SCALING LIMITS OF OVERSHOOTING LÉVY WALKS*

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In this paper, we obtain the scaling limits of one-dimensional overshooting Lévy walks. We also find the limiting processes for extensions of Lévy walks, in which the waiting times and jumps are related by power-law, exponential and logarithmic dependence. We find that limiting processes of overshooting Lévy walk are characterized by infinite mean-square-displacement. It also occurs that introducing different dependence between waiting times and jumps of Lévy walks results in subdiffusive properties.

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1. Introduction

A Lévy walk (LW) process, which is special type of well know Continuous-Time Random Walk (CTRW), has been found as an excellent tool for describing many phenomena, such as anomalous diffusion in complex systems [1, 2, 3, 4, 5, 6, 7, 8] through human travel [9, 10] and epidemic spreading [11, 12, 13] to the foraging patterns of micro-organisms and animals

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[14, 15, 16, 17, 18, 19, 20]. In classical CTRW process the position of random walker is described by the sequences of jumps and waiting times between them. Every jumps in this process is preceded by a waiting time and it is usually assumed that sequences of jumps and waiting times are independent and identically distributed (iid) sequences. Introducing a coupling between waiting times and corresponding jumps leads to the class of LWs processes. In the classical version of LW it is assumed that the jump and corresponding waiting times are equal, up to the constant.

Although a CTRW process was found as the powerful tool in describing many real-life phenomena, it occurs that sometimes it is beneficial to consider the so-called overshooting continuous-time random walk (OCTRW) [21, 24, 23, 22, 25, 26]. In OCTRW process, as opposed to CTRW, every waiting time is preceded by corresponding jump. Similarly, as in CTRW case, introducing spatiotemporal coupling in OCTRW model leads to the class of *overshooting Lévy walks.*

In this paper, we develop limit theory of overshooting Lévy walks and their extensions originating from OCTRW with heavy-tailed waiting times. Moreover, we investigate the mean square displacement (MSD) of the limiting processes.

1.1. Preliminaries

The CTRW process is given by the sequence of iid random vectors $\{(T_i, X_i)\}_{i\geq 1}$, where random variables T_i and X_i characterize the ith waiting time and ith jump of random walk, respectively. More precisely, let us define two counting processes which correspond to the sequence of waiting times $\{T_i\}_{i\geq 1}$. First, let the undershooting counting process $\{N(t)\}_{t\geq 0}$ representing the number of jumps of a particle up to time t be defined as follows

$$N(t) \overset{\text{def}}{=} \max\{n : T_1 + T_2 + \ldots + T_n \leq t\}\,, \tag{1}$$

and the overshooting counting process $\{\tilde{N}(t)\}_{t\geq 0}$ be defined as follows

$$\tilde{N}(t) \overset{\text{def}}{=} \min\{n : T_1 + T_2 + \ldots + T_n > t\}\,.$$

It is easy to observe that relation $\tilde{N}(t) = N(t) + 1$ holds for any $t \geq 0$.

Depending on the choice of the counting process we distinguish two fundamental models, namely a CTRW model

$$R(t) = \sum_{i=1}^{N(t)} X_i\,, \tag{2}$$

and OCTRW [22]

$$\tilde{R}(t) = \sum_{i=1}^{\tilde{N}(t)} X_i = \sum_{i=1}^{N(t)+1} X_i \,. \tag{3}$$

From definitions (2) and (3) one can conclude that a walker following CTRW process starts its motion at origin and waits there for a time T_1 before performing first jump. On the other hand, a walker following OCTRW process starts its motion by performing jump from the origin and then waits for time T_1 long in the new site. $R(t)$ describes the position of the random walker after the last jump before time t and $\tilde{R}(t)$ is the position at the first jump after time t. These both schemes leads to so-called 'wait–jump' and 'jump–wait' scenarios, respectively. CTRW process was originally used to model a diffusive particle, while OCTRW was found useful in modeling dielectric relaxation phenomena [24, 21] or financial instruments [23].

2. One-dimensional undershooting and overshooting Lévy walks and their extensions

2.1. One-dimensional undershooting and overshooting Lévy walks

To construct the Lévy walks originating from previously described random walks (2)–(3), we need to introduce additional assumptions on the waiting times and jumps. Let us assume that

A. the waiting times T_i are heavy-tailed distributed

$$P(T_i > t) \propto t^{-\alpha} \qquad \text{as} \quad t \to \infty \,, \tag{4}$$

with $\alpha \in (0, 1)$.

B. for any $i \geq 1$, $|X_i| = T_i$.

Random walks $R(t)$ and $\tilde{R}(t)$ satisfying both assumptions A and B are called *undershooting Lévy walk* (LW) and *overshooting Lévy walk* (OLW), respectively. Let us notice that in the physical literature only LWs are studied in details. In further analysis we will concentrate on OLW.

Notice that the sequence of waiting times T_i fulfilling assumption A belongs to the normal domain of attraction of some one-sided α-stable random variable with positive scale parameter σ. It means that

$$n^{-1/\alpha} \sum_{i=1}^{[nt]} T_i \Rightarrow S_\alpha(t) \tag{5}$$

as $n \to \infty$. Here, $S_\alpha(t)$ denotes an α-stable subordinator with the following Fourier transform [27]

$$\mathbb{E}\exp(ikS_\alpha(t)) = \exp(t\sigma^\alpha|k|^\alpha(i\mathrm{sgn}(k)\tan(\pi\alpha/2) - 1))\,, \tag{6}$$

where positive parameter σ is a scale parameter. The above used notation "$\Rightarrow$" denotes convergence in distribution[1]. In the context of scaling limits of LW and OLW processes, it is useful to introduce the so-called inverse α-stable subordinator S_α^{-1}, defined as [29, 30]

$$S_\alpha^{-1}(t) = \inf\{\tau > 0 : S_\alpha(\tau) > t\}\,.$$

The process S_α^{-1} is the scaling limit of the counting processes $N(t)$ and $\tilde{N}(t)$ with T_i satisfying condition (4), see [31]. It corresponds to the heavy-tailed waiting times in the underlying LW and OLW.

Now, let us consider an iid sequence $\{I_i\}_{i\geq 1}$ which is independent of the sequence of waiting times $\{T_i\}_{i\geq 1}$. The distribution of I_i is the following

$$P(I_i = 1) = p\,, \qquad P(I_i = -1) = q\,, \tag{7}$$

where $p+q = 1$ and $0 \leq p,\, q \leq 1$. Using the sequence $\{I_i\}_{i\geq 1}$, we can define the sequence of waiting times $\{X_i\}_{i\geq 1}$ such that $X_i = I_iT_i$. Therefore, sequence I_i determines the direction of jump X_i.

It is easy to check that the sequence $\{X_i\}_{i\geq 1}$ belongs to the normal domain of attraction of α-stable distribution

$$n^{-1/\alpha}\sum_{i=1}^{[nt]} X_i = n^{-1/\alpha}\sum_{i=1}^{[nt]} I_iT_i \Rightarrow L_\alpha(t)\,, \tag{8}$$

where $L_\alpha(t)$ is the α-stable motion with the following Fourier transform

$$\mathbb{E}\exp(ikL_\alpha(t)) = \exp(t\sigma^\alpha|k|^\alpha((p-q)\tan(\pi\alpha/2)\,\mathrm{sgn}(k) - 1))$$

with positive scale parameter σ.

In the following theorems we derive the scaling limit for the OLW processes and their extensions. First, let us notice that scaling limits for the undershooting LWs were studied in [32]. The next theorem extends scaling limit result to OLW process.

Theorem 1. *Let $\tilde{R}(t)$ be one-dimensional OLW process with the corresponding waiting times T_i satisfying (4) and jumps X_i such that $X_i = I_iT_i$ with I_i satisfying (7). Then*

$$\frac{\tilde{R}(nt)}{n} \Rightarrow L_\alpha\left(S_\alpha^{-1}(t)\right)$$

[1] More precisely, it denotes functional convergence in distribution in the J_1-Skorokhod topology [28] and it implies the convergence of finite-dimensional distributions.

as $n \to \infty$. Here, the processes $L_\alpha(t)$ and $S_\alpha(t)$ have the same instants of jumps, additionally the length of each jump of $L_\alpha(t)$ is equal to the length of corresponding jump of $S_\alpha(t)$.

Proof. See Appendix A. □

It has been shown in [32] that the limiting process in the LW scenario, which corresponds to Theorem 1, is characterized by the MSD satisfying the following property

$$\mathbb{E}L_\alpha^-\left(S_\alpha^{-1}(t)\right) \propto t^2\,,$$

as $t \to \infty$. This is no more true for the OLW process considered in Theorem 1. Let us observe that by conditional expected value we have

$$\begin{aligned}
\mathbb{E}\tilde{R}^2(t) &= \int_0^\infty P\left(\tilde{R}^2(t) > x\right) dx \\
&= \int_0^\infty \sum_{i=1}^\infty P\left(\tilde{R}^2(t) > x | \tilde{N}(t) = i\right) P\left(\tilde{N}(t) = i\right) dx \\
&\geq P\left(\tilde{N}(t) = 1\right) \int_0^\infty P\left(\tilde{R}^2(t) > x | \tilde{N}(t) = 1\right) dx \\
&= P\left(\tilde{N}(t) = 1\right) \int_0^\infty P\left(X_1^2 > x\right) dx = P\left(\tilde{N}(t) = 1\right) \mathbb{E}X_1^2 \qquad (9)
\end{aligned}$$

and since $\mathbb{E}X_1^2 = \infty$ we obtain that MSD of OLW process is infinite

$$\mathbb{E}\tilde{R}^2(t) = \infty.$$

In Fig. 1 we presented two exemplary trajectories of LW and OLW processes, as it is seen that the first jump in OLW scenario makes the MSD of the process infinite, although the velocity of the movement is constant.

The scaling limits of one-dimensional Lévy walk extensions has been recently developed in [32]. In this paper we give results for OLW, corresponding to the cases considered in [32]. We also present scaling limits for extensions of LW and OLW in the case of logarithmic dependence between waiting times and jumps.

Let us notice that OLW and LW processes with parameter $p = 1$ in (7) describe a situation when walker moves only upwards. These models are investigated in [33] and can play a crucial role in modeling dielectric relaxation phenomena in complex systems [24, 21].

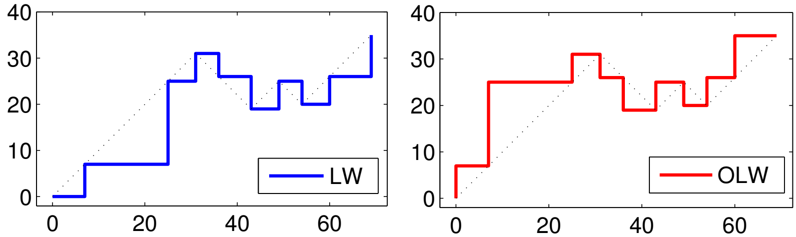

Fig. 1. Exemplary trajectories of LW (left panel) and OLW processes (right panel), in both cases waiting time equals to the jumps length. We can observe that for LW process every jump is preceded by a waiting time, while for OLW process every jump is followed by corresponding waiting time. Although in both cases the particle moves with constant velocity, the MSD of LW is finite, whereas the MSD of OLW is infinite.

2.2. Extension of one-dimensional undershooting and overshooting Lévy walks: power law dependence between waiting times and jumps

In this section, we derive scaling limit for some extensions of OLW. A first considered model introduce the power-law dependence between waiting times and jumps. The limiting process with such kind of dependence for LWs have been recently derived (Theorem 2 in [32]). The power-law and exponential dependence in the context of correlated CTRWs has been recently investigated in [34]. Below, we present the corresponding result in the case of OLW process.

Theorem 2. *Let $\tilde{R}(t)$ be one-dimensional OLW process with the corresponding waiting times T_i satisfying (4) and jumps X_i such that $X_i = I_i T_i^{\gamma}$, where $\gamma > \alpha$, with I_i satisfying (7). Then*

$$\frac{\tilde{R}(nt)}{n^{\gamma}} \Rightarrow L_{\alpha/\gamma}\left(S_{\alpha}^{-1}(t)\right)$$

as $n \to \infty$. Here, the processes $L_{\alpha/\gamma}(t)$ and $S_{\alpha}(t)$ have the same instants of jumps, additionally the length of each jump of $L_{\alpha/\gamma}(t)$ is equal to the length of corresponding jump of $S_{\alpha}(t)$ raised to the power γ.

Proof. See Appendix B. □

The MSD of the limiting process from Theorem 2 is also infinite. This fact is straightforward as we repeat calculation given in Eq. (9) with jump $X_i = I_i T_i^{\gamma}$.

2.3. Extension of one-dimensional undershooting and overshooting Lévy walks: exponential dependence between waiting times and jumps

The scaling limit process of LW process $R(t)$ with the corresponding waiting times T_i satisfying (4) and jumps X_i such that $X_i = I_i e^{-T_i}$ with I_i satisfying (7) with $p = q = 0.5$ is already known (Theorem 3. in [32]).

The next theorem establishes scaling limit result for corresponding OLW process.

Theorem 3. *Let $\tilde{R}(t)$ be one-dimensional OLW process with the corresponding waiting times T_i satisfying (4) and jumps X_i such that $X_i = I_i e^{-T_i}$ with I_i satisfying (7) with $p = q = 0.5$. Then*

$$\frac{\tilde{R}(nt)}{n^{\alpha/2}} \Rightarrow \sqrt{\mathbb{E}e^{-2T_1}}\, B\left(S_\alpha^{-1}(t)\right)$$

as $n \to \infty$, where $B(t)$ is a standard Brownian motion independent of process $S_\alpha(t)$.

Proof. See Appendix C. □

Let us notice that the limiting process is the same for both LW and OLW scenario, therefore it is characterized by the same MSD behavior

$$\mathbb{E}\left(\sqrt{\mathbb{E}e^{-2T_1}} B\left(S_\alpha^{-1}(t)\right)\right)^2 \propto t^\alpha\,,$$

as $t \to \infty$. Note that the probability density function of the limiting process $\sqrt{\mathbb{E}e^{-2T_1}}\, B(S_\alpha^{-1}(t))$ fulfills the fractional Fokker–Planck equation [1]

$$\frac{\partial w(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[K_\alpha \frac{\partial^2}{\partial x^2}\right] w(x,t)\,,$$

for appropriate anomalous diffusion constant K_α.

2.4. Extension of one-dimensional undershooting and overshooting Lévy walks: logarithmic dependence between waiting times and jumps

The next result for LW and OLW processes corresponds to the logarithmic dependence between waiting times and jumps. Namely, we assume that the waiting times T_i satisfying (4) and jumps X_i are related by the following formula: $X_i = I_i \ln(T_i + 1)$ with I_i satisfying (7). The following theorem have been found.

Theorem 4. *Let $\tilde{R}(t)$ and $R(t)$ be one-dimensional OLW and LW processes, respectively, with the corresponding waiting times T_i satisfying (4) and jumps X_i such that $X_i = I_i \ln(T_i+1)$ with I_i satisfying (7) with $p = q = 0.5$. Then*

$$\frac{\tilde{R}(nt)}{n^{\alpha/2}} \Rightarrow \sqrt{\mathbb{E}\ln^2(T_1+1)}\, B\left(S_\alpha^{-1}(t)\right) ,$$

and

$$\frac{R(nt)}{n^{\alpha/2}} \Rightarrow \sqrt{\mathbb{E}\ln^2(T_1+1)}\, B\left(S_\alpha^{-1}(t)\right)$$

as $n \to \infty$, where $B(t)$ is a standard Brownian motion independent of process $S_\alpha(t)$.

Proof. See Appendix D. □

Let us notice that the MSD of the limiting processes derived in Theorem 4 fulfills the following property

$$\mathbb{E}\left(\sqrt{\mathbb{E}\ln^2\left(T_1+1\right)}B\left(S_\alpha^{-1}(t)\right)\right)^2 \propto t^\alpha ,$$

since its probability density function satisfies the above mentioned fractional Fokker–Planck equation.

3. Conclusions

This paper concerns scaling limits of overshooting Lévy walk, which corresponds to 'jump–wait' scenario. We derived results devoted to the asymptotic behavior of some extensions of undershooting and overshooting Lévy walk processes. These extensions were constructed by assuming power-law, logarithmic and exponential dependence between waiting times and jumps. Depending on the choice of the function, the limiting process can have infinite second moment (power-law dependence) or display subdiffusive dynamics (exponential and logarithmic dependence). The derived explicit formulas for continuous-time limits of OLWs can be further applied to study their properties, such as first passage times, laws of large numbers and fractality.

Appendix A

Proof of Theorem 1

To prove Theorem 1 let us define the following array of jumps and waiting times of considered model

$$\{(X_{n,i}, T_{n,i})\}_{n,i\geq 1} = \left\{\left(n^{-1/\alpha} I_i T_i, n^{-1/\alpha} T_i\right)\right\}_{n,i\geq 1}$$

and sequence of OLWs generated by this array

$$\tilde{R}_n(t) = n^{-1/\alpha} \sum_{i=1}^{N_n(t)+1} I_i T_i \,,$$

where $N_n(t)$ is defined as

$$N_n(t) = \max\left\{k \geq 1 : n^{-1/\alpha} \sum_{i=1}^{k} T_i \leq t\right\}. \tag{A.1}$$

In the next part of the proof, we will find the limiting process of sequence $\tilde{R}_{n^\alpha}(t)$. Let us introduce the following sequences of processes

$$A_n(t) = \sum_{i=1}^{[nt]} X_{n,i} \,, \qquad E_n(t) = \sum_{i=1}^{[nt]} T_{n,i} \,.$$

From the assumptions (5) and (8), it follows that $A_n(t) \Rightarrow L_\alpha(t)$ and $E_n(t) \Rightarrow S_\alpha(t)$ respectively. In the next part of the proof, we will show the joint convergence of $(A_n(t), E_n(t)) \Rightarrow (L_\alpha(t), S_\alpha(t))$ based on Theorem 3.2.2 [35].

Let us observe that the array $\{(X_{n,i}, T_{n,i})\}$ fulfills condition (a) of Theorem 3.2.2 [35]. Namely, for any Borel sets $B_1 \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$, where $\mathbb{R}_+ = [0, \infty)$, we have that

$$nP(X_{n,1} \in B_1, T_{n,1} \in B_2) = nP\left(n^{-1/\alpha} I_1 T_1 \in B_1, n^{-1/\alpha} T_1 \in B_2\right)$$
$$= n\int_{B_2} P\left(I_1 u \in B_1\right) P\left(n^{-1/\alpha} T_1 \in du\right) = np\int_{B_2} \mathbb{I}(u \in B_1) P\left(n^{-1/\alpha} T_1 \in du\right)$$
$$+nq\int_{B_2} \mathbb{I}(-u \in B_1) P\left(n^{-1/\alpha} T_1 \in du\right) ,$$

and consequently

$$nP(X_{n,1} \in dx_1, T_{n,1} \in dx_2)$$
$$= \left(p\delta_{(x_2)}(dx_1) + q\delta_{(-x_2)}(dx_1)\right) nP\left(n^{-1/\alpha} T_1 \in dx_2\right) ,$$

where δ_a denotes the probability measure concentrated at point a. Due to the fact that $E_n(t) \Rightarrow S_\alpha(t)$ it follows that $E_n(1) \Rightarrow S_\alpha(1)$. Next, using (6) and Theorem 7.3.5 in [35] we have that Lévy measure of S_α is

$$\nu_{S_\alpha}((x,\infty)) = \frac{\sigma^\alpha}{\Gamma(1-\alpha)\cos(\pi\alpha/2)} x^{-\alpha}\,,$$

where $x > 0$. Therefore, by means of Theorem 3.2.2 in [35] we have that

$$nP(n^{-1/\alpha}T_1 > x) \longrightarrow \nu_{S_\alpha((x,\infty))} = \frac{\sigma^\alpha}{\Gamma(1-\alpha)\cos(\pi\alpha/2)} x^{-\alpha}\,, \tag{A.2}$$

as $n \to \infty$. Hence, by the relation (A.2), we obtain that

$$\begin{aligned} & nP(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \\ & = \left(p\delta_{(x_2)}(dx_1) + q\delta_{(-x_2)}(dx_1)\right) nP\left(n^{-1/\alpha}T_1 \in dx_2\right) \\ & \longrightarrow \left(p\delta_{(x_2)}(dx_1) + q\delta_{(-x_2)}(dx_1)\right)\nu_{S_\alpha}(dx_2) \end{aligned}$$

as $n \to \infty$. It is easy to check that measure $\nu_{(L_\alpha,S_\alpha)}$ defined in the following way

$$\nu_{(L_\alpha,S_\alpha)}(dx_1,dx_2) = \left(p\delta_{(x_2)}(dx_1) + q\delta_{(-x_2)}(dx_1)\right)\nu_{S_\alpha}(dx_2) \tag{A.3}$$

is indeed a Lévy measure. Moreover, processes S_α and L_α do not have Gaussian component in the Lévy–Khinchin representation, hence condition (b) of Theorem 3.2.2 [35] if fulfilled with $Q_{(L_\alpha,S_\alpha)}(\boldsymbol{t}) = \boldsymbol{0}$. Finally, based on Theorem 3.2.2 [35], we have that

$$(A_n(1), E_n(1)) \xrightarrow{d} (L_\alpha(1), S_\alpha(1))\,,$$

where $\xrightarrow{d}$ denotes one-dimensional convergence in distribution. Therefore, by Theorem 4.1 [36] we obtain

$$(A_n(t), E_n(t)) \Rightarrow (L_\alpha(t), S_\alpha(t))\,.$$

Next, using Theorem 3.6 [25], we have that

$$\tilde{R}_{n^\alpha}(t) \Rightarrow L_\alpha(S_\alpha^{-1}(t))\,. \tag{A.4}$$

Now, let us notice that $N_{n^\alpha}(t) = N(nt)$, since based on (1) we have that

$$N_{n^\alpha}(t) = \max\left\{k \geq 1 : (n^\alpha)^{-1/\alpha}\sum_{i=1}^{k} T_i \leq t\right\} = \max\left\{k \geq 1 : \sum_{i=1}^{k} T_i \leq tn\right\} \tag{A.5}$$

and let us observe also that

$$\tilde{R}_{n^\alpha}(t) = (n^\alpha)^{-1/\alpha} \sum_{i=1}^{N_{n^\alpha}(t)+1} I_i T_i = n^{-1} \sum_{i=1}^{N(nt)+1} I_i T_i = \frac{\tilde{R}(nt)}{n}\,. \tag{A.6}$$

Therefore, sequences $\tilde{R}_{n^\alpha}(t)$ and $\tilde{R}(nt)/n$ have the same limiting processes, thus based on (A.4) we obtain the final result

$$\frac{\tilde{R}(nt)}{n} \Rightarrow L_\alpha(S_\alpha^{-1}(t))\,.$$

Note that from formula (A.3) one easily conclude that process $L_\alpha(t)$ and $S_\alpha(t)$ have simultaneous jumps and the length of each jump of $L_\alpha(t)$ is equal to the length of corresponding jump of $S_\alpha(t)$.

Appendix B

Proof of Theorem 2

The proof of Theorem 2 has similar structure to the proof of Theorem 1. Let us define an array of jumps and waiting times corresponding to model considered in Theorem 2

$$\{(X_{n,i}, T_{n,i})\}_{n,i\geq 1} = \left\{\left(n^{-\gamma/\alpha} I_i T_i^\gamma, n^{-1/\alpha} T_i\right)\right\}_{n,i\geq 1} \tag{B.1}$$

and sequence of OLWs generated by this array

$$\tilde{R}_n(t) = n^{-\gamma/\alpha} \sum_{i=1}^{N_n(t)+1} I_i T_i^\gamma\,,$$

where $N_n(t)$ is defined as (A.1). In the next part of the proof, we will find the limiting process of sequence $\tilde{R}_{n^\alpha}(t)$. Let us introduce the sequences of processes corresponding to (B.1)

$$A_n(t) = \sum_{i=1}^{[nt]} X_{n,i}\,, \qquad E_n(t) = \sum_{i=1}^{[nt]} T_{n,i}\,.$$

It follows from the Appendix C in [32] and from the assumption (5), that $A_n(t) \Rightarrow L_{\alpha/\gamma}(t)$ and $E_n(t) \Rightarrow S_\alpha(t)$, respectively. In the next part of the proof, based on Theorem 3.2.2 [35], we will show the joint convergence of $(A_n(t), E_n(t)) \Rightarrow (L_{\alpha/\gamma}(t), S_\alpha(t))$.

Let us observe that array defined in (B.1) satisfy condition (a) of Theorem 3.2.2 [35]. Namely, for any Borel sets $B_1 \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$ we have that

$$nP(X_{n,1} \in B_1, T_{n,1} \in B_2) = nP\left(n^{-\gamma/\alpha} I_1 T_1^{\gamma} \in B_1, n^{-1/\alpha} T_1 \in B_2\right)$$
$$= n \int_{B_2} P\left(I_1 u^{\gamma} \in B_1\right) P\left(n^{-1/\alpha} T_1 \in du\right) = np \int_{B_2} \mathbb{I}(u^{\gamma} \in B_1) P\left(n^{-1/\alpha} T_1 \in du\right)$$
$$+ nq \int_{B_2} \mathbb{I}(-u^{\gamma} \in B_1) P\left(n^{-1/\alpha} T_1 \in du\right)$$

and consequently

$$\begin{aligned} & nP(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \\ &= \left(p\delta_{(x_2^{\gamma})}(dx_1) + q\delta_{(-x_2^{\gamma})}(dx_1)\right) nP\left(n^{-1/\alpha} T_1 \in dx_2\right) . \end{aligned}$$

Based on the property (A.2) we have that

$$nP(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \longrightarrow \left(p\delta_{(x_2^{\gamma})}(dx_1) + q\delta_{(-x_2^{\gamma})}(dx_1)\right) \nu_{S_\alpha}(dx_2) \tag{B.2}$$

as $n \to \infty$. It is easy to check that measure $\nu_{(L_{\alpha/\gamma}, S_\alpha)}$ defined in the following way

$$\nu_{(L_{\alpha/\gamma}, S_\alpha)}(dx_1, dx_2) = \left(p\delta_{(x_2^{\gamma})}(dx_1) + q\delta_{(-x_2^{\gamma})}(dx_1)\right) \nu_{S_\alpha}(dx_2) \tag{B.3}$$

is indeed a Lévy measure. Moreover, processes S_α and $L_{\alpha/\gamma}$ do not have Gaussian component in the Lévy–Khinchin representation, hence condition (b) of Theorem 3.2.2 [35] if fulfilled with $Q_{(L_{\alpha/\gamma}, S_\alpha)}(\boldsymbol{t}) = \mathbf{0}$. Finally, based on Theorem 3.2.2 [35], we have that $(A_n(1), E_n(1)) \xrightarrow{d} (L_{\alpha/\gamma}(1), S_\alpha(1))$ and by Theorem 4.1 [36], we prove the joint convergence $(A_n(t), E_n(t)) \Rightarrow (L_{\alpha/\gamma}(t), S_\alpha(t))$.

Now, as a result of Theorem 3.6 [25], we have that $\tilde{R}_{n^\alpha}(t) \Rightarrow L_{\alpha/\gamma}(S_\alpha^{-1}(t))$. Finally, using equality $N_{n^\alpha}(t) = N(nt)$ given in formula (A.5), we obtain that

$$\tilde{R}_{n^\alpha}(t) = (n^\alpha)^{-\gamma/\alpha} \sum_{i=1}^{N_{n^\alpha}(t)+1} I_i T_i^{\gamma} = n^{-\gamma} \sum_{i=1}^{N(nt)+1} I_i T_i^{\gamma} = \frac{\tilde{R}(nt)}{n^{\gamma}} ,$$

and therefore sequences $\tilde{R}_{n^\alpha}(t)$ and $\tilde{R}(nt)/n^{\gamma}$ have the same limiting processes. Hence $\tilde{R}_{n^\alpha}(t) \Rightarrow L_{\alpha/\gamma}(S_\alpha^{-1}(t))$ we obtain that

$$\frac{\tilde{R}(nt)}{n^{\gamma}} \Rightarrow L_{\alpha/\gamma}\left(S_\alpha^{-1}(t)\right) .$$

Note that from formula (B.3) one easily concludes that process $L_\alpha(t)$ and $S_\alpha(t)$ have simultaneous jumps and the length of each jump of $L_\alpha(t)$ is equal to the length of corresponding jump of $S_\alpha(t)$ raised to the power γ.

Appendix C

Proof of Theorem 3

The proof of Theorem 3 has similar structure to the proof of Theorem 1. Let us define the array of jumps and waiting times corresponding to considered model

$$\{(X_{n,i}, T_{n,i})\}_{n,i\geq 1} = \{(n^{-1/2} I_i e^{-T_i}, n^{-1/\alpha} T_i)\}_{n,i\geq 1} \tag{C.1}$$

and partial sequences of OLWs generated by this array

$$\tilde{R}_n(t) = n^{-1/2} \sum_{i=1}^{N_n(t)+1} I_i e^{-T_i} \,,$$

where $N_n(t)$ is defined as (A.1) and sequence $\{I_i\}_{i\geq 1}$ has distribution $P(I_i = 1) = P(I_i = -1) = \frac{1}{2}$. Let us introduce auxiliary sequences of processes corresponding to (C.1)

$$A_n(t) = \sum_{i=1}^{[nt]} X_{n,i} \,, \qquad E_n(t) = \sum_{i=1}^{[nt]} T_{n,i} \,.$$

In the next part of the proof, we will find the limiting process of sequence $\tilde{R}_{n^\alpha}(t)$.

First, based on Theorem 3.2.2 [35], we will show the joint convergence $(A_n(t), E_n(t)) \Rightarrow (\sqrt{\mathbb{E}e^{-2T_1}} B(t), S_\alpha(t))$. Let us observe that array defined in (C.1) satisfy condition (a) of Theorem 3.2.2 [35]. Namely, for any Borel sets $B_1 \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$ we have that

$$\begin{aligned}
&nP\left(X_{n,1} \in B_1, T_{n,1} \in B_2\right) = nP\left(n^{-1/2} I_1 e^{-T_1} \in B_1, n^{-1/\alpha} T_1 \in B_2\right) \\
&= n\int_{B_2} P\left(n^{-1/2} I_1 e^{-n^{1/\alpha} u} \in B_1\right) P\left(n^{-1/\alpha} T_1 \in du\right) \\
&= \tfrac{1}{2} n \int_{B_2} \mathbb{I}\left(n^{-1/2} e^{-n^{1/\alpha} u} \in B_1\right) P\left(n^{-1/\alpha} T_1 \in du\right) \\
&+ \tfrac{1}{2} n \int_{B_2} \mathbb{I}\left(-n^{-1/2} e^{-n^{1/\alpha} u} \in B_1\right) P\left(n^{-1/\alpha} T_1 \in du\right)
\end{aligned}$$

and based on the property (A.2) we have

$$nP\left(X_{n,1} \in dx_1, T_{n,1} \in dx_2\right) = \tfrac{1}{2}\delta_{\left(n^{-1/2}e^{-n^{1/\alpha}x_2}\right)}(dx_1)nP\left(n^{-1/\alpha}T_1 \in dx_2\right)$$
$$+ \tfrac{1}{2}\delta_{\left(-n^{-1/2}e^{-n^{1/\alpha}x_2}\right)}(dx_1)nP\left(n^{-1/\alpha}T_1 \in dx_2\right) \longrightarrow \delta_0(dx_1)\nu_{S_\alpha}(dx_2)\,,$$

as $n \to \infty$. It is easy to check that resulting measure

$$\nu_{(B,S_\alpha)}(dx_1, dx_2) = \delta_0(dx_1)\nu_{S_\alpha}(dx_2) \tag{C.2}$$

is indeed a Lévy measure. Observe that

$$n\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x}\rangle^2 P\left(X_{n,1} \in dx_1, T_{n,1} \in dx_2\right)$$
$$= \tfrac{1}{2}n\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \left(t_1 n^{-1/2}e^{-n^{1/\alpha}x_2} + t_2x_2\right)^2 \delta_{\left(n^{-1/2}e^{-n^{1/\alpha}x_2}\right)}(dx_1)P\left(n^{-1/\alpha}T_1 \in dx_2\right)$$
$$+ \tfrac{1}{2}n\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \left(-t_1 n^{-1/2}e^{-n^{1/\alpha}x_2} + t_2x_2\right)^2 \delta_{\left(-n^{-1/2}e^{-n^{1/\alpha}x_2}\right)}(dx_1)P\left(n^{-1/\alpha}T_1 \in dx_2\right)$$
$$= t_1^2 \int\limits_{\left\|\left(n^{-1/2}e^{-n^{1/\alpha}x_2}, x_2]\right)\right\|<\varepsilon} e^{-2n^{1/\alpha}x_2} P\left(n^{-1/\alpha}T_1 \in dx_2\right)$$
$$+ t_2^2 \int\limits_{\left\|\left(n^{-1/2}e^{-n^{1/\alpha}x_2}, x_2\right)\right\|<\varepsilon} x_2^2 nP\left(n^{-1/\alpha}T_1 \in dx_2\right) = t_1^2 C_{n,1} + t_2^2 C_{n,2}\,.$$

For arbitrary fixed $\varepsilon > 0$ and for all $n > 1/(\varepsilon^2 - x_2^2)$ we observe that

$$C_{n,1} = \int e^{-2n^{1/\alpha}x_2}\mathbb{I}\left(e^{-2n^{1/\alpha}x_2} < n\left(\varepsilon^2 - x_2^2\right)\right) P\left(n^{-1/\alpha}T_1 \in dx_2\right)$$
$$= \int e^{-2y}P(T_1 \in dy) = \mathbb{E}e^{-2T_1}\,,$$

and

$$C_{n,2} \le \int\limits_{x_2<\varepsilon} x_2^2 nP\left(n^{-1/\alpha}T_1 \in dx_2\right) \to \text{const} \times \varepsilon^{2-\alpha}\,,$$

as $n \to \infty$. Since

$$\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x} \rangle P(X_{n,1} \in dx_1, T_{n,1} \in dx_2)$$
$$= \int\limits_{\left\|\left(n^{-1/2} e^{-n^{1/\alpha} x_2}, x_2\right)\right\|<\varepsilon} t_2 x_2 P\left(n^{-1/\alpha} T_1 \in dx_2\right)$$

we have

$$0 \leq n \left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x} \rangle P\left(X_{n,1} \in dx_1, T_{n,1} \in dx_2\right) \right)^2$$
$$= \frac{1}{n} t_2^2 \left(n \int\limits_{\left\|\left(n^{-1/2} e^{-n^{1/\alpha} x_2}, x_2\right)\right\|<\varepsilon} x_2 P\left(n^{-1/\alpha} T_1 \in dx_2\right) \right)^2$$
$$\leq \frac{1}{n} t_2^2 \left(\int\limits_{x_2<\varepsilon} x_2 n P\left(n^{-1/\alpha} T_1 \in dx_2\right) \right)^2 \leq \frac{1}{n} C^2 t_2^2$$

for some constant $C < \infty$. The last inequality is due to the fact that

$$\int\limits_{x_2<\varepsilon} x_2 n P\left(n^{-1/\alpha} T_1 \in dx_2\right)$$

is finite. Therefore, we obtain

$$Q_{(B,S_\alpha)}(\boldsymbol{t}) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} n \left[\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x} \rangle^2 P(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \right.$$
$$\left. - \left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x} \rangle P(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \right)^2 \right]$$

$$= \lim_{\varepsilon\to 0}\liminf_{n\to\infty} n \left[\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t},\boldsymbol{x}\rangle^2 P(X_{n,1}\in dx_1, T_{n,1}\in dx_2) \right.$$

$$\left. - \left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t},\boldsymbol{x}\rangle P(X_{n,1}\in dx_1, T_{n,1}\in dx_2)\right)^2 \right] = t_1^2 \mathbb{E}e^{-2T_1}\,.$$

Finally, we checked that the condition (b) of Theorem 3.2.2 [35] is fulfilled with $Q_{(B,S_\alpha)}(\boldsymbol{t}) = t_1^2\mathbb{E}e^{-2T_1}$, which implies that following convergence $(A_n(1), E_n(1)) \xrightarrow{d} (\sqrt{\mathbb{E}e^{-2T_1}}B(1), S_\alpha(1))$ and then by Theorem 4.1 [36], we obtain that $(A_n(t), E_n(t)) \Rightarrow (\sqrt{\mathbb{E}e^{-2T_1}}B(t), S_\alpha(t))$. As a result of Theorem 3.6 [25], we have that $\tilde{R}_{n^\alpha}(t) \Rightarrow \sqrt{\mathbb{E}e^{-2T_1}}B(S_\alpha^{-1}(t))$. Finally, using equality $N_{n^\alpha}(t) = N(nt)$ given in (A.5) we obtain

$$\tilde{R}_{n^\alpha}(t) = (n^\alpha)^{-1/2}\sum_{i=1}^{N_{n^\alpha}(t)+1} I_i e^{-T_i} = n^{-\alpha/2}\sum_{i=1}^{N(nt)+1} I_i e^{-T_i} = \frac{\tilde{R}(nt)}{n^{\alpha/2}} \tag{C.3}$$

and therefore sequences $\tilde{R}_{n^\alpha}(t)$ and $\tilde{R}(nt)/n^{\alpha/2}$ have the same limiting processes. As $\tilde{R}_{n^\alpha}(t) \Rightarrow \sqrt{\mathbb{E}e^{-2T_1}}B(S_\alpha^{-1}(t))$ we prove that

$$\frac{\tilde{R}(nt)}{n^{\alpha/2}} \Rightarrow \sqrt{\mathbb{E}e^{-2T_1}}B\left(S_\alpha^{-1}(t)\right)\,.$$

Note that form formula (C.2) one easily concludes that process $B(t)$ and $S_\alpha(t)$ are independent.

Appendix D

Proof of Theorem 4

First, let us prove the result for OLW process, then by simple modification we will prove the corresponding result for LW process.

Let us define the following array of jumps and waiting times corresponding to the considered model

$$\{(X_{n,i}, T_{n,i})\}_{n,i\geq 1} = \left\{\left(n^{-1/2}I_i\ln(T_i+1), n^{-1/\alpha}T_i\right)\right\}_{n,i\geq 1} \tag{D.1}$$

and sequence of OLWs generated by this array

$$\tilde{R}_n(t) = n^{-1/2}\sum_{i=1}^{N_n(t)+1} I_i\ln(T_i+1)\,,$$

where $N_n(t)$ is defined as (A.1) and sequence $\{I_i\}_{i\geq 1}$ has distribution $P(I_i = 1) = P(I_i = -1) = \frac{1}{2}$.

Let us introduce the sequences of processes corresponding to (D.1)

$$A_n(t) = \sum_{i=1}^{[nt]} X_{n,i}\,, \qquad E_n(t) = \sum_{i=1}^{[nt]} T_{n,i}\,.$$

In the next part, based on Theorem 3.2.2 [35], we will show the joint convergence of $(A_n(t), E_n(t)) \Rightarrow (\sqrt{\mathbb{E}\ln^2(T_1+1)}B(t), S_\alpha(t))$.

Let us observe that array defined in (D.1) satisfy condition (a) of Theorem 3.2.2 [35]. Namely, for any Borel sets $B_1 \in \mathcal{B}(\mathbb{R}\backslash\{0\})$ and $B_2 \in \mathcal{B}(\mathbb{R}_+)$ we have that

$$\begin{aligned}
&nP(X_{n,1} \in B_1, T_{n,1} \in B_2) = nP\left(n^{-1/2} I_1 \ln(T_i+1) \in B_1, n^{-1/\alpha}T_1 \in B_2\right)\\
&= n\int_{B_2} P\left(n^{-1/2} I_1 \ln(n^{1/\alpha}u+1) \in B_1\right) P\left(n^{-1/\alpha}T_1 \in du\right)\\
&= n\frac{1}{2}\int_{B_2} \mathbb{I}\left(n^{-1/2}\ln(n^{1/\alpha}u+1) \in B_1\right) P\left(n^{-1/\alpha}T_1 \in du\right)\\
&+n\frac{1}{2}\int_{B_2} \mathbb{I}\left(-n^{-1/2}\ln(n^{1/\alpha}u+1) \in B_1\right) P\left(n^{-1/\alpha}T_1 \in du\right)
\end{aligned}$$

and based on property (A.2) we have that

$$\begin{aligned}
&nP(X_{n,1} \in dx_1, T_{n,1} \in dx_2)\\
&= \tfrac{1}{2}\delta_{\left(n^{-1/2}\ln\left(n^{1/\alpha}dx_2+1\right)\right)}(dx_1) nP\left(n^{-1/\alpha}T_1 \in dx_2\right)\\
&+\tfrac{1}{2}\delta_{\left(n^{-1/2}\ln\left(-n^{1/\alpha}dx_2+1\right)\right)}(dx_1) nP\left(n^{-1/\alpha}T_1 \in dx_2\right) \longrightarrow \delta_0(dx_1)\nu_{S_\alpha}(dx_2)\,,
\end{aligned}$$

as $n \to \infty$. As in Appendix C measure $\nu_{(B,S_\alpha)}$ is indeed a Lévy measure. Let us observe that

$$\begin{aligned}
&n \int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x}\rangle^2 P(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \\
&= \frac{n}{2} \int\limits_{||\boldsymbol{x}||<\varepsilon} \left(t_1 n^{-1/2} \ln\left(n^{1/\alpha} x_2 + 1\right) + t_2 x_2\right)^2 \\
&\times \delta_{\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right)\right)}(dx_1) P\left(n^{-1/\alpha} T_1 \in dx_2\right) \\
&+ \frac{n}{2} \int\limits_{||\boldsymbol{x}||<\varepsilon} \left(-t_1 n^{-1/2} \ln\left(n^{1/\alpha} x_2 + 1\right) + t_2 x_2\right)^2 \\
&\times \delta_{\left(-n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right)\right)}(dx_1) P\left(n^{-1/\alpha} T_1 \in dx_2\right) \\
&= n \int\limits_{\left\|\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right), x_2\right)\right\|<\varepsilon} \left[t_1^2 n^{-1} \ln^2\left(n^{1/\alpha} x_2 + 1\right) + t_2^2 x_2^2\right] P\left(n^{-1/\alpha} T_1 \in dx_2\right) \\
&= t_1^2 \int\limits_{\left\|\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right), x_2\right)\right\|<\varepsilon} \ln^2\left(n^{1/\alpha} x_2 + 1\right) P\left(n^{-1/\alpha} T_1 \in dx_2\right) \\
&+ t_2^2 \int\limits_{\left\|\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right), x_2\right)\right\|<\varepsilon} x_2^2 n P\left(n^{-1/\alpha} T_1 \in dx_2\right) = t_1^2 C_{n,1} + t_2^2 C_{n,2}\,.
\end{aligned}$$

For arbitrary fixed ε and for all $n > \left(\frac{x_2^2}{\varepsilon^2 - x_2^2}\right)^{\frac{\alpha}{\alpha-2}}$ we observe that

$$\begin{aligned}
&C_{n,1} \\
&= \int \ln^2\left(n^{1/\alpha} x_2 + 1\right) \mathbb{I}\left(\ln^2\left(n^{1/\alpha} x_2 + 1\right) < n^{2/\alpha} x_2^2\right) P\left(n^{-1/\alpha} T_1 \in dx_2\right) \\
&= \int \ln^2(y+1) P(T_1 \in dy) = \mathbb{E} \ln^2(T_1 + 1)\,,
\end{aligned}$$

and as $n \to \infty$

$$0 \le C_{n,2} \le \int\limits_{x_2<\varepsilon} x_2^2 n P\left(n^{-1/\alpha} T_1 \in dx_2\right) \to \text{const} \times \varepsilon^{2-\alpha}\,.$$

Moreover, we have that

$$
\begin{aligned}
& n\left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x}\rangle P(X_{n,1} \in dx_1, T_{n,1} \in dx_2)\right)^2 \\
&= n\left(\frac{1}{2}\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \left(t_1 n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right) + t_2 x_2\right)\right. \\
&\times \delta_{\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right)\right)}(dx_1) P\left(n^{-1/\alpha}T_1 \in dx_2\right) \\
&+ \frac{1}{2}\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \left(-t_1 n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right) + t_2 x_2\right) \\
&\left.\times \delta_{\left(-n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right)\right)}(dx_1) P\left(n^{-1/\alpha}T_1 \in dx_2\right)\right)^2 \\
&= n\left(\int\limits_{\left\|\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right), x_2\right)\right\|<\varepsilon} t_2 x_2 P\left(n^{-1/\alpha}T_1 \in dx_2\right)\right)^2 \\
&= \frac{1}{n}t_2^2\left(n\int\limits_{\left\|\left(n^{-1/2}\ln\left(n^{1/\alpha}x_2+1\right), x_2\right)\right\|<\varepsilon} x_2 P\left(n^{-1/\alpha}T_1 \in dx_2\right)\right)^2 \\
&\le \frac{1}{n}t_2^2\left(\int\limits_{x_2<\varepsilon} x_2 n P\left(n^{-1/\alpha}T_1 \in dx_2\right)\right)^2 = \frac{1}{n}t_2^2\left(\mathrm{const}\cdot\varepsilon^{1-\alpha}\right)^2 \to 0\,,
\end{aligned}
$$

as $n \to \infty$. Therefore, we have

$$
\begin{aligned}
Q_{(B,S_\alpha)}(\boldsymbol{t}) \;=\; \lim_{\varepsilon\to 0}\limsup_{n\to\infty} n \Bigg[& \int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x}\rangle^2 P(X_{n,1} \in dx_1, T_{n,1} \in dx_2) \\
& - \left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon} \langle \boldsymbol{t}, \boldsymbol{x}\rangle P(X_{n,1} \in dx_1, T_{n,1} \in dx_2)\right)^2 \Bigg]
\end{aligned}
$$

$$= \lim_{\varepsilon\to 0}\liminf_{n\to\infty} n\left[\int\limits_{\|\boldsymbol{x}\|<\varepsilon}\langle \boldsymbol{t},\boldsymbol{x}\rangle^2 P(X_{n,1}\in dx_1, T_{n,1}\in dx_2)\right.$$
$$\left. - \left(\int\limits_{\|\boldsymbol{x}\|<\varepsilon}\langle \boldsymbol{t},\boldsymbol{x}\rangle P(X_{n,1}\in dx_1, T_{n,1}\in dx_2)\right)^2\right]$$
$$= t_1^2\mathbb{E}\ln^2(T_1+1)\,.$$

We checked that the condition (b) of Theorem 3.2.2 [35] is fulfilled with $Q_{(B,S_\alpha)}(\boldsymbol{t}) = t_1^2\mathbb{E}\ln^2(T_1+1)$. This fact implies that $(A_n(1),E_n(1)) \xrightarrow{d} \left(\sqrt{\mathbb{E}\ln^2(T_1+1)}B(1),S_\alpha(1)\right)$, so by Theorem 4.1 [36] we can conclude the following joint convergence $(A_n(t),E_n(t)) \Rightarrow \left(\sqrt{\mathbb{E}\ln^2(T_1+1)}B(t),S_\alpha(t)\right)$. As a result of Theorem 3.6 [25], we have that $\tilde{R}_{n^\alpha}(t)\Rightarrow\sqrt{\mathbb{E}e^{-2T_1}}B(S_\alpha^{-1}(t))$. We can easily check that $N_{n^\alpha}(t) = N(nt)$ (see A.5) and that $\tilde{R}_{n^\alpha}(t) = \tilde{R}(nt)/n^{\alpha/2}$ (see C.3) and therefore we obtain that sequences $\tilde{R}_{n^\alpha}(t)$ and $\tilde{R}(nt)/n^{\alpha/2}$ have the same limiting processes.

As $\tilde{R}_{n^\alpha}(t)\Rightarrow\sqrt{\mathbb{E}\ln^2(T_1+1)}B(S_\alpha^{-1}(t))$ we prove that

$$\frac{\tilde{R}(nt)}{n^{\alpha/2}}\Rightarrow\sqrt{\mathbb{E}\ln^2(T_1+1)}B(S_\alpha^{-1}(t))\,.$$

The proof of the corresponding result for LW process has similar structure to the proof of OLW part. The main difference comes from the fact that LW process has different counting process than OLW process. For the array defined in (D.1) let us define corresponding sequence of LWs

$$R_n(t) = n^{-1/2}\sum_{i=1}^{N_n(t)} I_i\ln(T_i+1)\,,$$

where $N_n(t)$ is defined as (A.1). Since the following holds $N(nt) = N_{n^\alpha}(t)$ we can derive analogously to (C.3) that $R_{n^\alpha}(t) = R(nt)/n^{\alpha/2}$. Therefore, sequences $R_{n^\alpha}(t) = R(nt)/n^{\alpha/2}$ have the same limiting processes. By analogous calculations as in the OLW case, we find the limiting process of sequence $R_{n^\alpha}(t)$. Namely, it follows from the proof of OLW part that the sequences $A_n(t)$ and $E_n(t)$ corresponding to (D.1) converge jointly $(A_n(t),E_n(t))\Rightarrow(\sqrt{\mathbb{E}\ln^2(T_1+1)}B(t),S_\alpha(t))$. Finally, using Theorem 3.6 [25] we have the following result $R_{n^\alpha}(t) \Rightarrow \sqrt{\mathbb{E}\ln^2(T_1+1)}B(S_\alpha^{-1}(t))$ as the processes B and S_α^{-1} are continuous. Therefore, since $R(nt)/n^{\alpha/2}$ and $R_{n^\alpha}(t)$ have the same limiting process, we prove our statement

$$\frac{R(nt)}{n^{\alpha/2}} \Rightarrow \sqrt{\mathbb{E}\ln^2(T_1+1)}B((S_\alpha^{-1}(t))\,.$$

Note that the Lévy measure $\nu_{(B,S_\alpha)}$ is of the same form as in formula (C.2) and similarly one concludes the independence of process $B(t)$ and $S_\alpha(t)$.

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