# ANOMALOUS DIFFUSION — THE THINNING PROPERTY OF FRACTIONAL BROWNIAN MOTION*

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We show that thinning of increments of the fractional Brownian motion with Hurst exponent $H \neq 1/2$ breaks its $H$-self-similarity property. As a result, we obtain a new Gaussian process with stationary increments which is not the fractional Brownian motion for any $H$. Moreover, in the subdiffusion case ($H < 1/2$), the new process statistically resembles the classical Brownian motion ($H = 1/2$). To this end, we study analytically the second moment of such processes. Finally, Monte Carlo simulations show that the $H$ estimator obtained by mean square displacement is close to the Brownian motion case with $H = 1/2$. These results show that stationary data describing anomalous diffusion phenomenon can lead to different statistical conclusions for different resolution of measurement. Therefore, one should be very careful in statistical inference, especially in strong subdiffusion regimes ($H \approx 0$).

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## 1. Introduction

Anomalous diffusion has been observed in a wide variety of systems [1,2,3,4,5,6,7,8]. In recent years there has been great progress in the understanding of the different mathematical models that can lead to anomalous diffusion [5,9]. One of them is the fractional Brownian motion (FBM), which was introduced by A.N. Kolmogorov in 1940, see [10, 11]. It is the only Gaussian process with self-similarity property and stationary increments, see [12, 13]. The corresponding index of self-similarity $H \in (0, 1]$ is called

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the Hurst exponent. The FBM can serve as a possible model for anomalous diffusion phenomena [1, 2, 3, 4, 5]. Moreover, its increments are called the fractional Gaussian noise (FGN) which is a stationary sequence. It can model experimental measurements of stationary phenomena, see [14, 15, 16].

In this paper, we focus on the FGN derived from the corresponding FBM with some Hurst exponent $H \in (0,1]$. We consider the thinning operation of the FGN and the properties of the corresponding cumulative sum process. Such a new model, loses the self-similarity property and is no longer FBM for any $H \neq 1/2$.

The presented results indicate the important issue how to measure stationary experimental data. It turns out that less frequent measurement of the strong subdiffusion phenomenon can statistically lead to the classical Brownian motion (BM) case. That means the resolution of measurements can significantly influences on the process modeling observed phenomena. Therefore, one should be very careful when determine corresponding stochastic model to the stationary data.

## 2. Thinning of FBM increments

Let $\{X(k) := B_H(k) - B_H(k-1) : k = 1, 2, \ldots\}$ be a discrete stochastic process of increments of the FBM $\{B_H(t) : t \geq 0\}$ with the Hurst exponent $H$ and the covariance function

$$\mathrm{Cov}(B_H(t), B_H(s)) = \tfrac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \tag{1}$$

for any $s, t \geq 0$. The increment process $\{X(k) : k = 1, 2, \ldots\}$ is called the fractional Gaussian noise (FGN). It follows from covariance formula (1) that

$$\mathrm{Cov}(X(k), X(k+n)) = \tfrac{1}{2}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right). \tag{2}$$

Let us denote $\rho_H^X(n) := \mathrm{Cov}(X(k), X(k+n))$ to determine dependence only on $n$. Moreover, this covariance function has the following asymptotic behavior

$$\rho_H^X(n) \sim H(2H-1)n^{2H-2}, \qquad \text{as } n \to \infty. \tag{3}$$

It is important to mention, that one can consider FGN, where increments correspond to the time interval $h > 0$ (not necessarily with $h = 1$). Then, such process $\{X(k,h) := B_H(hk) - B_H(h(k-1)) : k = 1, 2, \ldots\}$, has an analogous covariance function

$$\rho_H^X(n,h) := \tfrac{1}{2}h^{2H}\left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right). \tag{4}$$

Let us consider the following issue of the thinned increment process $\{Y(k) : k = 1, 2, \ldots\}$, *i.e.*

$$Y(k) := X(2k-1), \qquad k = 1, 2, \ldots .$$

The process $\{Y(k) : k = 1, 2, \ldots\}$ is just the process of odd increments of $\{B_H(t) : t \geq 0\}$ corresponding to the time length 1. In the trajectory approach, one can think that the realizations of the thinned process come from increment process but with the loss of even–numbered observations. We present in Fig. 1 one trajectory of the FBM with Hurst exponent $H = 0.1$ and the corresponding trajectories of the FGN and the thinned process. Because it is not easy to notice the difference in the appearance of trajectories for FGN and the thinned process, we also present its cumulative sums. Moreover, even better way to manifest the different behavior of these processes is to display 2-dimensional trajectories consisting of cumulative sum process and such process released back in time. In details, let us assume that we have the trajectory of FGN $\{X(k) : k = 1, 2, \ldots, N\}$ of the length $N$. Then the cumulative sums sequence is just the FBM discrete trajectory $\{B_H(k) : k = 1, 2, \ldots, N\}$ and the mentioned 2-dimensional object is just $\{(B_H(k), B_H(N + 1 - k)) : k = 1, 2, \ldots, N\}$. We display analogous trajectories for corresponding thinned process, *i.e.* the thinned process $\{Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$, their cumulative sums $\{S(k) := Y(1) + Y(2) + \ldots + Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$ and 2-dimensional

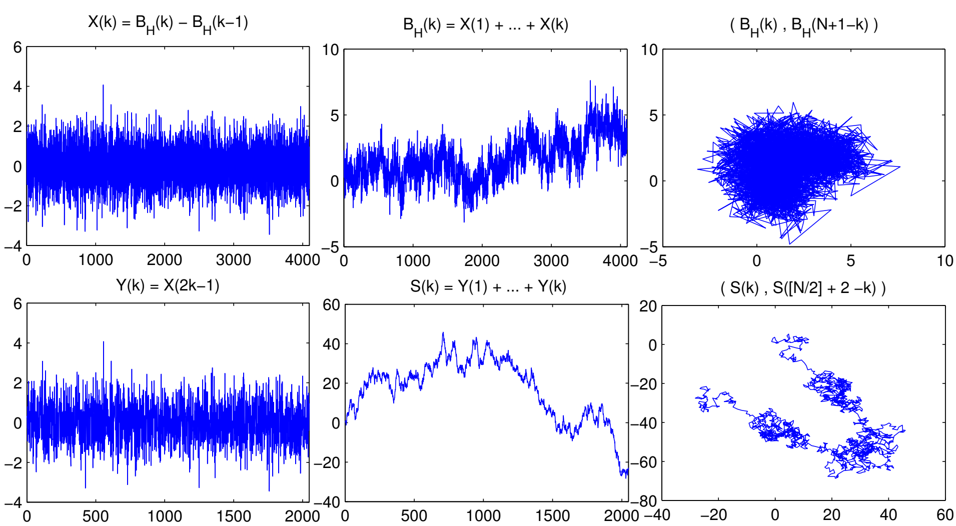


Fig. 1. Top row, from left to right: FGN trajectory $\{X(k) : k = 1, 2, \ldots, N\}$, corresponding FBM trajectory $\{B_H(k) : k = 1, 2, \ldots, N\}$ and 2-dimensional trajectory $\{(B_H(k), B_H(N+1-k)) : k = 1, 2, \ldots, N\}$. Bottom row, from left to right: thinned trajectory $\{Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$, corresponding cumulative sums trajectory $\{S(k) := Y(1) + Y(2) + \ldots + Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$ and 2-dimensional trajectory $\{(S(k), S(N + 1 - k)) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$. The Hurst exponent $H = 0.1$ and $N = 2^{12}$.

trajectory $\{(S(k), S(N+1-k)) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$. From Fig. 1, we see that for the case with $H = 0.1$, the thinned process behaves completely differently than original FGN. This is particularly evident for its cumulative sums and 2-dimensional trajectory. Moreover, the behavior looks more like Brownian diffusion case. This observation explains why the thinning transformation of the FBM can be very interesting and surprising.

Now we continue with some calculations related to the thinned FGN process. First, we focus on the covariance function of such process. Let us denote it as $\rho_H^Y(n) := \mathrm{Cov}(Y(k), Y(k+n))$. We easily get from (2)

$$\rho_H^Y(n) = \rho_H^X(2n) = \tfrac{1}{2}\left((2n+1)^{2H} + (2n-1)^{2H} - 2(2n)^{2H}\right) \tag{5}$$

and according to (3)

$$\rho_H^Y(n) \sim H(2H-1)2^{2H-2}n^{2H-2}\,, \qquad \text{as } n \to \infty\,.$$

So the asymptotic of the covariance function of thinned process $\{Y(k) : k = 1, 2, \ldots\}$ is the same power function as for $\{X(k) : k = 1, 2, \ldots\}$, but with new asymptotic constant $2^{2H-2}$. What is also very important, formulas (4) and (5) imply that the thinned process cannot be any FGN process even with any $H \neq 1/2$ and observed at time intervals of any length $h$. The only exception is the classical case of BM when $H = 1/2$. We present the covariance functions $\rho_H^X$ of FGN processes with different $H$ and the covariance functions $\rho_H^Y$ of the corresponding thinned processes, see Fig. 2. From Fig. 2 we see the biggest differences between covariance functions (of FGN and thinned process) for small lags. We present these differences more precisely in Fig. 3 and Fig. 4 (horizontal axis with logarithmic scale) separately for $H < 1/2$ and $H > 1/2$ cases. When $H > 1/2$ we see in Fig. 4 that the covariance of thinned process is just a little lower then for the original FGN process. On the contrary, for cases with $H < 1/2$ the difference is very drastic and covariance values for first lag are extremely increased for thinned process, see Fig. 3. Moreover, the smaller $H$ the greater value of $\rho_H^Y(1)$. After such observation, we can suspect that the thinning operation on FGN process changes its structure and properties, especially for small values of Hurst exponent $H \approx 0$.

Now we concentrate on the cumulative sum process of thinned process $\{Y(k) : k = 1, 2, \ldots\}$. For the FGN $\{X(k) : k = 1, 2, \ldots\}$ its cumulative sums determine just the FBM process $\{B_H(k) : k = 1, 2, \ldots\}$. Since, as mentioned earlier, the thinned process $\{Y(k) : k = 1, 2, \ldots\}$ is not the FGN (for any $H$ and time step $h$), the cumulative sums $\{S(k) : k = 1, 2, \ldots\}$ cannot be a discrete process coming from any FBM process. Now we give a simple argument showing that the $\{S(k) : k = 1, 2, \ldots\}$ process is not a FBM. It concerns the self-similarity property.

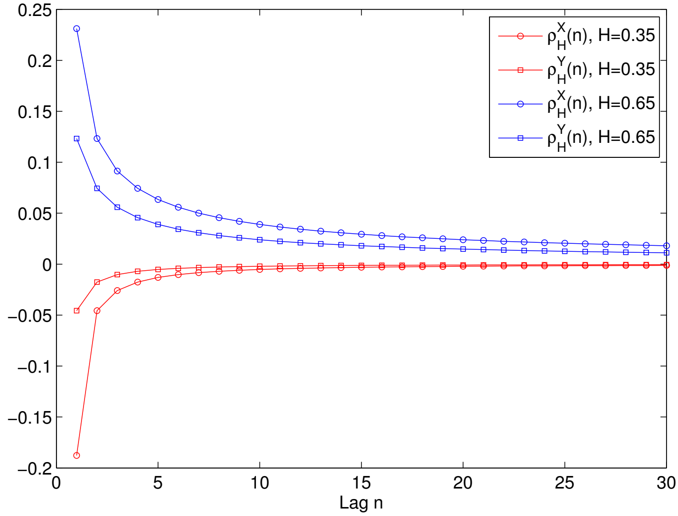


Fig. 2. The covariance functions $\rho_H^X$ and $\rho_H^Y$ of FGN (lines with circles) and corresponding thinned process (lines with squares) respectively. The case with $H = 0.35$ (light grey/red lines) and $H = 0.65$ (dark grey/blue lines).

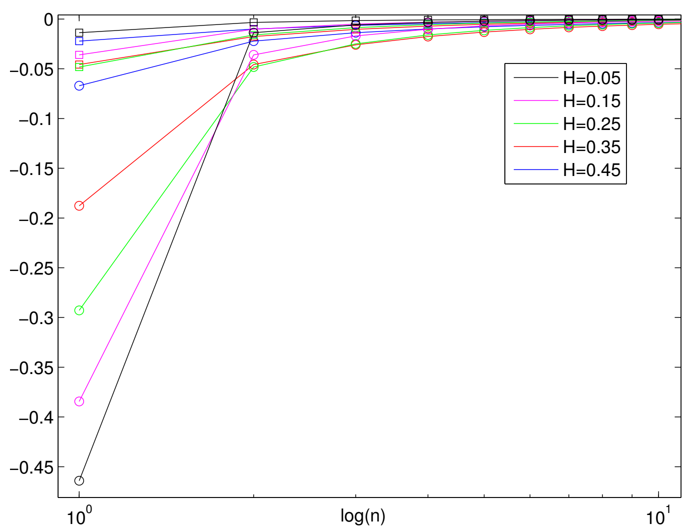


Fig. 3. The covariance functions $\rho_H^X$ and $\rho_H^Y$ of FGN (lines with circles) and corresponding thinned process (lines with squares) respectively. The cases with $H < 1/2$.

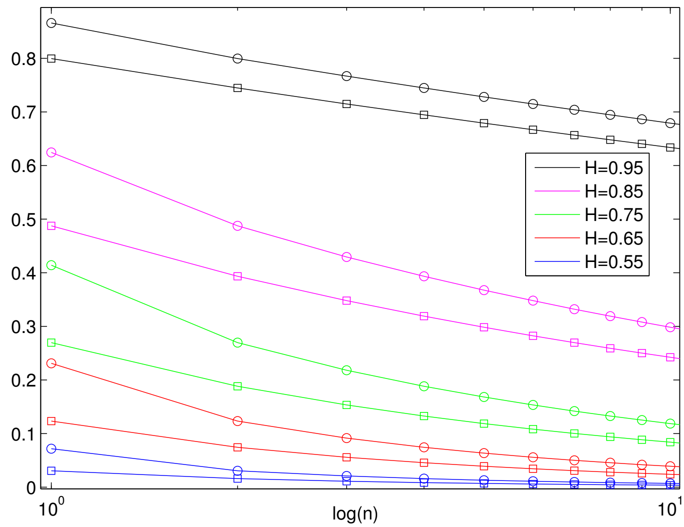


Fig. 4. The covariance functions $\rho_H^X$ and $\rho_H^Y$ of FGN (lines with circles) and corresponding thinned process (lines with squares) respectively. The cases with $H > 1/2$.

Let us suppose that the process $\{S(k) : k = 1, 2, \ldots\}$ is self-similar with some Hurst exponent $\tilde{H} \in (0, 1)$. Then from $\tilde{H}$ self-similarity property we have that

$$S(k) = k^{\tilde{H}} S(1)\,,$$

and so

$$E(S(k)^2) = k^{2\tilde{H}}\,.$$

But, on the other hand, we have

$$\begin{aligned} E\left(S(k)^2\right) &= E\left(\sum_{i=1}^{k} B_H(2i-1) - \sum_{i=1}^{k} B_H(2i-2)\right)^2 \\ &= k + 2(k+1) * \rho_H(2(k+1))(k-2)\,, \end{aligned} \tag{6}$$

where $*$ means the finite convolution

$$a(k) * b(k)(k) := \sum_{i=1}^{k} a(i) b(k-i)\,. \tag{7}$$

Hence

$$k^{2\tilde{H}} = k + 2(k+1) * \rho_H(2(k+1))(k-2)\,,$$

and consequently,

$$\tilde{H} = \frac{\ln\left(k + 2(k+1) * \rho_H(2(k+1))(k-2)\right)}{2\ln k}\,.$$

This quantity is constant for any $k$ only when $H = 1/2$ (because if $H = 1/2$ then $\rho_H(n) = 0$ for all $n$) and then $\tilde{H} = 1/2$. Thus the process $\{S(k) : k = 1, 2, \ldots\}$ is not self-similar, except the case with $H = 1/2$. That means the thinning operation on FGN process breaks the self-similarity structure of the corresponding FBM process. Because the process $\{S(k) : k = 1, 2, \ldots\}$ is Gaussian with stationary increments $\{Y(k) : k = 1, 2, \ldots\}$, it is not FBM for any Hurst exponent $H \in (0, 1)$. Only in the classical BM case with $H = 1/2$ the thinning leads again to BM and preserves $1/2$ self-similarity.

## 3. Second moment behavior

Let us now focus on the second moment of the process $\{S(k) : k = 1, 2, \ldots\}$. The formula (6) can be rewritten, by applying (7), as

$$E\left(S(k)^2\right) = k + 2\sum_{i=1}^{k-1}(k-i)\rho_H(2i)\,. \tag{8}$$

It is straightforward to notice that when $H \searrow 0$ then

$$E(S(k)^2) \approx k = E\left(B(k)^2\right)\,,$$

where $\{B(t) : t \geq 0\}$ is a standard BM. That means, the second moment of the process $\{S(k) : k = 1, 2, \ldots\}$, which is the cumulative sum process of odd increments of FBM with very small $H$, is very close to the second moment of BM. Therefore, the thinning operation on the increments of extreme subdiffusion process changes the second moment of their cumulative sums as the second moment of pure Brownian diffusion.

Now, let us consider how, in general, the thinning operation change the second moment of the process. Let us study two following sequences using formula (8)

$$M(k) := \min_{H\in(0,1)} E(S(k)^2)\,, \qquad k = 2, 3, \ldots,$$

and

$$H(k) := \arg\min_{H\in(0,1)} E\left(S(k)^2\right)\,, \qquad k = 2, 3, \ldots\,.$$

Observe that $M(1) \equiv 1$ for any $H \in (0, 1)$. These sequences are marked in Fig. 5.

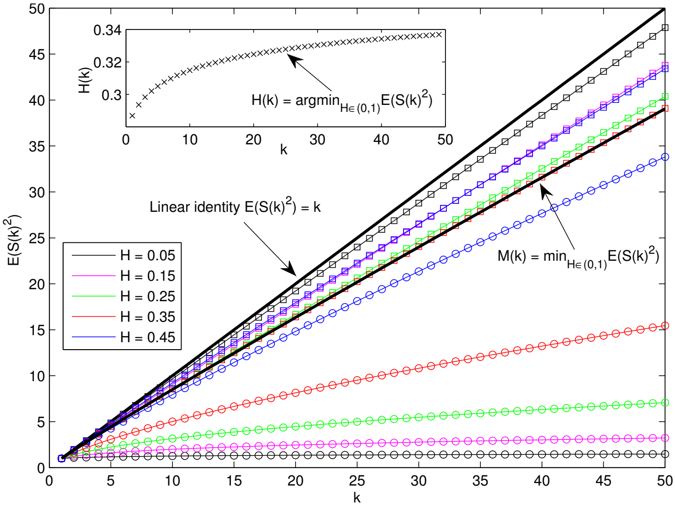


Fig. 5. The second moments $E(B_H(k)^2)$ and $E(S(k)^2)$ of FBM (lines with circles) and corresponding cumulative sum process $\{S(k) : k = 1, 2, \ldots\}$ (lines with squares) respectively. The cases with $H < 1/2$.

When $H < 1/2$, then $\rho_H(k) < 0$ for all $k$ and according to formula (8) we subtract something from linear component $k$ of $E(S(k)^2)$. In Fig. 5, we present the second moment $E(S(k)^2)$ of the process $\{S(k) : k = 1, 2, \ldots\}$ and the second moment $E(B_H(k)^2) = k^{2H}$ of the corresponding FBM. We also mark the sequence $\{M(k) : k = 2, 3, \ldots\}$ showing the minimum values (with respect to $H$) of the second moment $E(S(k)^2)$ of the thinned process. From Fig. 5 we see that the second moment $E(S(k)^2)$ of the thinned process behaves like BM or FBM with $H \nearrow 1/2$. Moreover, the smallest values of $E(S(k)^2)$ are obtained for $H > 0.25$, see Fig. 5.

Summarizing, in the subdiffusion case of FBM with Hurst exponent $H < 1/2$ the thinning operation on FGN leads to the cumulative sum process $\{S(k) : k = 1, 2, \ldots\}$ with the second moment similar to the Brownian diffusion or subdiffusion case with $H$ close to $1/2$.

When $H > 1/2$, then $\rho_H(k) > 0$ for all $k$ and according to formula (8) the second moment $E(S(k)^2)$ grows faster than linearly (Brownian diffusion case). In Fig. 6, we present the second moment $E(S(k)^2)$ of the process $\{S(k) : k = 1, 2, \ldots\}$ and the second moment $E(B_H(k)^2) = k^{2H}$ of the corresponding FBM for cases with $H > 1/2$. From Fig. 6 we see that the second moment $E(S(k)^2)$ of the thinned process is a little lower then for corresponding FBM but still greater then $1/2$. Therefore, in the superdiffusion case of FBM with Hurst exponent $H > 1/2$ the thinning operation on FGN leads to the cumulative sum process $\{S(k) : k = 1, 2, \ldots\}$ which exhibits weaker superdiffusion behavior (smaller $H$) of its second moment.

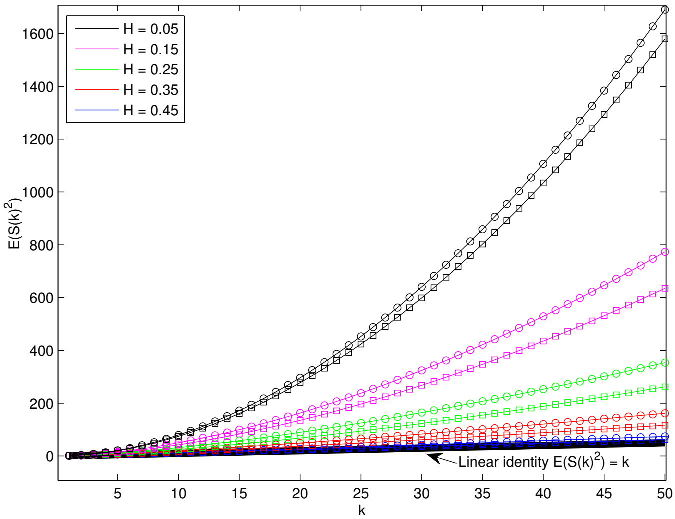


Fig. 6. The second moments $E(B_H(k)^2)$ and $E(S(k)^2)$ of FBM (lines with circles) and corresponding cumulative sum process $\{S(k) : k = 1, 2, \ldots\}$ (lines with squares) respectively. The cases with $H > 1/2$.

## 4. Sample MSD

The complementary issue is to show the difference between the estimated $H$ values for the process $\{S(k) : k = 1, 2, \ldots\}$ and the true value of $H$. For such goal we consider the sample mean square displacement (MSD), see [17]. Let $\{X(k) : k = 1, 2, \ldots, N\}$ be a sample of length $N$. Then the sample MSD is defined by

$$M_N(\tau) = \frac{1}{N-\tau} \sum_{k=1}^{N-\tau} (X(k+\tau) - X(k))^2 \, .$$

If $N$ is large and $\tau$ small and the sample comes from FBM, then

$$M_N(\tau) \overset{d}{\sim} \tau^{2d+1} \, ,$$

where $d = H - 1/2$ and $\overset{d}{\sim}$ means similarity in distribution. One can consider $\ln(M_N(\tau))$ and get the estimate of $H$ as the half of the slope value for the line $(\ln(n), \ln(M_N(\tau)))$, see [18].

We conduct the following simulation experiment. For each value of Hurst exponent $H \in \{0 + i \times 0.01 : i = 1, 2, \ldots, 99\}$ we simulate 100 trajectories of FGN (each of length $N = 2^{14}$). We thin each of FGN trajectory obtaining the trajectory of the thinned process $\{Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$. Then for the corresponding cumulative sum process $\{S(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$ for thinned process $\{Y(k) : k = 1, 2, \ldots, \lceil N/2 \rceil\}$, we compute the sample MSD and estimator of Hurst exponent for each of 100 trajectories corresponding one value of $H$. In Fig. 7 we present the boxplots of estimated $H$ values for this process. One can see that the biggest difference between the estimated $H$ values for the process $\{S(k) : k = 1, 2, \ldots\}$ and the true value of $H$ occur especially for cases with $H < 1/2$. Thus the thinning operation extremely changes the memory structure of the strong subdiffusion processes ($H \approx 0$) and as a result one gets the subdiffusion processes with $H \approx 1/2$ similar to the pure Brownian diffusion.

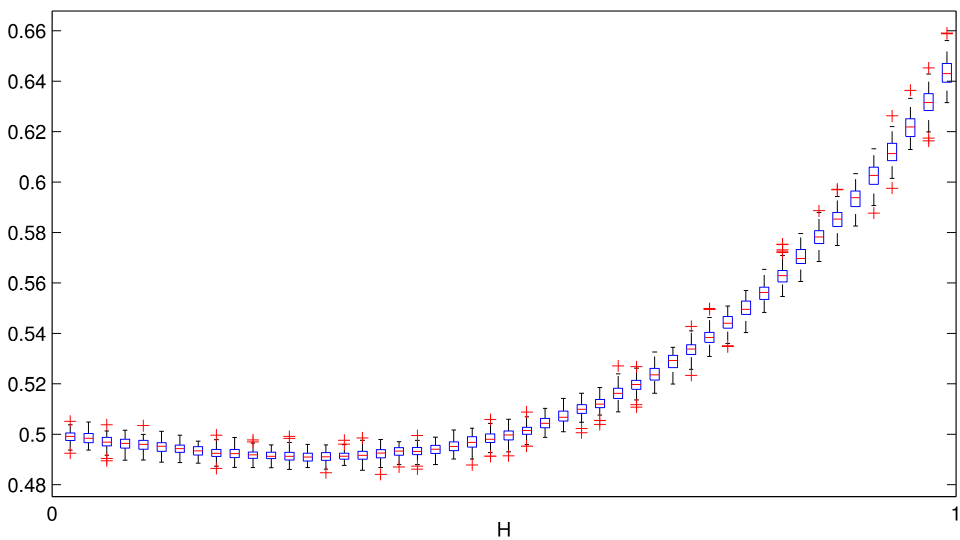


Fig. 7. The boxplots of estimated values of Hurst exponent $H$, obtained via the MSD method for cumulative sum process $\{S(k) : k = 1, 2, \ldots, \ldots, \lceil N/2 \rceil\}$. The length of corresponding FBM trajectories is $N = 2^{14}$.

## 5. Conclusions

We have considered thinning of the FGN sequence which is the increment process of FBM. Such operation leads to the thinned Gaussian process $\{Y(k)\}$ with the same asymptotic of the covariance function, but with a new asymptotic constant. By analysis of the second moment we have proved that for $H \neq 1/2$ the cumulative sum process $S(k)$ of the thinned process is a new Gaussian process with stationary increments but without $H$-self-similarity property. Therefore, it is not an FBM, except for the classical Brownian motion case with $H = 1/2$.

Numerically, we have obtained the lower bounds (with respect to index $H$) for the second moment of such new cumulative sum process. For subdiffusion cases with $H < 1/2$, the second moment behaves like for Brownian diffusion or weak subdiffusion regime. For $H > 1/2$ the second moment is a little lower than for the corresponding FBM but still greater than 1/2. Hence the significant difference between cumulative sum process of FGN and the corresponding FBM holds mainly for strong subdiffusion cases.

We believe that these results are important for experimentalists. Stationary data describing the collection of measurements of an anomalous diffusion phenomenon can lead to different statistical conclusions for different resolution of measurements. One should be very careful in statistical inference, especially in strong subdiffusion regimes when the less frequent measurements can show classical diffusion behavior.

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