

MODELING OF SHORT TERM INTEREST RATE BASED ON TEMPERED FRACTIONAL LANGEVIN EQUATION*

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We develop a new class of continuous-time models based on the solutions of tempered fractional Langevin equations for Ornstein–Uhlenbeck process driven by Lévy noise. We present methods of simulation of sample paths of such processes. We show how to use such models in modeling short term interest rate. We develop tempered Vasiček interest rate model by finding explicit solutions of tempered fractional Langevin equations.

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1. Introduction

The concept of the interest rate r is strictly connected with our everyday life and it tells us how the present value of money is related to its value in future. Considering the interest-rate products one often has to drop the assumption of deterministic behavior of interest rates and start to deal with them in a stochastic setup. The probabilistic nature of interest rates complicates valuing even the simplest cash flow streams. Moreover, the need for pricing interest rate products is one of main goals for actuaries and financial analysts. Therefore, we are still looking for models that describe reality in more accurate way. In our research, we are often led to the field of statistical physics and dynamics of complex physical systems. The methods used there are very useful in the analysis of various economic processes.

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Present paper uses physical models and some new methods from fractional calculus to develop a new class of continuous-time models which can be helpful in modeling short term interest rate.

One of the most pronounced examples of processes that has been of fundamental importance both in mathematics and physics is Ornstein–Uhlenbeck (O–U) process

$$dr(t) = (a - br(t))dt + \sigma dW(t), \quad r(0) = r_0, \quad (1)$$

where $a, b, \sigma > 0$ and r_0 are constant coefficients. a/b is the long term mean level, b is spread of reversion to the mean and σ is standard deviation. $W(t)$ is a Brownian motion. Eq. (1) can be also written in term of Langevin equation

$$\frac{dr(t)}{dt} = (a - br(t)) + \sigma \xi(t), \quad (2)$$

where $\xi(t)$ is the Brownian noise *i.e.* the derivative of Brownian motion in the distributional sense.

O–U was introduced originally by Uhlenbeck and Ornstein [1] as a model for the velocity process in the Brownian diffusion. O–U process provides also a stationary solution for the velocity in classical Klein–Kramers dynamics [2, 3]. In finance, Ornstein–Uhlenbeck process is known as a Vasiček short term interest rate model [4]. One can name dozens of examples of applications of this model to financial data of interest rates, currency exchange rates or commodity prices ([5] and references therein). The characteristic feature of this model is that it exhibits the mean-reversion, namely the process in the long-time period is pulled to the mean level. Such mean-reversion is in accordance with economic phenomenon that interest rates in the long time period stays the same constant average value.

Popularity of Ornstein–Uhlenbeck process resulted in its many generalizations. These generalizations are motivated by the facts that the assumption of normality for many observed data is not satisfied. Empirical observations confirm heavy-tailed or leptokurtic distribution of price changes see *e.g.* [6] and references therein. Thus, Mandelbrot [7] and Fama [8] proposed the α -stable distribution instead of Gaussian law to describe asset returns. Stable laws have found applications in many fields, one can name here finance [9], physics [10] and electrical engineering [11]. The O–U process with α -stable noise was analyzed in [12, 13] as a model for financial data description. Apart from α -stable noise, one can also use fractional Brownian dynamics [14], or recently introduced tempered α -stable distributions [15] in order to capture anomalous character of financial dynamics.

In this paper, we introduce fractional generalizations of O–U process. Namely, in Eq. (2) we replace the ordinary derivative with the general tempered Riemann–Liouville derivative [16]. Moreover, we use general Lévy noise input

$$\zeta(t) = \frac{dL(t)}{dt},$$

instead of Brownian one. A real valued stochastic process $L(t), t \geq 0$ is called Lévy process if it is continuous in probability, has independent and stationary increments and $L(0) = 0$, see [17]. Examples of such process are α -stable, tempered α -stable, Linnik, Gaussian, generalized inverse Gaussian. The similar approach with Riemann–Liouville derivative and Brownian white noise is presented in [18]. The class of fractional differential equations driven by Lévy noise is presented in [19, 20, 21, 22].

This article is structured as follows. In Sec. 2 we present general definitions and properties of tempered fractional derivatives. Section 3 is devoted to solutions of tempered fractional equations with Lévy noise. In Sec. 4 we apply earlier considerations and develop general tempered Vasiček model for short term interest rate. We discuss basic properties of this model. Section 5 concludes the paper.

2. Tempered fractional derivatives

Fractional derivatives are usually connected in physics with stable–Lévy processes and the subordination techniques [23]. Recently Rosiński [24] and Cartea, del-Castillo-Negrete [25] introduced independently a new class of tempered stable processes. Tempered α -stable processes possess very important feature, namely, they have finite moments of all orders but, at the same time, they resemble stable laws in many aspects (see [24] for details). We can name many fields of applications of such processes, finance [26, 27], biology [28], physics in the description of anomalous diffusion (especially when one observes the transition from the initial subdiffusive character of motion in short times to standard diffusion in long times) [29, 30, 31]. Since tempered α -stable processes are extension of α -stable ones, the tempered derivatives are a natural extension of fractional ones.

Before considering general tempered fractional derivatives, let us recall definitions of Riemann–Liouville fractional derivatives and integrals [32]. For integrable function f and $\alpha > 0$ the fractional Riemann–Liouville integral is defined as

$${}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(t') (t - t')^{\alpha-1} dt', \quad (3)$$

where $t \in (0, T)$. For suitable good functions $f(t)$ and $0 < \alpha < 1$, the Riemann–Liouville fractional derivative is defined as [32]

$${}_0D_t^\alpha f(t) = \left(\frac{\partial}{\partial t} \right) {}_0I_t^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{f(t')}{(t-t')^\alpha} dt'. \quad (4)$$

One can observe that for $\alpha = 1$ we obtain classical ordinary derivative *i.e.* ${}_0D_t^1 f(t) = f'(t)$. Following [16] the fractional tempered derivative of the Riemann–Liouville type is defined as

$${}_0D_t^{\alpha, \lambda} f(t) = e^{-\lambda t} {}_0D_t^\alpha e^{\lambda t} f(t) - \lambda^\alpha f(t), \quad (5)$$

where $\lambda > 0$ is a truncation parameter. For $\lambda = 0$ one obtains fractional derivative and for $\alpha = 1, \lambda = 0$ we recover classical derivative.

Dealing with tempered fractional derivatives is strictly connected with various properties of Laplace transform. Let us recall that the Laplace transform of a function $f(t)$ is defined as

$$\mathcal{L}(f(t)) = \tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt.$$

An extensively useful property in fractional calculus of fractional derivatives is its convolution property. The convolution of two functions $f(t)$ and $h(t)$ is defined as

$$(f * h)(t) = \int_0^t f(t-s) h(s) ds.$$

The Laplace transform of convolution of two functions yields

$$\mathcal{L}(f * h) = \mathcal{L}(f) \mathcal{L}(h).$$

The Laplace transform of Eq. (4) yields [32]

$$\mathcal{L}\left({}_0D_t^\alpha f(t)\right) = s^\alpha \tilde{f}(s) - {}_0I_{0+}^{1-\alpha} f(0_+). \quad (6)$$

Thus, assuming that $f(0) = 0$, the Laplace transform of the Eq. (5) is given by the formula

$$\mathcal{L}\left({}_0D_t^{\alpha, \lambda} f(t)\right) = (s + \lambda)^\alpha \tilde{f}(s) - \lambda^\alpha \tilde{f}(s). \quad (7)$$

3. Tempered fractional Langevin equations

In this section, we present some generalizations of the Langevin equation for O–U process defined in Eq. (2). For simplicity we assume that $a = 0$. As will be shown later, the solutions of tempered fractional Langevin equations are of the form of stochastic integrals with Lévy integrator. Our investigations generalize the previous results [18, 19] since we apply tempered fractional derivatives. Fractional Langevin equation with fractional Riemann–Liouville derivative was considered in [33] to study the anomalous diffusion of a free particle coupled to a fractal heat bath. Generalized Langevin equation with the fractional derivative and nonlocal dissipative force was analyzed in [34]. Such equations serve as extensions of classical Langevin equations and can be used in modeling various complex physical systems. For example, using fractional derivatives one can model subdiffusion $0 < \alpha < 1$ or superdiffusion $1 < \alpha < 2$ [35]. On the other hand, tempered fractional derivatives can be useful in modeling of intermediate situations between normal and anomalous diffusion (see [30, 31] and references therein).

Let us formulate the first generalization of Langevin equation of the form

$${}_0D_t^{\alpha, \lambda} r(t) + br(t) = \sigma \zeta(t), \quad 1 > \alpha > 0, b > 0, \sigma > 0, \quad (8)$$

where the standard derivative is replaced by the tempered one.

Theorem 3.1 *The solution of Eq. (8) with the initial condition $r(0) = 0$ has the following form*

$$r(t) = \int_0^t \sigma e^{-\lambda(t-s)} (t-s)^{\alpha-1} E_{\alpha, \alpha}(-(b-\lambda^\alpha)(t-s)^\alpha) dL(s). \quad (9)$$

Proof.

$$\begin{aligned}
 {}_0D_t^{\alpha,\lambda}r(t) &= \mathcal{L}^{-1}\left((s+\lambda)^\alpha\mathcal{L}(r(t))-\lambda^\alpha\mathcal{L}(r(t))\right) \\
 &= \mathcal{L}^{-1}\left(((s+\lambda)^\alpha-\lambda^\alpha)\mathcal{L}(\sigma e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-(b-\lambda^\alpha)t^\alpha))\mathcal{L}(\zeta(t))\right) \\
 &= \mathcal{L}^{-1}\left(\sigma\left(1-\frac{b}{(s+\lambda)^\alpha+(b-\lambda^\alpha)}\right)\widetilde{\zeta(s)}\right) \\
 &= \sigma\zeta(t)-\mathcal{L}^{-1}\left(\frac{\sigma b\widetilde{\zeta(s)}}{(s+\lambda^\alpha)+(b-\lambda^\alpha)}\right) \\
 &= \sigma\zeta(t)-\mathcal{L}^{-1}\left(\mathcal{L}(b\sigma e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-(b-\lambda^\alpha)t^\alpha))\mathcal{L}(\zeta(t))\right) \\
 &= \sigma\zeta(t)-b\sigma e^{-\lambda t}t^{\alpha-1}E_{\alpha,\alpha}(-(b-\lambda^\alpha)t^\alpha)*\zeta(t) \\
 &= \sigma\zeta(t)-br(t).
 \end{aligned}$$

To show that the solution satisfies the initial condition one can use its explicit representation Eq. (9). This completes the *proof*.

Another generalization of Langevin equation for O–U process has two tempered fractional derivatives

$$A {}_0D_t^{\alpha,\lambda}r(t) + B {}_0D_t^{\beta,\lambda}r(t) + br(t) = \sigma\zeta(t), \quad A, B \neq 0, b > 0, 1 > \alpha > \beta > 0. \quad (10)$$

Theorem 3.2 *The solution of Eq. (10) with the initial condition $r(0) = 0$ has the following form*

$$\begin{aligned}
 r(t) &= \sum_{k=0}^{\infty} \int_0^t \sigma e^{-\lambda(t-s)} \frac{1}{A} \frac{(-1)^k}{k!} \left(\frac{b - A\lambda^\alpha - B\lambda^\beta}{A} \right)^k \\
 &\quad \times \left((t-s)^{\alpha(k+1)-1} E_{\alpha-\beta, \alpha+\beta k}^{(k)} \left(-\frac{B}{A}(t-s)^{\alpha-\beta} \right) \right) dL(s). \quad (11)
 \end{aligned}$$

Proof. Let us first observe that the Laplace transform of the Eq. (10) yields

$$A(s+\lambda)^\alpha\widetilde{r}(s) + B(s+\lambda)^\beta\widetilde{r}(s) - A\lambda^\alpha\widetilde{r}(s) - B\lambda^\beta\widetilde{r}(s) + b\widetilde{r}(s) = \sigma\widetilde{\zeta}(s)$$

thus

$$\widetilde{r}(s) = \frac{\sigma\widetilde{\zeta}(s)}{A(s+\lambda)^\alpha + B(s+\lambda)^\beta - A\lambda^\alpha - B\lambda^\beta + b}.$$

We have that

$$\begin{aligned} & \frac{1}{A(s + \lambda)^\alpha + B(s + \lambda)^\beta - A\lambda^\alpha - B\lambda^\beta + b} \\ &= \sum_{k=0}^{\infty} \frac{(b - A\lambda^\alpha - B\lambda^\beta)^k}{A^{k+1}} (-1)^k \frac{(s + \lambda)^{-\beta(k+1)}}{[(s + \lambda)^{\alpha-\beta} + \frac{B}{A}]^{k+1}}. \end{aligned}$$

Then we have that the Laplace transform of r equals

$$\tilde{r}(s) = \sum_{k=0}^{\infty} \frac{(b - A\lambda^\alpha - B\lambda^\beta)^k}{A^{k+1}} (-1)^k \frac{(s + \lambda)^{-\beta(k+1)}}{[(s + \lambda)^{\alpha-\beta} + \frac{B}{A}]^{k+1}} \sigma \tilde{\zeta}(s).$$

Inverting the above and using the fact, see [32],

$$\mathcal{L} \left(t^{\alpha n + \beta - 1} \left(\frac{\partial}{\partial u} \right)^n E_{\alpha, \beta}(ut^\alpha) \right) = \frac{n! s^{\alpha - \beta}}{(s^\alpha - u)^{n+1}}$$

we obtain desired solution. To show that the solution satisfies the initial condition one can use its explicit representation Eq. (11). One can easily extend Theorem 3.2 to arbitrary number of tempered fractional derivatives.

We will need one more generalization of Vasiček model, namely

$$A \frac{dr(t)}{dt} + B {}_0 D_t^{\alpha, \lambda} r(t) + br(t) = \sigma \zeta(t), \quad A \neq 0, B, b \geq 0, 1 > \alpha > \beta > 0. \quad (12)$$

Theorem 3.3 *The solution of Eq. (12) with the initial condition $r(0) = 0$ has the following form*

$$\begin{aligned} r(t) &= \sum_{k=0}^{\infty} \int_0^t \sigma e^{-\lambda(t-s)} \frac{1}{A} \frac{(-1)^k}{k!} \left(\frac{b - B\lambda^\alpha - A\lambda}{A} \right)^k \\ &\quad \times \left((t-s)^k E_{1-\alpha, 1+\alpha k}^{(k)} \left(-\frac{B}{A} (t-s)^{1-\alpha} \right) \right) dL(s). \end{aligned} \quad (13)$$

Proof of this theorem is similar to Theorem 3.2.

4. Application to finance: tempered Vasiček model for short interest rate

The classical Vasiček model [4] was one of the first diffusion-based interest rate models proposed in the literature [5]. It assumes that the interest

rate evolves as an Ornstein–Uhlenbeck process Eq. (1) with constant coefficients under the risk-neutral measure. Direct integration of Eq. (1) gives us the exact formula for $r(t)$

$$r(t) = r_0 \exp(-bt) + \frac{a}{b}(1 - \exp(-bt)) + \sigma \int_0^t \exp(-b(t-u)) dW(u).$$

As one can easily infer we have

$$\begin{aligned} E(r(t)) &= r_0 \exp(-bt) + \frac{a}{b}(1 - \exp(-bt)), \\ \text{Var}(r(t)) &= \frac{\sigma^2}{2b}(1 - \exp(-2bt)). \end{aligned}$$

The main consequence of the above is that the short rate r is mean reverting, since the expected value tends to a/b when $t \rightarrow \infty$.

The basic interest rate contract is a T -maturity zero-coupon bond, see [5]. It guarantees its holder the payment of one unit currency at time T . We will denote the value of the bond at time $t < T$ by $B(t, T)$. Note that $B(T, T) = 1$. One can observe that the price of a zero-coupon bond is strictly dependent on maturity T . According to yield-to-maturity hypothesis [36] the value of a zero coupon bond is

$$B(t, T) = \exp \left\{ - \int_t^T E(r(s) | \mathcal{F}_t) ds \right\}, \quad \forall t \in [0, T]. \quad (14)$$

Expectation in Eq. (14) is conditioned on some σ -field $\mathcal{F}_t$, which represents the knowledge of an investor prior to time t [36, 37]. For the yield to maturity $Y(t, T)$ and forward interest rate $f(t, T)$ (see [36]) we have the following formulas

$$Y(t, T) = -\frac{1}{T-t} \ln B(t, T) = \frac{1}{T-t} \int_t^T E(r(s) | \mathcal{F}_t) ds, \quad (15)$$

$$f(t, T) = -\frac{\partial \ln B(t, T)}{\partial T} = E(r_T | \mathcal{F}_t), \quad \forall t \in [0, T]. \quad (16)$$

$Y(t, T)$ is a rate of return on a bond if it was held until the maturity and $f(t, T)$ is an interest rate specified now for a loan that will take place at a specified future date. In what follows, we consider the following tempered fractional generalization of Vasiček model

$$A \frac{dr(t)}{dt} + B {}_0 D_t^{\alpha, \lambda} r(t) = a + \sigma \zeta(t), \quad A, B \neq 0, a, \sigma > 0, 1 > \alpha > 0. \quad (17)$$

In special case, namely $\alpha = 0, \lambda = 0$ we obtain classical O-U process (1). Due to lineality, the general solution of Eq. (17) is the sum of two components $r(t) = r_D(t) + r_S(t)$, namely the solution of deterministic part

$$A \frac{dr_D(t)}{dt} + B {}_0D_t^{\alpha, \lambda} r_D(t) = a, \quad A, B \neq 0, a > 0, 1 > \alpha > 0 \quad (18)$$

and the solution of the stochastic part

$$A \frac{dr_S(t)}{dt} + B {}_0D_t^{\alpha, \lambda} r_S(t) = \sigma \zeta(t), \quad A, B \neq 0, \sigma > 0, 1 > \alpha > 0. \quad (19)$$

Let us first calculate the general solution of the deterministic tempered fractional differential equation Eq. (18). To do this we formulate the following theorem.

Theorem 4.1 *The general solution of the Eq. (18) with the assumption $A = -B\lambda^{\alpha-1}$, has the following form*

$$r_D(t) = \frac{c}{A} e^{-\lambda t} E_{1-\alpha, 1} \left(-t^{1-\alpha} \frac{B}{A} \right) + \int_0^t \frac{1}{A} e^{-\lambda(t-x)} E_{1-\alpha, 1} \left(-\frac{B}{A} (t-x)^{1-\alpha} \right) a \, dx, \quad (20)$$

where $c \in \mathbb{R}$ is some constant.

Proof. Let us first observe that the solution of Eq. (18) is a sum of solutions of two separate equations. Namely the solution of the equation

$$A \frac{dr(t)}{dt} + B {}_0D_t^{\alpha, \lambda} r(t) = 0 \quad (21)$$

and

$$A \frac{dr(t)}{dt} + B {}_0D_t^{\alpha, \lambda} r(t) = a. \quad (22)$$

Using results of Theorem 3.3 one can easily find that deterministic part has the form as in Eq. (20). This completes the *proof*.

The solution of the stochastic part can be obtained via result presented in Theorem 3.3, thus the general solution of Eq. (17) has the form

$$r(t) = r_D(t) + \int_0^t \frac{\sigma}{A} e^{-\lambda(t-s)} E_{1-\alpha, 1} \left(-(t-s)^{1-\alpha} \frac{B}{A} \right) dL(s). \quad (23)$$

In Fig. 1 we present sample realization of $r_S(t)$, which was generated by the method presented in Appendix A.

Now using Eq. (23) we can formulate the following theorem, which gives formulas for yield to maturity, forward rates and bond prices in tempered fractional Vasiček model.

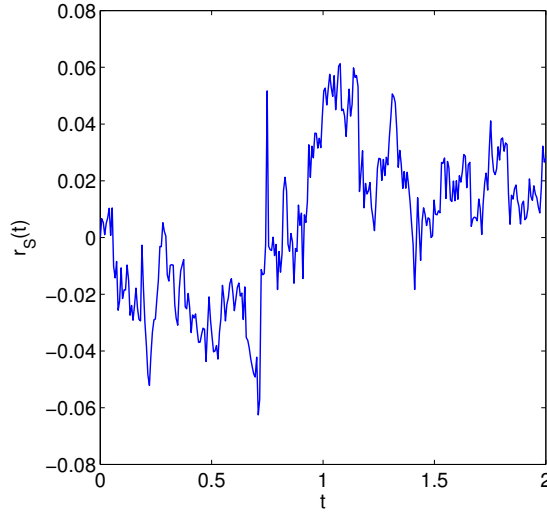


Fig. 1. Sample realization of $r_S(t)$ with 1.8-stable Lévy noise. Parameters are: $\alpha = 0.8$, $\sigma = 0.6$, $A, B = 1$.

Theorem 4.2 *In tempered fractional Vasicek model given in Eq. (17) with additional assumption $A = -B\lambda^{\alpha-1}$, yield-to-maturity forward rate, and bond price are given by the following formulas*

$$\begin{aligned}
 Y(t, T) &= \frac{1}{T-t} \left(\int_t^T r_D(s) ds + \int_0^T \phi(u-t, t) du \right. \\
 &\quad \left. + \int_t^T \int_0^{u-t} K'(u-t-s) \phi(s, t) ds du \right), \\
 f(t, T) &= r_D(T) + \phi(T-t, t) + \int_0^{T-t} K'(T-t-s) \phi(s, t) ds, \\
 B(t, T) &= e^{-\int_t^T r_D(s) ds - \int_0^T \phi(u-t, t) du - \int_t^T \int_0^{u-t} K'(u-t-s) \phi(s, t) ds du},
 \end{aligned}$$

where $K(t)$ is equal

$$K(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{B\lambda^{\alpha}}{A} \right)^k t^k E_{1-\alpha, 1+\alpha k}^{(k)} \left(-\frac{B}{A} t^{1-\alpha} \right)$$

and

$$\phi(u, t) = r_S(t) + \frac{B}{A} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-s)^{-\alpha} - (t+u-s)^{-\alpha}) r_S(s) ds \right).$$

Proof of this theorem is based on calculation of the following conditional expectation for the stochastic part of Eq. (17).

$$\psi(u, t) = E(r_S(t+u) | \mathcal{F}_t). \quad (24)$$

Recalling that $\zeta(t)$ is the derivative of Lévy process $L(t)$, one can observe that the solution of the stochastic part of Eq. (19) is also the solution of the following integral equation

$$r_S(t) = \frac{\sigma}{A} L(t) - \frac{B}{A} ({}_0I_t^{1-\alpha} r_S(t) - {}_0I_t^1 \lambda^\alpha r_S(t)).$$

Then we have

$$\begin{aligned} \psi(u, t) &= E \left(\frac{\sigma}{A} L(t+u) - \frac{B}{A} ({}_0I_{t+u}^{1-\alpha} r_S(t+u) - {}_0I_{t+u}^1 \lambda^\alpha r_S(t)) \mid \mathcal{F}_t \right) \\ &= E \left(\frac{\sigma}{A} L(t+u) - \frac{B}{A} ({}_0I_{t+u}^{1-\alpha} r_S(t+u) - {}_0I_{t+u}^1 \lambda^\alpha r_S(t)) + r_S(t) - r_S(t) \mid \mathcal{F}_t \right) \\ &= r_S(t) + \frac{B}{A} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^t ((t-s)^{-\alpha} - (t+u-s)^{-\alpha}) r_S(s) ds \right) \\ &\quad - \frac{B}{A} \left(\frac{1}{\Gamma(1-\alpha)} \int_t^{t+u} ((t+u-s)^{-\alpha}) E(r_S(s) | \mathcal{F}_t) ds - \int_t^{t+u} \lambda^\alpha E(r_S(s) | \mathcal{F}_t) ds \right) \\ &= \phi(u, t) - \frac{B}{A} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^u ((u-k)^{-\alpha}) E(r_S(t+k) | \mathcal{F}_t) dk - \int_0^u \lambda^\alpha E(r_S(t+k) | \mathcal{F}_t) ds \right) \\ &= \phi(u, t) - \frac{B}{A} \left(\frac{1}{\Gamma(1-\alpha)} \int_0^u ((u-k)^{-\alpha}) \psi(k, t) dk - \int_0^u \lambda^\alpha \psi(k, t) ds \right). \end{aligned}$$

We thus obtained integral equation of the form

$$\psi(u, t) = \phi(u, t) - \frac{B}{A} ({}_0I_u^{1-\alpha} \psi(u, t) - {}_0I_u^1 \lambda^\alpha \psi(u, t)). \quad (25)$$

The solution of the above is given by [19]

$$\psi(u, t) = \phi(u, t) + \int_0^u K'(u-s) \phi(s, t) ds,$$

where

$$K(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{B\lambda^\alpha}{A} \right)^k t^k E_{1-\alpha, 1+\alpha k}^{(k)} \left(-\frac{B}{A} t^{1-\alpha} \right).$$

Thus, conditional expectation for r is given by

$$E(r(t+u)|\mathcal{F}_t) = r_D(t+u) + \phi(u, t) + \int_0^u K(u-s)\phi(s, t)ds. \quad (26)$$

Now application of the above in formulas (14), (15) and (16) completes the *proof*.

The above results can be further applied to model real-life data using tempered Vasiček model. This will be the subject of another paper.

5. Conclusions

Classical Langevin equation is a model of normal diffusion. On the other hand, by application of Riemann–Liouville fractional derivatives one can model sub- and superdiffusion. In this paper, we go one step further and apply tempered fractional derivatives. Due to the fact that tempered fractional derivatives occupy intermediate place between ordinary and fractional derivatives, they can serve as a proper tool in describing situations between normal and anomalous diffusion.

In this paper, we introduced several extensions of the classical Langevin equation. These extensions are based on application of recently introduced tempered fractional derivatives. We derived explicit solutions of tempered fractional Langevin equations and showed how to simulate their paths. We also introduced extension of classical Vasiček model for short term interest rate, providing another tool in valuing interest-rate products. We derived formulas for yield-to-maturity, forward rate and bond price in this model.

We believe that the proposed models can serve as another useful tool both in physical and financial modeling.

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Appendix A

Simulation of processes of the general form

$$Y(t) = \int_0^t K(t-s) dL(s) \quad (\text{A.1})$$

is based on the following Theorem [38].

Theorem A.1 *Let Y be of the form (A.1) and let us assume that the function $K(t)$ is of the form*

$$K(t) = \int_0^\infty e^{-\nu t} \mu(d\nu), \quad (\text{A.2})$$

where μ is a finite borel measure on $[0, \infty)$. Then we can write $Y(t)$ in the form

$$Y(t) = \int_0^\infty X(\nu, t) \mu(d\nu),$$

where $X(\nu, t)$ is a solution of the following stochastic differential equation

$$dX(\nu, t) = -\nu X(\nu, t) dt + dL(t), \quad X(\nu, 0) = 0. \quad (\text{A.3})$$

Therefore, crucial is to write function $K(t)$ in the form (A.2). It turns out that the function

$$K(t) = \frac{1}{b} t^{\beta-1} E_{\beta-\alpha, \beta} \left(-\frac{c}{b} t^{\beta-\alpha} \right), \quad \beta \leq 1,$$

can be written as

$$K(t) = \frac{1}{\pi} \int_0^\infty \frac{c\nu^\alpha \sin(\alpha\pi) + b\nu^\beta \sin(\beta\pi)}{b^2\nu^{2\beta} + c^2\nu^{2\alpha} + 2bc \cos(\pi(\beta - \alpha))\nu^{\alpha+\beta}} e^{-t\nu} d\nu.$$

Thus for the function

$$K(t) = e^{-\lambda t} \frac{1}{\pi} \int_0^\infty \frac{c\nu^\alpha \sin(\alpha\pi) + b\nu^\beta \sin(\beta\pi)}{b^2\nu^{2\beta} + c^2\nu^{2\alpha} + 2bc \cos(\pi(\beta - \alpha))\nu^{\alpha+\beta}} e^{-t\nu} d\nu,$$

after the change of variables $\nu + \lambda \rightarrow u$ we have

$$K(t) = \frac{1}{\pi} \int_{1[\lambda, \infty]}^\infty \frac{c(u-\lambda)^\alpha \sin(\alpha\pi) + b(u-\lambda)^\beta \sin(\beta\pi)}{b^2(u-\lambda)^{2\beta} + c^2(u-\lambda)^{2\alpha} + 2bc \cos(\pi(\beta - \alpha))(u-\lambda)^{\alpha+\beta}} e^{-tu} du.$$

Using this representation we can approximate process in Eq. (A.1) through the following steps

- Define $S \subset [\lambda, \infty]$, $S = [r^{-m}, r^n]$, where $m, n \in \mathbb{N}$ and $r > 1$.
- Approximate process $X(\nu, t)$ given by the Eq. (A.3) in the following way

$$X_{\Delta}(\nu, k\Delta) = e^{-u\Delta} X_{\Delta}(\nu, (k-1)\Delta) + L(k\Delta) - L((k-1)\Delta).$$

- Approximate $Y(t)$ by

$$Y(t) = \sum_{i=-m}^{n-1} X_{\Delta}(r^i, t) \mu(A_i),$$

where $A_i = [r^i, r^{i+1}]$.

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