HOW TO IDENTIFY THE PROPER MODEL?*

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One of the important steps towards constructing an appropriate mathematical model for the real-life data is to determine the structure of dependence. A conventional way of gaining information concerning the dependence structure (in the second-order case) of a given set of observations is estimating the autocovariance or the autocorrelation function (ACF) that can provide useful guidance in the choice of satisfactory model or family of models. As in some cases, calculations of ACF for the real-life data may turn out to be insufficient to solve the model selection problem, we propose to consider the autocorrelation function of the squared series as well. Using this approach, in this paper we investigate the dependence structure for several cases of time series models. In order to illustrate theoretical results, we calibrate one of the examined process to real data set that presents CO_2 concentration in the indoor air.

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1. Motivation

One of the important steps towards constructing an appropriate mathematical model for the real-life data is to determine the structure of dependence. A conventional way of gaining information concerning the dependence structure (in the second-order case) of a given set of observations is estimating one of the most popular measure of dependence — the autocovariance (ACVF) or the autocorrelation (ACF) functions — and this can provide useful guidance in the choice of satisfactory model or family

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of models, [1, 2, 3, 4]. This traditional method is well-established in time series textbooks, see *e.g.* [5, 6, 7], and widely used in various areas, *e.g.* physics [3, 8, 9], electrical engineering [10], in economics (finance) [11, 12], meteorology [13, 14].

However, the inspection of the correlations for the data can lead sometimes to the wrong preliminary choice of the model. For example, if we look at the sample autocorrelation functions on the top left and bottom left panels of Fig. 1 we can guess that these two sets of data were drawn from the same time series model. But we do not make this mistake if we compare the sample autocorrelations for the squared data. Indeed, as we can see from the right panels of Fig. 1, the sample ACFs (circles) for the squares behave in a quite different manner for the considered samples. Therefore, in this paper we propose to consider not only the autocovariance or the autocorrelation function but the higher order dependence, namely the dependence structure for the squared series, as well.

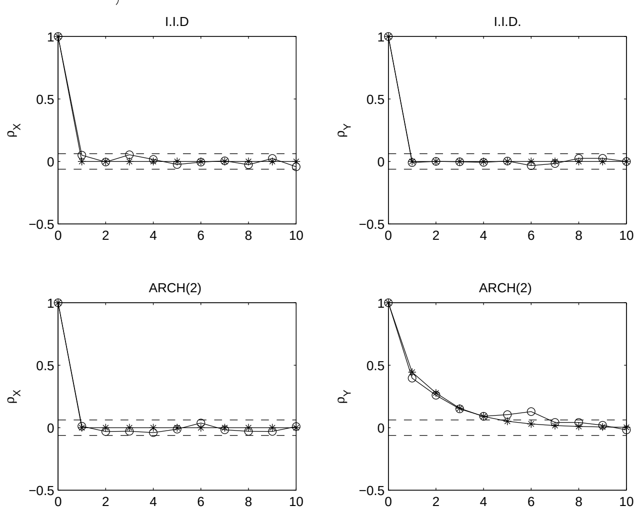

Fig. 1. The theoretical ACF (stars) and sample ACF (circles) for X_n (left panels) and $Y_n = X_n^2$ (right panels), where X_n is i.i.d. standard Gaussian noise (top panels) and X_n is ARCH(2) model with $\phi_1 = 0.4$, $\phi_2 = 0.1$ and standard Gaussian noise (bottom panels).

In Sec. 2 we recall the well-known definition of the autocovariance and the autocorrelation functions and note that it is sufficient to restrict our attention to the autocorrelation function. Then our aim is to show the differences in the behavior of the ACF for the process and its squares for assorted models. Therefore, in Sec. 3 we consider some time series models and in order to keep the results simple and illustrative, we focus on particular, low order models. We study examples of Autoregressive Moving Average (ARMA) models as they belong to the class of linear processes and they are often used for modeling empirical time series. Although linear processes are appropriate for describing many real-life phenomena, they do not capture some of the features of financial time series (*e.g.* periods of high and low volatility tend to persist in the market longer than can be explained by linear processes). Therefore, ARCH (Autoregressive Conditionally Heteroskedastic) and GARCH (Generalized ARCH) processes were developed in order to model the behavior of stock, stock index, currency, *etc.* and thus we discuss two examples of such models. In the same section, we consider also a specific SARIMA model, we provide explicit expressions for the autocorrelation functions and we show that the autocorrelation functions might prove a helpful guide in selecting a possible model for the real-life data that exhibits seasonal behavior. Section 4 contains real data analysis in the context of presented methodology. Finally, Sec. 5 gives a few concluding remarks.

2. Measures of dependence for finite-variance time series

If we want to describe the dependence structure of finite-variance time series $\{X_n\}$ we can use one of the most popular measures of dependence, *i.e.* the autocovariance function or the autocorrelation function

$$\begin{aligned}\mathrm{cov}(X_n, X_m) &= \mathbb{E}(X_n X_m) - \mathbb{E}X_n \mathbb{E}X_m\,,\\ \mathrm{corr}(X_n, Y_m) &= \frac{\mathrm{cov}(X_n, X_m)}{\sqrt{\mathbb{V}\mathrm{ar}(X_n)}\sqrt{\mathbb{V}\mathrm{ar}(X_m)}}\,.\end{aligned}$$

It is worth noting that, contrary to the case of some measures defined for infinite-variance time series (see *e.g.* [19]), both measures considered here are symmetric. And the rescaled (standardized) autocovariance gives the autocorrelation. Moreover, it is obvious that for stationary time series $\{X_n\}$ both the autocovariance and the autocorrelation functions depend only on the difference between n and m and, therefore, we can write

$$\mathrm{cov}_X(m-n) = \mathrm{cov}(X_n, X_m)\,, \qquad \mathrm{corr}_X(m-n) = \mathrm{corr}(X_n, X_m)\,.$$

Usually in model building it is convenient to use the autocorrelation function. Therefore, from now on, we will study only this measure. Of course, if we have the formula for the autocorrelation function for the stationary time series, we can get the autocovariance function by multiplication by $\mathbb{V}\mathrm{ar}(X_n)$.

We will denote by $\rho_X(k)$ the theoretical ACF for stationary time series $\{X_n\}$ and by $\hat{\rho}_X(k)$ the sample autocorrelation function (sample ACF) for n observations $x_1, x_2, \ldots, x_n$ given by

$$\hat{\rho}_X(k) = \frac{\sum_{i=1}^{n-|k|}(x_i - \bar{x})(x_{i+|k|} - \bar{x})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}\,,$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

As we will study not only the dependence structure of the time series but of the squared time series as well, let us assume that $\mathbb{E}X_n^4$ exists for every n, X_n^2 is stationary and denote $Y_n = X_n^2$. Moreover, as a very wide class of stationary processes can be generated by using i.i.d. noise (independent identically distributed random variables) as the forcing terms in a set of equations we will use the symbol $\{\xi_n\}$ to denote a series of zero-mean i.i.d. random variables with finite fourth moment. We take the notation $\sigma^2 = \mathbb{E}\xi_n^2$, $\gamma = \mathbb{E}\xi_n^4$.

3. The ACF for time series models

In this section, let us consider a few time series models. For each of them we provide the formula for the ACF for the time series together with the ACF for its squares.

3.1. Independent identically distributed variables

Let us first consider time series of zero-mean independent identically distributed random variables with finite fourth moment, *i.e.* $X_n = \xi_n$. It is clear that this series is stationary and that autocorrelation functions are given by

$$\rho_X(k) = \mathbb{I}_{k=0}\,, \qquad \rho_Y(k) = \mathbb{I}_{k=0}\,.$$

This means that not only for $\{X_n\}$ the autocorrelations are zero (except $k = 0$) but the same result we have for the squared process $\{Y_n\}$. And now we can look at top panels of Fig. 1 again because these graphs were plotted for i.i.d. standard Gaussian noise.

3.2. MA(1) model

The sequence $\{X_n\}$ is said to be MA(1) (Moving Average) process if for every n

$$X_n = \xi_n + a\xi_{n-1}\,, \tag{3.1}$$

where the innovations ξ_n are defined as in Sec. 2. MA processes found many practical applications, *i.e.* they were used to model human vocal-tract system using natural speech signals, [15]. The other applications the reader can find in [16].

It is clear that this process is stationary and we can show that

$$\rho_X(k) \;=\; \frac{\mathbb{I}_{k=0} + a^2\mathbb{I}_{k=0} + a\mathbb{I}_{k=1}}{1 + a^2}\,,$$

$$\rho_Y(k) = \begin{cases} 1 & \text{for } k = 0\,, \\ \frac{a^2\gamma - \sigma^4 a^2}{\gamma(1+a^4)+\sigma^4(4a^2-1-a^4)} & \text{for } k = 1\,, \\ 0 & \text{for } k > 1\,. \end{cases}$$

This means that for both MA(1) and squared MA(1) models the autocorrelation function is equal to zero except $k = 0$ and $k = 1$. Note also that if $a > 0$ then $\rho_X(1)$ is greater than zero while for $a < 0$, $\rho_X(1)$ falls below zero.

Let us consider an example of MA(1) model, *i.e.*, let $\{X_n\}$ be given by (3.1) with $a = 0.8$ and standard Gaussian noise, *i.e.* $\xi_n \overset{\text{iid}}{\sim} N(0,1)$. The theoretical and sample ACF for $\{X_n\}$ and $\{X_n^2\}$ are presented on the top panels of Fig. 2. It is easily seen that for $k > 1$ the theoretical ACF is equal to zero and sample ACF is placed within confidence intervals so it should be treated as equal to zero.

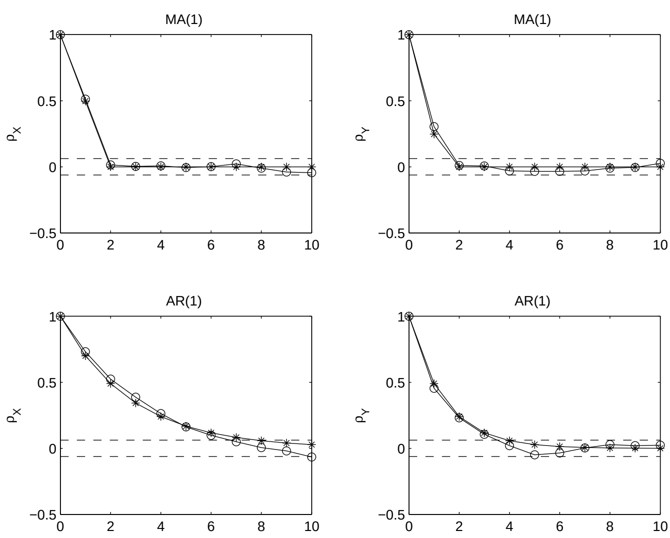

Fig. 2. The theoretical ACF (stars) and sample ACF (circles) for X_n (left panels) and $Y_n = X_n^2$ (right panels), where X_n is MA(1) model with $a = 0.8$ and standard Gaussian noise (top panels) and X_n is AR(1) model with $b = 0.7$ and standard Gaussian noise (bottom panels).

It is worth noting that one can obtain similar result for MA(q) models but then the measures are not equal to zero only for $k \leq q$. Therefore, if the sample autocorrelation functions of both data and squared data are zero except a few first lags one can try to describe the data with MA models.

3.3. AR(1) models

The sequence $\{X_n\}$ given by

$$X_n - bX_{n-1} = \xi_n \,, \tag{3.2}$$

where the innovations ξ_n are defined as in Sec. 2, is called AR(1) (Autoregressive) model. We assume that $0 < |b| < 1$ in order to get the so-called causal model and then $X_n = \sum_{j=0}^{\infty} b^j \xi_{n-j}$. For this model we have

$$\rho_X(k) = b^k \,,$$

and this means that for $b > 0$ this measure decays exponentially as k tends to infinity. However, for $b < 0$ the autocorrelation function oscillates and it is convenient to investigate its behavior by taking even and odd ns separately — the ACF tends exponentially to zero for each of these two subsequences of ns. The autocorrelation function for squared AR(1) model tends to zero even faster (although still exponentially) as it is given by

$$\rho_Y(k) = b^{2k} \,.$$

The AR-type models were used in many areas of interest. For example in [17] the authors propose to use such processes to analyze variabilities of heart rate and systolic blood pressure while in [18] the AR system describes DNA sequences.

And now let us consider an example of AR(1) model, *i.e.* let $\{X_n\}$ be given by (3.2) with $b = 0.7$ and $\xi_n \overset{\text{iid}}{\sim} N(0,1)$. According to our results, the theoretical and sample ACFs for $\{X_n\}$ and $\{X_n^2\}$ tend to zero quite quickly, see the bottom left and the bottom right panel of Fig. 2. Moreover, as AR(1) can be treated as MA(∞) model, one can use the results obtained for AR(1) model to imagine the behavior of ACF for some infinite order moving average processes.

So when we deal with the real-life data and the sample ACF tends to zero rapidly it could suggest the use of ARMA model as for this type of models the autocorrelation function is bounded by functions exponentially tending to zero.

3.4. ARCH(2) models

Although ARMA processes are useful in modeling real-life data of various kinds, they belong to the class of linear processes and they do not capture the structure of financial data. Therefore, the class of ARCH processes was introduced in [20] to allow the conditional variance of a time series process to depend on past information. ARCH models have become popular in the past few years as they provide a good description of financial time series, [21, 22, 23, 24]. They are nonlinear stochastic processes, their distributions are heavy-tailed with time-dependent conditional variance and they model clustering of volatility.

Let us consider the ARCH(2) process with Gaussian innovations defined by the equations

$$X_n = \sqrt{h_n}\xi_n\,, \qquad h_n = \theta + \psi_1 X_{n-1}^2 + \psi_2 X_{n-2}^2\,, \tag{3.3}$$

where $\xi_n \overset{\text{iid}}{\sim} N(0,1)$ and $\theta, \psi_2 > 0$, $\psi_1 \geq 0$. For such model the conditional variance h_n of X_n depends on the past through the two most recent values of X_n^2. We assume that $\psi_2 + 3\psi_1^2 + 3\psi_2^2 + 3\psi_1^2\psi_2 - 3\psi_2^3 < 1$, as this condition guarantees the existence of the fourth moment for the process $\{X_t\}$, see [25]. As it was shown in [25] the following formula holds

$$\begin{aligned}
\rho_X(k) &= \mathbb{I}_{k=0}\,,\\
\rho_Y(k) &= \begin{cases} 1 & \text{for } k=0\,,\\[2ex] \frac{\psi_1}{1-\psi_2} & \text{for } k=1\,,\\[2ex] \frac{\psi_2+\psi_1^2-\psi_2^2}{1-\psi_2} & \text{for } k=2\,,\\[2ex] \psi_1\rho_Y(k-1)+\psi_2\rho_Y(k-2) & \text{for } k>2\,. \end{cases}
\end{aligned}$$

So the most important fact is that for ARCH(2) model the ACF is zero (only $\rho_X(0) = 1$) and this means that the X_ns are uncorrelated (but not independent). Therefore, if we look only at the sample ACF of the data generated from ARCH(2) model, see *e.g.* the bottom left panel of Fig. 1, we can come to the wrong conclusion that we deal with i.i.d. noise, *cf.* the top left panel of Fig. 1. Fortunately, if we check the behavior of ACF for squared time series we notice the significant difference between the ARCH(2) process and i.i.d. variables, compare left panels of Fig. 1.

The same results are also true for the general ARCH(q) models, *i.e.* the X_ns are uncorrelated while the ACF is not equal to zero for the squared time series.

3.5. GARCH(1,1) models

In empirical applications of the ARCH model a relatively long lag in the conditional variance equation is often called for. In order to avoid problems with negative variance parameter estimates, a more general class of processes, GARCH process was introduced in [26], allowing for a much more flexible lag structure. These kind of models seem to describe stock, stock index and currency returns very well, so they are now widely used to model financial time series (see [23, 24, 27] and references therein). The Gaussian GARCH(1,1) model is defined by the equations

$$X_n = \sqrt{h_n}\xi_n\,, \qquad h_n = \theta + \phi h_{n-1} + \psi X_{n-1}^2\,, \tag{3.4}$$

where $\xi_n \overset{\text{iid}}{\sim} N(0,1)$ and $\theta, \phi, \psi > 0$. We assume that $3\psi^2 + 2\psi\phi + \phi^2 < 1$, as this condition guarantees the existence of the fourth moment for the process $\{X_n\}$, see [25]. For the considered model we have

$$\begin{aligned} \rho_X(k) &= \mathbb{I}_{k=0}\,, \\ \rho_Y(k) &= \begin{cases} 1 & \text{for } k = 0\,, \\ \frac{\psi(1-\psi\phi-\phi^2)}{1-2\psi\phi-\phi^2} & \text{for } k = 1\,, \\ (\psi+\phi)^{k-1}\rho_Y(1) & \text{for } k > 1\,, \end{cases} \end{aligned}$$

see [25]. As in the case of ARCH model, for GARCH(1,1) model the ACF is zero and this means that the X_ns are uncorrelated (but not independent).

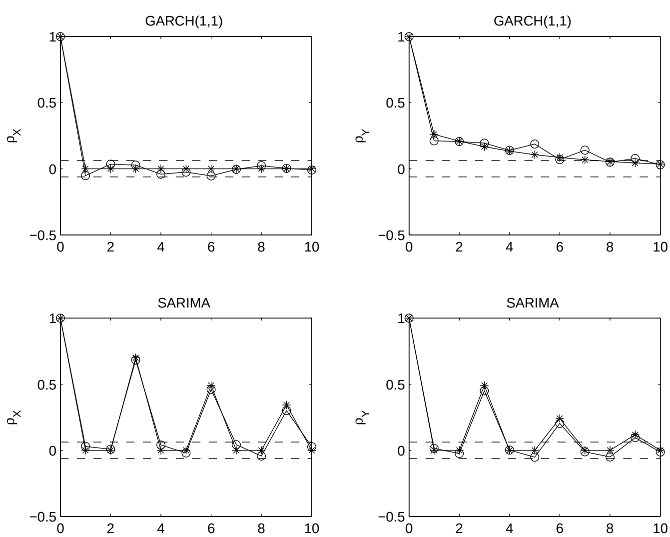

Fig. 3. The theoretical ACF (stars) and sample ACF (circles) for X_n (left panels) and $Y_n = X_n^2$ (right panels), where X_n is GARCH(1,1) model with $\phi = 0.6, \psi = 0.2$ and standard Gaussian noise (top panels) and X_n is SARIMA(0,0,0)×(1,0,0) with period $s = 3$ and $\phi = 0.7$ and standard Gaussian noise (bottom panels).

The ACF for $\{X_n^2\}$ is not equal to zero and it tends to zero exponentially. We can observe this behavior on the top panels of Fig. 3, where the example of GARCH(1,1) is given. The ACF for the time series is zero while the same function for squared time series is greater than zero but tends to zero quite quickly.

For the general GARCH(p,q) models we get similar results, *i.e.* the X_ns are uncorrelated while the ACF is not equal to zero for the squared time series. Therefore, if we analyze the data and the sample ACF is zero for all lags, we should look at the ACF of the squared data in order to get more information.

3.6. SARIMA models

Many real-life phenomena exhibit seasonal behavior. The traditional time series method to deal with them is the classical decomposition where the model of the observed series incorporates trend, seasonality and random noise. However, in modeling empirical data it might not be reasonable to assume, as in the classical decomposition, that the seasonal component repeats itself precisely in the same way cycle after cycle. SARIMA (Seasonal Autoregressive Integrated Moving Average) models allow for randomness in the seasonal pattern from one cycle to the next, see [5]. A SARIMA$(p,d,q)\times(P,D,Q)$ model with period s it is a process $\{X_n\}$ that satisfies the equation

$$\phi(B)\varPhi(B^s)(1-B)^d(1-B^s)^D X_n = \theta(B)\varTheta(B^s)\xi_n\,,$$

where $\phi(z) = 1-\phi_1 z-\ldots-\phi_p z^p$, $\varPhi(z) = 1-\varPhi_1 z-\ldots-\phi_P z^P$, $\theta(z) = 1+\theta_1 z-\ldots-\theta_q z^q$, $\varTheta(z) = 1+\varTheta_1 z-\ldots-\varTheta_Q z^Q$ and B is a backward shift operator.

SARIMA processes were used to model real time series that exhibit seasonality, such as energy demand and prices [28, 29], meteorological and malaria variables [30] and data related to water quality [31].

Let us consider SARIMA$(0,0,0)\times(1,0,0)$ system with period $s=3$, standard normal innovations and let us assume that $0<\phi<1$. In this case, the equation for X_n takes the following form

$$X_n - \phi X_{n-3} = \xi_n\,. \tag{3.5}$$

The solution of this equation is $X_n = \sum_{j=0}^{\infty}\phi^j\xi_{n-3j}$ and this implies that the unique solution of equation (3.5) is stationary. Moreover, $\mathbb{E}X_n = 0$ and

$$\rho_X(k) = \begin{cases} \phi^{k/3} & \text{when } \; k \mod 3 = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For the process $Y_n = X_n^2$ we determined the explicit formula for the autocorrelation function

$$\rho_Y(k) = \begin{cases} \phi^{2k/3} & \text{when } \ k \mod 3 = 0\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The considered model is called 'seasonal' with period $s = 3$ and the ACF function reflects its character. The autocorrelation functions for both X_n and Y_n oscillate and the period is also 3 — the behavior of the theoretical and sample ACF for the considered model with $\phi = 0.7$ can be observed on the bottom panels of Fig. 3.

4. Applications

In order to illustrate the theoretical results presented above in this section, we analyze real data set that describes CO_2 concentration in the air in open space of huge company. The time series presents 2-minutes observations from working hours from Tuesday (12.04.2011, 5:36–20:26) and Friday (15.04.2011, 5:36–20:00) because data reported on those days exhibit completely different behavior than this reported on Monday, Wednesday and Thursday. Before further analysis, we remove deterministic trend that in this case is a polynomial of order 2. In Fig. 4 we present the examined (after removing the trend) (top panel) and squared (bottom panel) time series.

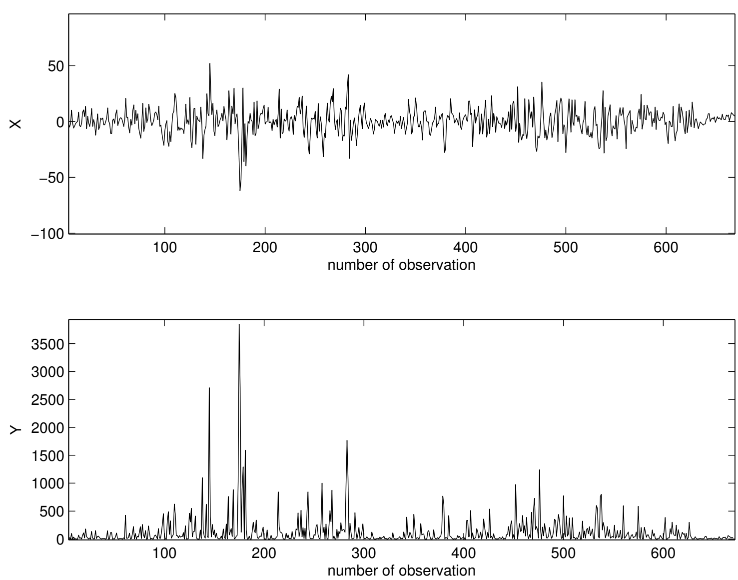

Fig. 4. The real data set after removing deterministic trend (top panel) and squared time series (bottom panel).

Because inspection of the correlation for real set of observations can lead sometimes to the wrong preliminary choice of the model, therefore, in the first step of our analysis we examine the autocorrelation functions of the time series as well as the squared data. As we observe in Fig. 5, the sample ACFs tend to zero rapidly, therefore, we can suggest the use of ARMA-type model. For simplicity, we take under consideration only AR(p) processes.

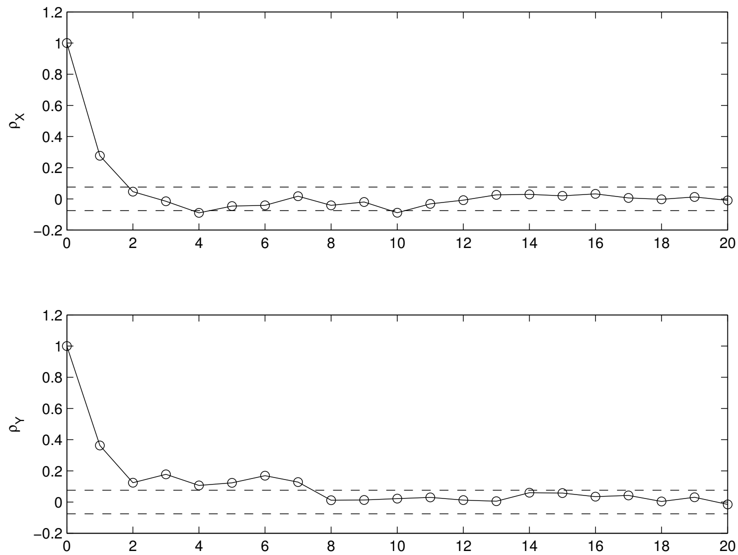

Fig. 5. The sample ACF for X_n — real data set (top panel) and $Y_n = X_n^2$ (bottom panel).

Next, by using least squares method we estimate the parameters of AR(p) models. On the basis of final prediction error (FPE) [32] we select the best p parameter that in this case is equal to 1. The estimated b parameter of AR(1) model given in (3.2) is equal to 0.2771. We also examine the residuals. The variance ratio test applied to cumulative sums of residuals suggest that this series is a random walk, that means the residuals constitute i.i.d. sample. Moreover, we test also the distribution of the residuals by using procedure presented in [33] based on the cumulative distribution function and recognize that residuals come from the NIG distribution, *i.e.* distribution with the following density function, [34, 35]

$$f_{\mathrm{NIG}}(x) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}\,, \qquad x \in R\,, \quad (4.6)$$

where K_1 is a modified Bessel function of the third kind and $\delta > 0$, $0 \leq |\beta| < \alpha$, $\mu \in R$. By using the maximum likelihood method [35], we obtain the following estimates of the parameters α, β, δ and μ

$$\alpha = 0.08\,, \qquad \beta = -0.0030\,, \qquad \delta = 9.7572\,, \qquad \mu = 0.5193\,.$$

5. Conclusions

In this paper, we recall that in the second-order case commonly applied method of gaining information concerning the dependence structure of a given set of observations by estimating the autocorrelation (or autocovariance) function can provide useful guidance in the choice of satisfactory model or family of models. However, the results reported above, clearly suggest that considering the autocorrelation function for the squares might be helpful in distinguishing between different types of time series models and prove even a better guide in selecting a set of possible model candidates when modeling real-life data. This method is worth applying as it is very simple. Indeed, in order to get the sample ACF for the squares one does not need to prepare new computer programs — it is sufficient to restrict to any well-know routine for ACF and use it for squared observations.

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