n=1}^\infty\frac{\lambda^n\langle S_\Psi^n(t)\rangle}{n!}\le e^{pt}\sum_{n=1}^\infty\frac{\lambda^n\Gamma(n)}{\Psi^n(p)n!}=e^{pt}\sum_{n=1}^\infty\frac{\lambda^n}{\Psi^n(p)n}<\infty \tag{C.3}$$

for large enough p (recall that $\Psi(p)\to\infty$ as $p\to\infty$). This shows existence and finiteness of exponential moments of inverse subordinators. Conditioning on $\sigma(S_\Psi(y):y\ge 0)$ we obtain

$$\mathbb{E}(\exp\{\lambda B(S_\Psi(T))\})=\mathbb{E}\left(\frac{\lambda^2}{2}S_\Psi(T)\right)<\infty\,.$$

From the Doob's maximal inequality we get that

$$\mathbb{E}\left(\sup_{t\le T}\exp\{\lambda B(S_\Psi(t))\}\right)^2\le 4\mathbb{E}(\exp\{2\lambda B(S_\Psi(T))\})<\infty\,.$$

Thus by Hölder inequality, the above results with combination with (C.2) yield

$$\mathbb{E}^{\mathbb{Q}_\epsilon}\left(\sup_{t\ge 0}Z^{S_\Psi(T)}(t)\right)<\infty\,.$$

Thus we obtain that $Z^{S_\Psi(T)}(t)$ is a uniformly integrable martingale. From the martingale representation theorem it follows that there exist a random variable X such that

$$Z_\Psi(t)=Z^{S_\Psi(T)}(S_\Psi(t))=\mathbb{E}^{\mathbb{Q}_\epsilon}\left(X|\mathcal{H}_{S_\Psi(t)}\right)\,.$$

As conclusion we get that $Z_\Psi(t)$ is a $(\mathcal{H}_{S_\Psi(t)},\mathbb{Q}_\epsilon)$-martingale. Since $\mathcal{F}_t\subseteq\mathcal{H}_{S_\Psi(t)}$ it is also a $(\mathcal{F}_t,\mathbb{Q}_\epsilon)$-martingale.

Appendix D

Proof of Theorem 3.3. The formula follows from the fact that

$$\begin{aligned} C_{\Psi}(X_0, K, T, \sigma, r) &= e^{-rT}\left\langle (Z_{\Psi}(T) - K)^+ \right\rangle_{\mathbb{Q}} \\ &= e^{-rT}\left\langle \exp\left\{-\gamma B(S_{\Psi}(T)) - \frac{\gamma^2}{2} S_{\Psi}(T)\right\}(Z_{\Psi}(T) - K)^+ \right\rangle, \end{aligned} \quad \text{(D.1)}$$

where $\langle\cdot\rangle_{\mathbb{Q}}$ denotes the expectation with respect to martingale measure $\mathbb{Q}$. Conditioning on $S_{\Psi}(T)$ we easily obtain formula (13).

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