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DYNAMICS OF CONFINED LÉVY FLIGHTS IN TERMS OF (LÉVY) SEMIGROUPS*

Piotr Garbaczewski, Vladimir A. Stephanovich

Institute of Physics, University of Opole, 45-052 Opole, Poland

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The master equation for a probability density function (PDF) driven by Lévy noise, if conditioned to conform with the principle of detailed balance, admits a transformation to a contractive strongly continuous semigroup dynamics. Given *a priori* a functional form of the semigroup potential, we address the ground-state reconstruction problem for generic Lévy-stable semigroups, for *all* values of the stability index $\mu \in (0,2)$. That is known to resolve the problem about an invariant PDF for confined Lévy flights. Jeopardies of the procedure are discussed, with a focus on: *(i)* when an invariant PDF actually is an asymptotic one, *(ii)* subtleties of the PDF μ-dependence in the vicinity and sharply *at* the boundaries 0 and 2 of the stability interval, where jump-type scenarios cease to be valid.

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1. Conceptual background

Under Lévy noise model, we understand any member of a subclass of uni-variate stable probability distributions determined by a characteristic exponent $-F(p) = -|p|^\mu$ of $\langle \exp(ipX)\rangle$, with $0 < \mu < 2$. The induced free jump-type dynamics, $\langle \exp(ipX_t)\rangle = \exp[-tF(p)]$, where $t \geq 0$, is conventionally interpreted in terms of Lévy flights and quantified by means of a pseudo-differential (fractional) equation for a corresponding time-dependent probability density function (PDF)

$$\partial_t \rho = -|\Delta|^{\mu/2}\rho = \int [w_\mu(x|y)\rho(y) - w_\mu(y|x)\rho(x)]dy\,. \tag{1}$$

The jump rate $w_\mu(x|y) \propto 1/|x-y|^{1+\mu}$ is a symmetric function, $w_\mu(x|y) = w_\mu(y|x)$.

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The "free" fractional Fokker–Planck equation (1) has no stationary solutions. On the other hand, asymptotic invariant PDFs for confined Lévy flights are known to arise in the standard Langevin modeling of an external forces impact on the Lévy-stable noise, [1]. A disadvantage of that approach is a non-existence of Boltzmann-type (thermal) equilibria, [2,3,4,5,6,7]. We note that the reference stable laws generically have no moments of the order higher than one (and may as well have none at all). To the contrary, PDFs for confined Lévy flights may have an arbitrary, not necessarily finite number of moments.

Interestingly, if we enforce [4, 7] the principle of detailed balance to hold true, by a suitable modification of transition rates $w(x|y) \to w_U(x|y)$ (with U playing the role of an external microscopic potential), asymptotic Boltzmann-type equilibria of the form $\rho(x) \sim \exp[-U(x)]$ in principle become admissible. The price paid is that the standard Langevin modeling of confined Lévy flights becomes inadequate.

We generalize the master equation (1) to encompass non-symmetric jump rates as follows, $w_\mu(x|y) \to w_\mu^U(x|y) \neq w_\mu^U(y|x)$

$$w_\mu^U(x|y) = w_\mu(x|y)\, \exp\left(\frac{U(y)-U(x)}{2}\right)\,, \tag{2}$$

where $U(x)$ is a continuous function on R. With $w_\mu^U(x|y)$ replacing $w_\mu(x|y)$, Eq. (1) takes the form

$$\partial_t \rho = -[\exp(-U/2)]\, |\Delta|^{\mu/2}[\exp(U/2)\rho] + \rho \exp(U/2)|\Delta|^{\mu/2}\exp(-U/2)\,. \tag{3}$$

For a suitable (to secure normalization) choice of $U(x)$, $\rho_{\rm eq}(x) \propto \exp[-U(x)]$ is a stationary solution of Eq. (3). The detailed balance principle necessarily follows

$$w_U(x|y)\rho_{\rm eq}(y) = w_U(y|x)\rho_{\rm eq}(x)\,. \tag{4}$$

The master equation (3) cannot be derived within the multiplicative or additive Langevin modeling. As mentioned above, the Eq. (3) at least on formal grounds, admits a transformation to a strongly continuous semigroup dynamics. That is akin to a mapping of the standard Fokker–Planck equation into the generalized diffusion (semigroup) equation which is widely exploited in the context of diffusion-type processes (there *e.g.* the Fokker–Planck operator is mapped into a symmetric operator, which furthermore needs to be extended to a self-adjoint one). In the Brownian case, the Langevin and semigroup derivations, while interpreted in terms of $\rho(x,t)$, refer to the same diffusion-type process. On the contrary, this equivalence *does not* persist in the case of confined Lévy flights.

The passage from Eq. (3) for confined Lévy flights to the semigroup dynamics is accomplished by means of a redefinition

$$\rho(x,t) = \rho_*^{1/2}(x)\Psi(x,t)\,, \tag{5}$$

where $\rho_*(x) = \rho_{\rm eq}(x) = Z^{-1}\exp[-U(x)]$ is an asymptotic invariant PDF of the jump-type process, while the dynamics of a *real* positive-definite function $\Psi(x,t)$ follows the semigroup pattern $[\exp(-\hat{H}t)\,\Psi](x,0) = \Psi(x,t)$ for $t \geq 0$, *cf.* [7, 5, 6]. Here, we have introduced the Lévy–Schrödinger Hamiltonian operator with an external potential

$$\hat{H}_\mu \equiv |\Delta|^{\mu/2} + \mathcal{V}(x)\,. \tag{6}$$

Suitable properties of $\mathcal{V}$ need to be assumed, so that *(i)* $-\hat{H}_\mu$ is a legitimate (self-adjoint) generator of a (strongly continuous, contractive) semigroup $\exp(-t\hat{H}_\mu)$, *(ii)* asymptotically, as $t \to \infty$ we get $\Psi(x,t) \to \Psi_*(x) \sim \rho_*^{1/2}(x)$ and $\Psi_*(x)$ is a unique ground state of $\hat{H}$. Said otherwise, once we have *a priori* selected an invariant probability density $\rho_{\rm eq}(x) \doteq \rho_*(x) \propto \exp[-U(x)]$ in Eq. (3), to justify its interpretation as an asymptotic PDF of a well defined jump-type process we must have guaranteed an existence of an associated contractive semigroup dynamics for which $\Psi_*(x) \sim \rho_*^{1/2}(x)$.

Looking for stationary solutions of the semigroup equation $\partial_t\Psi = -\hat{H}_\mu\Psi$, we realize that if a square root of a positive invariant PDF $\rho_*(x)$ is an asymptotic outcome of the dynamics $\Psi \to \rho_*^{1/2}$, then the resulting fractional Sturm–Liouville equation $\hat{H}_\mu\rho_*^{1/2} = 0$ stands for a *compatibility condition* imposed upon the functional form of $\mathcal{V}(x)$

$$\mathcal{V} = -\frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}\,. \tag{7}$$

We note that the inferred semigroup dynamics provides a solution for the Lévy stable *targeting problem*, with a predefined invariant PDF. It is an identification $\rho(x,t) = \rho_*^{1/2}(x)\Psi(x,t)$ that does the job. We have discussed this issue in some detail in our previous publications, [3, 5, 6].

Inversely, if we choose *a priori* a concrete potential function $\mathcal{V}(x)$, then an ultimate functional form of an invariant PDF $\rho_*(x)$ (actually $\rho_*^{1/2}(x)$), if in existence, needs to come out from the above compatibility condition. However, a solvability of (7) with respect to $\rho_*^{1/2}$ merely associates an *invariant* PDF ρ_* with a pre-defined semigroup potential $\mathcal{V}$. In regard to the *asymptotic regime* $\Psi(x,t) \to \rho_*^{1/2}$ we need much more. Namely, $\rho_*^{1/2}(x)$ must belong to the domain of a self-adjoint operator $\hat{H}$ and a proper identification of such domain may sometimes become tricky.

In the present paper, we address the latter problem along with jeopardies involved, for a predefined function $\mathcal{V}(x)$, while admitting all stability index values $0 < \mu < 2$ in the compatibility condition (7).

Remark 1: For clarity of presentation and self-explanatory features of discussion, we add few comments about the stochastic process in question. Let $\hat{H}$ be a self-adjoint operator in a suitable Hilbert space domain. Additionally, let $\mathcal{V} = \mathcal{V}(x)$ be a bounded from below continuous function. Then, the integral kernel $k(y,s,x,t) = \{\exp[-(t-s)\hat{H}]\}(y,x)$, $s < t$, of the semigroup operator $\exp(-t\hat{H})$ is positive and jointly continuous in all variables. The semigroup dynamics reads: $\Psi(x,t) = \int \Psi(y,s)\, k(y,s,x,t)\, dy$ so that for all $0 \leq s < t$ we can reproduce the dynamical pattern of behavior, actually set by Eq. (3), but now in terms of Markovian transition probability densities $p(x,s,y,t)$: $\rho(x,t) = \rho_*^{1/2}(x)\Psi(x,t) = \int p(y,s,x,t)\rho(y,s)dy$, where $p(y,s,x,t) = k(y,s,x,t)\, \rho_*^{1/2}(x)/\rho_*^{1/2}(y)$. An asymptotic behavior of $\Psi(x,t) \to \rho_*^{1/2}(x)$ implies $\rho(x,t) \to \rho_*(x)$ as $t \to \infty$.

Remark 2: The spectral theory of fractional operators of the form (6) has received a broad coverage in the mathematical [8, 9, 10, 11, 12] and mathematical physics literature [13, 14]. Various rigorous estimates pertaining to the decay of the eigenfunctions at spatial infinities, quantify the number of moments of the associated PDFs for different classes of potential functions $\mathcal{V}(x)$. As well, fractional versions of the Feynman–Kac formula for an integral kernel of the semigroup operator have an ample coverage therein.

2. μ-family of semigroup potentials for a predefined invariant PDF and the $\mu \in (0,2)$ boundary issue

For a pseudo-differential operator $|\Delta|^{\mu/2}$, the action on a function from its domain is greatly simplified in the physics-oriented research. Normally, with $\nu_\mu(dx)$ standing for the Lévy measure, we have

$$\left(|\Delta|^{\mu/2} f\right)(x) = -\int_R \left[f(x+y) - f(x) - \frac{y\,\nabla f(x)}{1+y^2}\right] \nu_\mu(dy)\,. \tag{8}$$

In majority of physical problems, one actually considers the Cauchy principal value of the integral

$$\left(|\Delta|^{\mu/2} f\right)(x) = -\int [f(x+y) - f(x)]\nu_\mu(dy)\,, \tag{9}$$

which upon changing an integration variable $y \to z = x + y$, may be reproduced in the form, see *e.g.* [7]

$$-\left(|\Delta|^{\mu/2} f\right)(x) = \frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi} \int \frac{f(z) - f(x)}{|z-x|^{1+\mu}}\, dz\,. \tag{10}$$

Let us investigate the properties of the $-|\Delta|^{\mu/2}f(x)$ by means of the Fourier image of $f(x)$. We employ a redefinition of Eq. (10)

$$-|\Delta|^{\mu/2}f(x) = \frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi}\int\limits_{-\infty}^{\infty} dy\frac{f(x+y)-f(x)}{|y|^{1+\mu}} \tag{11}$$

which yields

$$-|\Delta|^{\mu/2}f(x) = \frac{\Gamma(1+\mu)\sin\frac{\pi\mu}{2}}{\pi\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(k)e^{-\imath kx}dk\int\limits_{-\infty}^{\infty}\frac{(e^{-\imath ky}-1)dy}{|y|^{1+\mu}}\,. \tag{12}$$

The integral over dy can be calculated as follows

$$\int\limits_{-\infty}^{\infty}\frac{\left(e^{-\imath ky}-1\right)dy}{|y|^{1+\mu}} = 2|k|^{\mu}\Gamma(-\mu)\cos\frac{\pi\mu}{2}\,. \tag{13}$$

It is seen that at the limiting value $\mu = 0$, $\Gamma(0)$ is divergent, so that the integral (13) is divergent as well. The same happens for $\mu = 2$, in view of the divergence of $\Gamma(-2)$. However, irrespective of how close to 0 or 2 the label $\mu > 0$ is, the integral (13) is convergent.

It is interesting to observe that the divergence of the Fourier integral, as μ approaches 0 or 2, becomes compensated, if we substitute it back to Eq. (12) and next consider the limiting behavior of the result

$$\begin{aligned} -|\Delta|^{\mu/2}f(x) &= \frac{2\Gamma(1+\mu)\Gamma(-\mu)\sin\frac{\pi\mu}{2}\cos\frac{\pi\mu}{2}}{\pi\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}|k|^{\mu}f(k)e^{-\imath kx}dk \\ &= -\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}|k|^{\mu}f(k)e^{-\imath kx}dk\,. \end{aligned} \tag{14}$$

Here we use the identity

$$\Gamma(1+\mu)\Gamma(-\mu) = -\frac{\pi}{\sin\pi\mu}\,. \tag{15}$$

In view of the above divergence obstacle, the integral representation (10) is invalid *at* the boundaries of the stability interval. However, the range of validity of its ultimate Fourier version, *e.g.* the right-hand side of Eq. (14), can be safely extended to the boundary values 0 and 2.

In fact, it is $-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|k|^{\mu}f(k)e^{-\imath kx}dk \equiv -\partial_{\mu}/\partial|x|^{\mu} \equiv (-\Delta)^{\mu/2}$ that is commonly interpreted in the literature as a definition of the fractional derivative of the μth order. On formal grounds, this definition encompasses the boundary cases $(-\Delta)^0 \equiv 1$ and $-|\Delta|^{2/2} \equiv \Delta$.

2.1. Heavy-tailed case

Let us first consider a specific (long time asymptotic) PDF (member of the so-called Cauchy family, [3]) $\rho(x) = 2/[\pi(1+x^2)^2]$, such that $\rho^{1/2}(x) = \sqrt{\frac{2}{\pi}}\frac{1}{1+x^2}$ and its Fourier transform reads $\rho^{1/2}(k) = e^{-|k|}$. In view of

$$\int_{-\infty}^{\infty} e^{-\imath kx} f(|k|)dk \equiv 2\int_{0}^{\infty} \cos(kx) f(k) dk$$

we have from compatibility condition (7)

$$\begin{aligned} \mathcal{V}_\mu(x) &= -\left(1+x^2\right)\int_0^\infty k^\mu e^{-k}\cos kx\ dk \\ &= -\left(1+x^2\right)^{\frac{1-\mu}{2}}\,\varGamma(1+\mu)\cos\left[(1+\mu)\arctan x\right]\,, \quad 0<\mu<2\,. \end{aligned} \tag{16}$$

Expression (16) permits to reproduce easily a number of $\mathcal{V}_\mu$ for different values of $\mu \in (0,2)$. All $\mathcal{V}_\mu$, $\mu \in (0,2)$, derive from a pre-defined (quadratic Cauchy) asymptotic PDF. However, we find most interesting the following three limiting cases. Namely, for $\mu = 1$, we get the result derived previously in Ref. [5]

$$\mathcal{V}_1(x) = \frac{x^2-1}{x^2+1}\,. \tag{17}$$

In the vicinity of the boundary value, $\mu = 2$, the expression (16) yields

$$\begin{aligned} \mathcal{V}_{\mu\to 2,2}(x) \ &\approx\ \frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^2} - \frac{\mu-2}{\left(1+x^2\right)^2}\Bigg[2x\left(x^2-3\right)\arctan x \\ &\quad + \left(3x^2-1\right)\left(2\gamma-3+\ln\left(1+x^2\right)\right)\Bigg]\,, \end{aligned} \tag{18}$$

where $\gamma \approx 0.577216$ is Euler constant. Hence, for the boundary value $\mu = 2$ we get

$$\mathcal{V}_2(x) = \frac{2\left(3x^2-1\right)}{\left(1+x^2\right)^2} \tag{19}$$

which stays in conformity with a naive expectation that $-|\varDelta| \equiv \varDelta$. Indeed, on formal grounds, we readily get (19) via $\mathcal{V}_2(x)\rho^{1/2}(x) = +d^2\rho^{1/2}(x)/dx^2$, as expected.

For $\mu = 0$, in view of $\arctan x = \arccos(1/\sqrt{1+x^2})$, from (16) we recover $\mathcal{V}_0(x) = -1$ which is compatible with $\mathcal{V}_0(x) = -\frac{|\varDelta|^0\rho_*^{1/2}}{\rho_*^{1/2}} = -1$ upon an identification $|\varDelta|^0 \equiv 1$.

2.2. Gaussian PDF

Let us consider the invariant PDF in a Gaussian form $\rho_* = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$ whose square root $\rho_*^{1/2}$ has the Fourier image

$$(\rho^*)^{1/2}(k) \equiv f(k) = \frac{1}{\sqrt{\sigma}\sqrt[4]{2\pi}} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} e^{-\frac{x^2}{4\sigma^2}} e^{ikx} dx = \sqrt{\sigma}\sqrt[4]{\frac{2}{\pi}} e^{-k^2\sigma^2} . \quad (20)$$

Accordingly,

$$\begin{aligned} \mathcal{V}_\mu(x) &= -\frac{1}{(\rho^*)^{1/2}(x)} \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (\rho^*)^{1/2}(k) |k|^\mu e^{-ikx} dk \\ &= -\frac{2\sqrt{\sigma}}{(\rho^*)^{1/2}(x)\sqrt[4]{2\pi}\sqrt{\pi}} \int\limits_0^\infty |k|^\mu \cos kx \; e^{-k^2\sigma^2} dk \, . \end{aligned} \quad (21)$$

Clearly, at $\mu = 2$ we arrive at

$$\mathcal{V}_2(x) = \frac{1}{2\sigma^2}\left(\frac{x^2}{2\sigma^2} - 1\right) , \quad (22)$$

whose equivalent derivation is provided by setting $-|\Delta| \equiv +\Delta$ in Eq. (7). Indeed, we verify by inspection that $\mathcal{V}_2(x) = \frac{f''(x)}{f(x)}$.

The case of $\mu = 0$ can be handled as before with the outcomes $\mathcal{V}_0(x) = -1$ and $|\Delta|^0 \equiv 1$. To this end we observe that (set $\mu = 0$ in Eq. (13)) $\int_0^\infty \cos kx \; e^{-k^2\sigma^2} dk = \frac{\sqrt{\pi}}{2\sigma} \exp(-\frac{x^2}{4\sigma^2})$, while $(\rho^*)^{1/2} = \frac{\sqrt{\sigma}}{\sqrt[4]{2\pi}} \exp(-\frac{x^2}{4\sigma^2})$.

3. μ-family of PDFs for a predefined $\mathcal{V}(x)$: generalities

Presently, we shall proceed in reverse. If we choose *a priori* a specific potential function $\mathcal{V}(x)$, then an ultimate functional form of an invariant PDF $\rho_*(x)$ (actually $\rho_*^{1/2}(x)$), *if in existence*, needs to come out from the compatibility condition (7). For any functional form of the confining potential $\mathcal{V}(x)$, Eq. (7) imposes the following identity to be obeyed by an invariant (terminal) PDF (we denote $\rho_*^{1/2}(x) \equiv f(x)$)

$$\mathcal{V}(x) f(x) = -|\Delta|^{\mu/2} f(x) \, , \quad (23)$$

where $0 < \mu < 2$. Remembering that we consider $\mathcal{V}(x)$ to be a continuous and bounded from below function (may be unbounded from above), we turn over to the standard Fourier transform method with

$$f(k)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(x)e^{\imath kx}dx,\ \ f(x)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} f(k)e^{-\imath kx}dk\,.$$

Denoting the Fourier image of the right-hand side of Eq. (23) as u_k, we obtain

$$u_k=-|k|^{\mu}f(k)\,. \qquad (24)$$

Equating Fourier images of both sides of Eq. (23), provided they exist, yields

$$u_k=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\mathcal{V}(x)f(x)e^{\imath kx}dx=\frac{1}{2\pi}\int\int\limits_{-\infty}^{\infty}\mathcal{V}(x)e^{\imath x(k-k')}f(k')dk'dx\,. \qquad (25)$$

In this case, the Fourier image $f(k)$ of a solution $f(x)$ to Eq. (23) is defined by following integral equation of a convolution type

$$f(k)=-\frac{1}{|k|^{\mu}\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\mathcal{V}(k-k')f(k')dk'\,. \qquad (26)$$

Here we employ the identity $\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{\imath x(k-k')}dx=\delta(k-k')$, valid in the sense of distributions.

Our confining assumption implies that the function $\mathcal{V}_{\mu}(x)$ typically grows at infinities so that its Fourier image may not exist, unless distributionally. The same jeopardy appears in the case of a product $\mathcal{V}(x)f(x)$, (25). Modulo those obstacles, we shall demonstrate that, if interpreted in the sense of distributions, the general solution of Eq. (26) can be expressed in terms of δ-function and its derivatives.

Let us restrict further consideration to even functions $\mathcal{V}$ (in this case the terminal PDF is an even function as well) and for clarity of presentation assume them to be rational functions. The Taylor series comprise even powers of x only

$$\mathcal{V}(x)=\mathcal{V}(0)+\mathcal{V}_{\mu}''(0)\frac{x^2}{2!}+\mathcal{V}_{\mu}^{(4)}(0)\frac{x^4}{4!}+\dots\,. \qquad (27)$$

The Fourier image of (27), *cf.* (25) and (26), yields

$$\begin{aligned}\mathcal{V}(k) &= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\left[\mathcal{V}(0)+\mathcal{V}''(0)\frac{x^2}{2!}+\mathcal{V}^{(4)}(0)\frac{x^4}{4!}\right]e^{\imath kx}dx\\ &\equiv \sqrt{2\pi}\left[\mathcal{V}(0)\delta(k)-\frac{\mathcal{V}''(0)}{2}\delta''(k)+\frac{\mathcal{V}^{(4)}(0)}{4!}\delta^{(4)}(k)-\dots\right], \qquad (28)\end{aligned}$$

i.e. it has the form of the infinite series of even derivatives of the Dirac δ-function. We note that if $\mathcal{V}(x)$ is a simple (even) polynomial (example: $\mathcal{V}(x) = ax^2 + bx^4$), the above series are finite.

Accordingly, we end up with the following differential equation of the infinite even order for the Fourier image $f(-k) = f(k)$ of $\rho_*^{1/2}(x) \equiv f(x)$

$$\frac{\mathcal{V}''(0)}{2}\frac{d^2 f(k)}{dk^2} - \frac{\mathcal{V}^{(4)}(0)}{4!}\frac{d^4 f(k)}{dk^4} + \cdots = \Big[k^\mu + \mathcal{V}(0)\Big] f(k)\,, \qquad k \geq 0\,. \quad (29)$$

We choose the following initial conditions for Eq. (29)

$$f(k=0) \equiv \int\limits_{-\infty}^{\infty} f(x)dx = A\,, \quad f^{(2n-1)}(k=0) = 0\,, \quad n = 1, 2, 3, \ldots\,. \quad (30)$$

Note that this imposes an integrability condition on $\rho_*^{1/2}(x)$ on $\mathcal{R}$. Here, by $f^{(2n-1)}(k=0)$, we denote the odd derivatives of $f(k)$ at $k = 0$.

The integration constant A is not completely arbitrary and should be consistent with the normalization condition $\int_{-\infty}^{\infty} f^2(x)dx \equiv \int_{-\infty}^{\infty} \rho_*(x)dx = 1$. In view of the Parceval identity, we have

$$\int\limits_{-\infty}^{\infty} f^2(x)dx = \int\limits_{-\infty}^{\infty} f^2(k)dk \equiv 2\int\limits_{0}^{\infty} f^2(k)dk = 1\,. \quad (31)$$

This means that $f(k=0) = A$ must be compatible with $\int_0^\infty f^2(k)dk = 1/2$.

One should not expect an easy analytic outcome of the solution of the infinite order differential equation (29). In most cases of interest, the infinite series can be truncated, but generically a numerical assistance is unavoidable. The practical strategy of finding (at worst approximately for truncated series and an arbitrary functional shape of $\mathcal{V}(x)$) an $L^2(\mathcal{R})$ integrable non-negative ground state of the *a priori* prescribed semigroup, and hence the terminal PDF $\rho_*(x)$, can be summarized as follows.

- Expand $\mathcal{V}(x)$ in power series. The number of terms in the series should be chosen so as to obtain a sufficiently good approximation of the potential.

- Solve the differential equation (29) with initial conditions (30). If $\mathcal{V}(x)$ is a polynomial function, there are good chances to solve this equation analytically. Otherwise we should reiterate to numerics. Check a compatibility of $f(k=0) = A$ with the normalization condition.

- Analytically or numerically take the inverse Fourier transform to obtain a non-negative function $f(x)$, to be interpreted as $\rho_*^{1/2}(x)$.
- Check, whether the obtained $f(x)$, $x^2 f(x)$ and $|\Delta|^{\mu/2} f(x)$ are absolutely integrable (to guarantee that they actually are Fourier transformable). Additionally, check that they are square integrable functions and $L^2(R)$ normalize the resultant $f(x)$.
- Everything is being accomplished under an assumption that $|\Delta|^{\mu/2} + \mathcal{V}(x)$ is not merely symmetric, but a self-adjoint operator in a suitable domain that is dense in $L^2(R)$. A tacit assumption is that $\rho_*^{1/2}(x)$ actually *is* in the domain of $\hat{H}$. However, this point is quite delicate and involves jeopardies, as we shall see in what follows. The operators considered are always symmetric. An issue of their self-adjoint extensions is by no means trivial. Only a self-adjoint generator allows to interpret the derived PDF as the terminal (asymptotic invariant) one for an associated semigroup dynamics

4. Case study of the PDF reconstruction for $\mathcal{V} = x^2/2$ and $\mu \in (0, 2)$

We shall consider an exemplary analytic (in part computer-assisted) realization of the previously outlined inverse procedure. Our motivations come from Ref. [5], where the Cauchy oscillator has been addressed.

We recall that by analogy with the familiar harmonic oscillator Hamiltonian (related to the Ornstein–Uhlenbeck process) $\hat{H} \equiv -D\Delta + \left(\frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2}\right)$, whose non-negative spectrum starts from 0, we have considered, [5, 12], the Cauchy oscillator problem in the form $\hat{H}_{1/2} \equiv \lambda|\nabla| + \left(\frac{\kappa}{2}\, x^2 - \mathcal{E}_0\right)$, with $\mathcal{E}_0$ left unspecified. The compatibility condition (7) appears in the form $\left(\frac{\kappa}{2}\, x^2 - \mathcal{E}_0\right) \rho_*^{1/2} = -\lambda\, |\nabla|\, \rho_*^{1/2}$. Its Fourier transform reads

$$-\frac{\kappa}{2} \Delta_p \tilde{f} + \gamma |p| \tilde{f} = \mathcal{E}_0 \tilde{f}\,, \tag{32}$$

where $\tilde{f}(p)$ stands for the Fourier transform of $f = \rho_*^{1/2}(x)$.

By changing an independent variable p to $k = (p - \sigma)/\zeta$, next denoting $\psi(k) = \tilde{f}(p)$ with the identifications $\sigma = \mathcal{E}_0/\gamma$ and $\zeta = (4\kappa/\gamma)^{1/3}$, we may rewrite the above eigenvalue problem (with $\mathcal{E}_0$ standing for an eigenvalue) in the form of the following ordinary differential equation

$$\frac{d^2\psi(k)}{dk^2} = 2|k|\psi(k)\,, \tag{33}$$

whose solutions can be represented in terms of Airy functions, *cf.* Ref. [5]. We note in passing that a slightly different scaling was originally employed in Ref. [5]: $\zeta = (\kappa/2\gamma)^{1/3} \rightarrow \frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$.

Equation (33) is, in fact, a departure point for our subsequent discussion. Quite at variance with the standard (text-book) harmonic oscillator intuitions and our own discussion of the Cauchy oscillator spectral problem [5], just out of curiosity and as a useful exercise let us consider the compatibility condition for $\mathcal{V} = x^2/2$ proper, *e.g. without* any additive counterterms of the form $-\mathcal{E}_0$

$$\frac{x^2}{2}\rho_*^{1/2} = -|\Delta|^{\mu/2}\rho_*^{1/2}\,, \qquad 0 < \mu < 2\,. \tag{34}$$

That amounts to assigning the "improper" eigenvalue zero as the bottom eigenvalue of the corresponding spectral problem.

We take Fourier images of both sides of Eq.(34) to obtain

$$u_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{x^2}{2}f(x)e^{\imath kx}dx = -\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{\partial^2}{\partial k^2}\int_{-\infty}^{\infty}f(x)e^{\imath kx}dx \equiv -\frac{1}{2}\frac{\partial^2 f(k)}{\partial k^2}\,. \tag{35}$$

Accordingly, we have

$$\frac{d^2 f(k)}{dk^2} = 2|k|^{\mu}f(k)\,, \tag{36}$$

which, for $\mu = 1$, yields the above Eq. (33).

To find a solution to Eq. (36), our main idea here (and possibly for an arbitrary potential) stems from the approach we have originally adopted for Eqs. (32) and (33), [5, 19] and based on the exploitation of Airy functions. We note that both the (Fourier) integral definition of the fractional operator and the solution construction procedure involves the Cauchy principal value integral and the continuity (gluing) conditions in the vicinity and ultimately at the point $0 \in R$.

We should find the decaying solution of the corresponding differential equation (36) in the k-space, on the positive semi-axis ($k > 0$), and an oscillatory one on the negative semi-axis ($k < 0$). Then we need to shift the obtained solution to the right so that the first maximum k_{m} of the oscillatory part is located at $k = 0$. After "chopping" the rest of the oscillating part one has to reflect the remaining piece about the vertical axis to get an even "bell-shaped" function.

The obtained k-space solution should be Fourier-inverted (properly — here we encounter a crucial difference with the reasoning of [5] for the Cauchy oscillator) and squared, while keeping in mind the $L^2(R)$ normalization issue. This procedure is expected to determine the desired invariant (and a candidate for terminal) PDF in the x-space.

Remark 3: We need to explain how our workings with the $+k_{\mathrm{m}}$ shift are to be understood, since (36) itself is *not* translationally invariant. Eq. (36) has the form $d^2f/dk^2 = G(k)f(k)$, where $G(k)$ can, in principle, be an arbitrary function. We execute $k \to k + k_{\mathrm{m}}$ everywhere in (36), so arriving at $d^2f/dk^2 = G(k + k_{\mathrm{m}})f(k + k_{\mathrm{m}})$. Obviously, with $k' = k + k_{\mathrm{m}}$ we have $d^2f/dk'^2 = G(k')f(k')$, where $k' \in R$ and the same as previously boundary data at infinities. This function is actually Fourier inverted with respect to the k'-label. To see clearly, why such precaution needs to be kept in mind, let us notice that $f(x) = \int_0^\infty f(k')\cos(k'x)dk'$ maps back (36) into (34). On the other hand, to map (33) into the spectral problem for $\hat{H}_{1/2} \equiv \lambda|\nabla| + \left(\frac{\kappa}{2}\,x^2 - \mathcal{E}_0\right)$, we need to execute the inverse Fourier transform with respect to p in $\psi(k) = \psi[k(p)] = \tilde{f}(p)$.

The solutions of Eq. (36) have different forms for $k > 0$ and $k < 0$ respectively, [16]. Namely, for $k \geq 0$ we have

$$f(k) = \sqrt{k}\left[C_{11}I_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right) + C_{12}K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right)\right], \qquad q = \frac{1}{2}(\mu + 2)\,, \tag{37}$$

while for $k < 0$ there holds

$$f(k) = \sqrt{|k|}\left[C_{21}J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22}N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right]. \tag{38}$$

Here, $J_\nu(x)$ and $N_\nu(x)$ are Bessel functions and $I_\nu(x)$ and $K_\nu(x)$ are modified Bessel functions, see [17]. The asymptotics of $I_\nu(x)$ and $K_\nu(x)$ at $x \to \infty$ read [17]

$$I_\nu(x) \approx \frac{e^x}{\sqrt{2\pi x}}\,, \qquad K_\nu(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x}\,, \tag{39}$$

while as $k \to -\infty$ the asymptotics of the functions $J_\nu(x)$ and $N_\nu(x)$ is oscillatory [17]. This means that to obtain a localized PDF, we should leave in (37) the term with $K_{\frac{1}{2q}}$ only, so that $f(k)$ assumes the following form

$$f(k) = \begin{cases} C_{12}\sqrt{k}K_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}k^q\right)\,, & k \geq 0\,, \\ \sqrt{|k|}\left[C_{21}J_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right) + C_{22}N_{\frac{1}{2q}}\left(\frac{\sqrt{2}}{q}|k|^q\right)\right], & k < 0\,. \end{cases} \tag{40}$$

As the equation (36) is of the second order, we should impose the continuity conditions at $k = 0$ for a solution and its first derivative. We note, that the numerical solution of equation (36) directly involves the value of function and its first derivative at $k = 0$. That implies (see Appendix A for more details)

$$f(k) = C\sqrt{|k|}\begin{cases} K_\nu(u)\,, & k \geq 0\,, \\ \frac{\pi}{2}\left[\cot\frac{\pi\nu}{2}J_\nu(u) - N_\nu(u)\right]\,, & k < 0\,, \end{cases} \tag{41}$$

where $C \equiv C_{12}$,

$$\nu = \frac{1}{2q} \equiv \frac{1}{\mu+2}\,, \qquad u = \frac{\sqrt{2}}{q}|k|^q \equiv \frac{2\sqrt{2}}{\mu+2}|k|^{1+\frac{\mu}{2}}\,. \tag{42}$$

We note here that for $\mu = 1$ we obtain from (41)

$$f(k) = C\sqrt{k}K_{\frac{1}{3}}\left(\frac{2\sqrt{2}}{3}k^{\frac{3}{2}}\right) = C\frac{\pi\sqrt{3}}{2^{\frac{1}{6}}}\mathrm{Ai}\left(2^{\frac{1}{3}}\ k\right)\,, \tag{43}$$

known from Ref. [5], where the eigenvalue problem for the Cauchy oscillator has been solved, see also Ref. [12].

The next step is to find the position k_{m} of the first maximum of an oscillating part, next shift the solution to the right by k_{m}, reflect the solution with respect to the y axis and "chop" the rest of oscillating parts. By equating to zero the first derivative of an oscillating contribution to (41), after some algebra we get the following equation

$$N_{\nu-1}(u) - \cot\frac{\pi\nu}{2}J_{\nu-1}(u) = 0\,, \tag{44}$$

where ν and u are defined by (42). Solutions of this equation can be tabulated, see Table I. The "raw" solutions (41) are shown (along with the position of k_{m}) in Fig. 1.

The normalization condition allows us to fix the admissible values of hitherto unspecified constant C. Namely, we have

$$\begin{aligned} C^2\int_{-\infty}^{\infty} f^2(k)dk &= 2C^2\int_0^{\infty} f^2(k)dk \\ &= 2C^2\left[\int_0^{-k_{\mathrm{m}}} f_1^2(k)dk + \int_{-k_{\mathrm{m}}}^{\infty} f_2^2(k)dk\right] = 1\,, \end{aligned} \tag{45}$$

where f_1 and f_2 denote the oscillatory and decaying parts of Eq. (41) respectively. This integration can be performed numerically and results are reproduced in the right column of the Table I. The final step of the procedure is to invert the k-space solutions to x-space and square them to obtain the invariant PDF. Except for special μ cases, this procedure can be accomplished only numerically.

TABLE I

Roots $k_{\mathrm{m}}(\mu)$ of Eq. (44) corresponding to first maximum of oscillatory part of (41) for different μ (middle column) and normalization constants $C(\mu)$ (right column).

μ	$k_{\mathrm{m}}(\mu)$	$C(\mu)$
0.0	$-0.55536 = -\frac{\pi}{4\sqrt{2}}$	$0.597135 = \sqrt{\frac{2}{\pi}}\left(1+\frac{\pi}{4}\right)^{-1/2}$
0.2	-0.621962	0.500134
0.4	-0.679458	0.429855
0.6	-0.729002	0.376894
0.8	-0.771717	0.335701
1.0	-0.808617	0.302823
1.2	-0.840577	0.276010
1.4	-0.868346	0.253745
1.6	-0.892550	0.234970
1.8	-0.913716	0.218927
2.0	-0.932286	0.205597

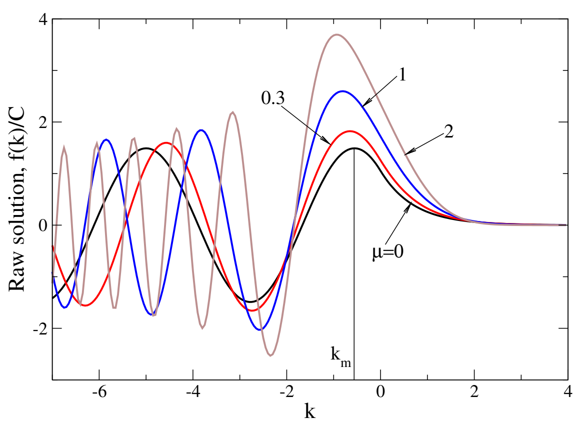

Fig. 1. Raw solutions of Eq. (41). Figures correspond to μ values. Solution for $\mu = 1$ corresponds to Airy function (43). Formal solution for $\mu = 0$ and the position of the first maximum of the oscillatory part k_{m} are shown as an example.

5. Limiting (mis)behavior at the boundaries of $(0, 2) \ni \mu$ and the zero (bottom) eigenvalue issue

The stability interval $\mu \in (0, 2)$ is an open set. However, since μ can be chosen to be arbitrarily close, respectively to 0 or 2, simply out of curiosity, it is not useless to address a hitherto unexplored issue, of what is actually going on in the limiting behaviors of $\mu \downarrow 0$ and $\mu \uparrow 2$. This scenario can be consistently executed on the level of Fourier transforms.

We note that the operator $-|\Delta|^{\mu/2}$, as defined by Eq. (10), is a pseudo-differential (Riesz) operator and the integral therein needs to be taken as its Cauchy principal value. Hence both in x and k-spaces the point $0 \in R$ is particularly distinguished.

On formal grounds, as exemplified in Sec. 2, a naive expectation would amount to literal setting of $\mu = 0$ or $\mu = 2$ instead of the "normal" stability index values $\mu \in (0, 2)$. Then, the operator $-|\Delta|^{\mu/2}$ would turn over into $-|\Delta|^0 \equiv -1$ and $-|\Delta| \equiv \Delta$ respectively. However, this compelling picture turns out to be problematic, except for rather special cases, like those considered above in Sec. 2.

We recall that the core of our solution method lies in Fourier transforming of the compatibility condition (7)

$$\frac{x^2}{2}\rho_*^{1/2} = -|\Delta|^{\mu/2}\rho_*^{1/2} \to \frac{d^2 f(k)}{dk^2} = 2|k|^\mu f(k) \tag{46}$$

and actually solving the obtained differential equation in k-space.

Given the solution, the unsettled problem is the validity of the inverse Fourier transform of the outcome. That means that not only $f(k)$ itself, but also $|k|^\mu f(k)$ and specifically $\frac{d^2 f(k)}{dk^2}$ need to be Fourier invertible to reproduce the x-space version of the problem. One should as well be aware that with a resolved $\rho_*^{1/2}$, we must have secured the validity of the direct Fourier transform, that is explicit in Eq. (46).

5.1. *Case of* $\mu = 0$

We seriously address a Fourier transformed equation (46) and employ again Eq. (36), while blindly setting $\mu = 0$ therein. The "raw" solution of (36) can be obtained from (40). It has the form

$$f(k) = \begin{cases} C_{12} e^{-k\sqrt{2}}\,, & k \geq 0\,, \\ C_{21}\cos k\sqrt{2} + C_{22}\sin k\sqrt{2}\,, & k < 0\,. \end{cases} \tag{47}$$

The continuity condition at $k = 0$ for $f(k)$ reads

$$C_{12} = C_{21} \tag{48}$$

and for the derivative

$$C_{22} = -C_{12} = -C_{21}\,, \tag{49}$$

so that (we set $C_{12} \equiv C$)

$$f(k) = C \begin{cases} e^{-k\sqrt{2}}\,, & k \geq 0\,, \\ \cos k\sqrt{2} - \sin k\sqrt{2} \equiv \sqrt{2}\cos\left(k\sqrt{2} + \frac{\pi}{4}\right)\,, & k < 0\,. \end{cases} \tag{50}$$

The first maximum of oscillatory part is located at

$$k_{\rm m} = -\frac{\pi}{4\sqrt{2}} \approx -0.555360367 \tag{51}$$

in accordance with Table I. The "raw" solution (50) is shown (along with the position of $k_{\rm m}$) in Fig. 1.

Now we shift the whole solution to the right and "chop" the unnecessary piece of an oscillatory part

$$f(k) = C \begin{cases} \sqrt{2}\cos k\sqrt{2}\,, & 0 \leq k \leq -k_{\rm m}\,, \\ e^{-(k+k_{\rm m})\sqrt{2}}\,, & k > -k_{\rm m}\,. \end{cases} \tag{52}$$

The normalization condition (45) reads

$$2C^2 \left[2\int\limits_0^{-k_{\rm m}} \cos^2 k\sqrt{2}dk + \int\limits_{-k_{\rm m}}^{\infty} e^{-2\sqrt{2}(k+k_{\rm m})}dk \right] = 1 \tag{53}$$

so that $C = \frac{1}{\sqrt[4]{2}\sqrt{1+\frac{\pi}{4}}} \approx 0.629325$. This normalization coefficient is different from that in Table I, because the transition from Bessel functions of index $1/2$ to elementary ones introduces an auxiliary coefficient $\sqrt{\pi}/2^{3/4}$. Thus, we have $C^2_{\rm table} = (2\sqrt{2}/\pi)C^2 = (2/\pi)/(1+\pi/4)$. This minor difference is insignificant with respect to the normalizability of the pertinent eigenfunction, as an overall coefficient before $f(k)$ is just C, (53).

Now we invert the Fourier transform to get

$$\begin{aligned} f(x) &= \frac{2}{\sqrt{2\pi}} \int\limits_0^{\infty} f(k)\cos\ kx\ dk \\ &= C\sqrt{\frac{2}{\pi}} \left[\sqrt{2} \int\limits_0^{-k_{\rm m}} \cos\ k\sqrt{2}\ \cos\ kx\ dk + e^{-k_{\rm m}\sqrt{2}} \int\limits_{-k_{\rm m}}^{\infty} e^{-k\sqrt{2}} \cos\ kx\ dk \right] \\ &= C\sqrt{\frac{2}{\pi}} \frac{4}{x^4-4} \left(x\ \sin\frac{\pi x}{4\sqrt{2}} - \sqrt{2}\cos\frac{\pi x}{4\sqrt{2}} \right)\,, \qquad C = \frac{1}{\sqrt[4]{2}\sqrt{1+\frac{\pi}{4}}}\,. \end{aligned} \tag{54}$$

We note that at the point $x = \pm\sqrt{2}$, the function $f(x)$ looks divergent. However, since the terms in the numerator yield the same zero as in the denominator, the potentially dangerous divergence is removed when one approaches $x = \pm\sqrt{2}$. The function $f(x)$ decays at spatial infinities as $1/x^3$ and is positive for $|x| < 5$. For $|x| > 5$ we encounter oscillations so that both zeroes and negative values are developed. Thus, it is clear the function $f(x)$ cannot be interpreted as an (arithmetic) square root of a probability density $\rho_*(x)$. By this reason alone the obtained solution is incongruent with the x-space version of the compatibility condition.

More than that, an inversion of the Fourier transformed equation Eq. (46) back to the x-space leads to a contradiction, since $\frac{x^2}{2}f(x) \neq -f(x)$. The roots of this discrepancy are obvious. The function (54) at spatial infinities behaves as $f(x) \sim \frac{1}{x^3}\sin\frac{\pi x}{4\sqrt{2}}$. Consequently, $x^2 f(x) \sim \frac{1}{x}\sin\frac{\pi x}{4\sqrt{2}}$ is not absolutely integrable on R and thus not Fourier transformable.

5.2. *Case of* $\mu = 2$

Concerning $\mu = 2$, we recall the eigenvalue equation $(-\Delta + \frac{x^2}{2} - E_0)$ $\rho_*^{1/2} = 0$, where $E_0 = 1/2$ is the lowest eigenvalue for a quantum harmonic oscillator in the units $\hbar^2/2m = 1$, $\omega^2 = m$, where m is a particle mass and ω is an oscillator frequency. That gives rise to the Gaussian ground state function as a celebrated textbook result.

From this point of view, the adopted (Sec. 4) form of the compatibility condition (possibly, with a notable exception of $\mu = 1$) might seem puzzling. Specifically, the standard eigenvalue equation for the harmonic oscillator is plainly incompatible with the $(-\Delta + \frac{x^2}{2})\rho_*^{1/2} = 0$ demand, expected (per force) to correspond to the case of $\mu = 2$ in the considerations of Sec. 4, *cf.* Eq. (46).

Nonetheless, the latter equation seems to admit a non-Gaussian heavy-tailed solution, which is derivable from Eqs. (37)–(41). In particular, an asymptotic ($k \to \infty$) behavior of

$$K_{1/2q}\left(\frac{\sqrt{2}}{q}k^q\right) \sim k^{-q/2}\exp\left[-\frac{\sqrt{2}}{q}k^q\right] \tag{55}$$

with $q = (\mu+2)/2$, $\mu \in (0, 2]$ yields an asymptotics in the k-space for $\mu = 2$, *cf.* (41), as follows: $f(k) \sim (1/\sqrt{k})\exp[-k^2/\sqrt{2}]$. That secures an absolute convergence of the inverse Fourier integral.

We have no sufficient analytic tools in hands to analyze various features of the ultimate x-space solution. However, the performed numerical (computer assisted) analysis proves that both $f(x)$ and $x^2 f(x)$ are absolutely integrable.

The same occurs for $[x^2 f(x)]^2$. The pertinent $f(x) \equiv \rho_*^{1/2}(x)$ has an inverse polynomial decay at infinities, decaying faster than $1/x^3$. In the vicinity of $x = 0$ the obtained function mimics a Gaussian. Those properties are visualized in Figs. 2 and 3.

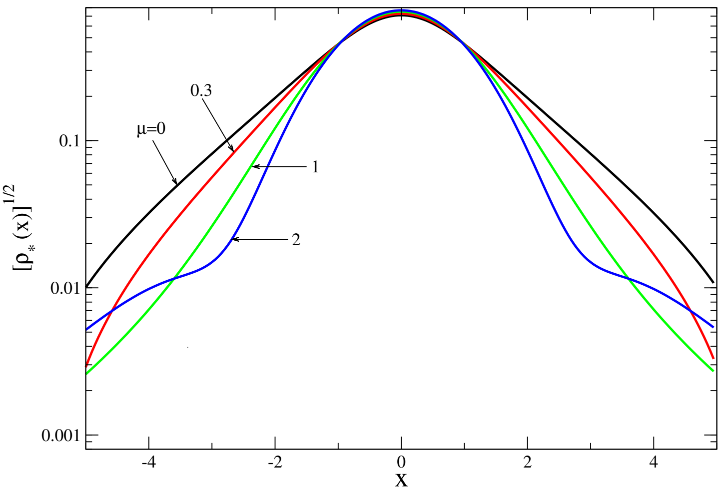

Fig. 2. $f(x) = \rho_*^{1/2}(x)$ for $\mu = 0$, 0.3, 1, 2. The logarithmic scale is employed to better visualize the behavior on the "tails".

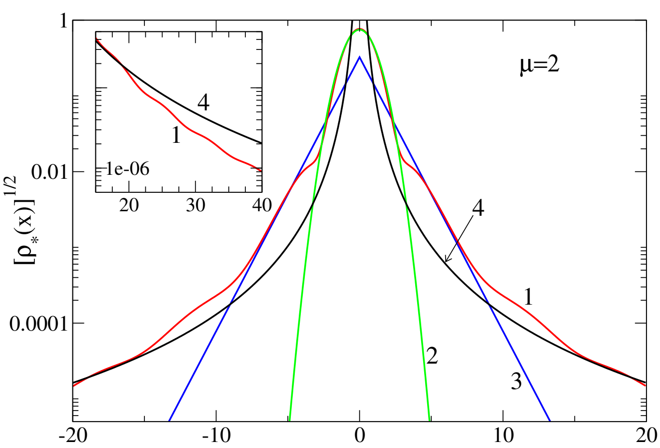

Fig. 3. Different asymptotic fits (curves 2, 3, 4) of $f(x) = (\rho_*(x))^{1/2}$ (curve 1) for $\mu = 2$. Figures near curves: 2 — $0.749 \exp(-x^2/2)$, 3 — $0.15 \exp(-|x|/1.2)$, 4 — $0.13/x^3$. Inset shows that $f(x)$ decays faster then $1/x^3$.

The annoying point that should be clearly spelled out in connection with our derivation procedure may pertain to the point $0 \in R$ and the fact that we have actually (albeit tacitly) defined the Laplacian not on the whole of R, but on $\{R\backslash 0\}$. A subsequent comment [21] indicates where the peculiarity of our solution may possibly be rooted.

Comment 1: To make a clear distinction with a textbook quantum mechanical reasoning for the harmonic oscillator problem, let us point out that what we have actually obtained appears to be related to the (normally discarded) "improper" eigenvalue $n = -1/2$ in the familiar $E_n = \hbar\omega(n + 1/2)$ formula. Here, we emphasize the textbook restriction upon the harmonic oscillator eigenvalues $n \geq -1/2$, whose interpretation is: "all states corresponding to $n < -1/2$ need to vanish identically", [20]. A sufficient condition to this end is that the lowest energy eigenfunction vanishes under the action of the annihilation operator. This actually selects $n = 0$ as the bottom spectral label and $n \in (N \bigcup 0)$. Our reasoning demonstrates that it is not a necessary condition, and actually, an "improper" eigenvalue $n = -1/2$ is admissible as a legitimate spectral label in the harmonic oscillator problem. However in this case, a specific choice of the boundary data (implicit in the construction procedure but hitherto not explicitly spelled out) appears to enter the game. That may have an effect on the spectral features (including that of the proper domain definition and the self-adjointness issue) of the original problem, see *e.g.* the next comment.

Comment 2, [21]: Generically, the Laplacian $-\Delta$, if defined on a Schwartz space $S(R)$ (fast decaying functions, continuous and with continuous derivatives), is non-negative and essentially self-adjoint operator. Therefore, its closure, which is a self-adjoint operator, is non-negative as well. Its continuous spectrum extends from 0 to $+\infty$.

However, if we define the Laplacian on $S(\{R\backslash 0\})$, the operator remains non-negative but ceases to be essentially self-adjoint. Its deficiency indices read $(2, 2)$, implying an existence of a two-parameter family of self-adjoint extensions. There is a well known in the literature example of such extensions determined by the boundary data $f'(+0) - f'(-0) = \alpha f(0)$. On the operator level so point-wise restricted operator $-\Delta_\alpha$ may be symbolically represented $-\Delta + \alpha\delta(x)$, where $\delta(x)$ stands for Dirac delta distribution.

For $\alpha = 0$ we have a "normal" Laplacian. The case of $\alpha \to +\infty$ sets Dirichlet boundary condition at 0 (an impenetrable barrier). If $\alpha > 0$, the operator $-\Delta_\alpha$ has a non-negative absolutely continuous spectrum.

The case of $\alpha < 0$ is particularly interesting, since then $-\Delta_\alpha$ has one non-degenerate isolated negative eigenvalue $-\alpha^2/2$ with an eigenfunction $|\alpha|^{1/2} \exp(-|\alpha x|/2)$, see *e.g.* [22]. The remaining part of the spectrum begins at 0 and is absolutely continuous up to $+\infty$.

An operator $-\Delta + x^2$, defined on $S(\{R\backslash 0\})$, is regarded as positive and symmetric, but is not essentially self-adjoint. The deficiency indices are also $(2, 2)$. Assuming the previous boundary conditions, we turn over to the one-parameter family of self-adjoint extensions $-\Delta_\alpha + x^2$, assuming likewise $\alpha < 0$.

The choice of $\alpha = 0$ leads to the standard harmonic oscillator problem. while for negative α there appears a fairly non-standard spectral solution described in detail in Ref. [22]. The problem has exactly one negative eigenvalue plus a positive discrete spectrum of the dimensionless form $\epsilon = \nu + 1/2$.

Given negative α, the eigenvalues ϵ come out from a transcendental equation $\nu - \alpha \Gamma(1-\nu/2)/\Gamma(1/2-\nu/2) = 0$. The value $\epsilon = 0$ appears for $\nu = -1/2$ *i.e.* for $\alpha = -\Gamma(3/4)/\Gamma(5/4)$. The corresponding eigenfunction shows up a $\sim x^{-1/2} \exp(-x^2/2)$ decay at $+\infty$, [22].

6. Conclusions

To conclude, here we have presented an outline of the general formalism for how to find an invariant PDF for a pre-defined semigroup potential and all stability index values $0 < \mu < 2$. We have considered the generic case of symmetric even PDFs, for which a potential $\mathcal{V}(x)$ is an even function. That was dictated by known spectral properties of fractional operators associated with a symmetric stable noise. We have reduced the problem of finding a terminal PDF to that of finding a solution of the ordinary differential equation with infinite number of terms in momentum space. Such differential equation, even if hard to handle analytically, can rather easily be solved with a numerical assistance. For polynomial potentials the number of terms becomes finite and the pertinent equation can be solved analytically.

The outlined procedure has been explicitly performed for the family of Lévy stable oscillators, with a common quadratic semigroup potential $\mathcal{V}(x) = x^2/2$, but with *no* standard bottom energy eigenvalue renormalization.

In this case, a solution of the corresponding differential equation in k-space has been obtained by employing a suitable continuity procedure at $k = 0$. After Fourier-inversion and squaring the result, this yields an ultimate functional form of the desired invariant (and possibly terminal) PDF of the jump-type process, for arbitrary $\mu \in (0, 2)$.

We have analyzed a limiting behavior of solutions in the vicinity and at the boundaries $\mu = 0$ and 2 of an open stability interval $(0, 2)$. In the case of $\mu = 0$, we have derived an explicit analytical form of the solution in k- and (by Fourier inversion) in the x-spaces. The pertinent function shows a nontrivial oscillating behavior and thus is definitely not a square root of any PDF. Moreover, with an explicit analytic form of $f(x)$ one easily proves that the $x^2 f(x)$ is not-Fourier transformable, hence the crucial connection between the x-space and k-space solutions, Eq. (46) has been lost.

We have shown that for $\mu = 2$ a positive $\rho_*^{1/2}(x)$ is obtained. This invariant solution definitely corresponds to the "improper eigenvalue" $\epsilon = 0$ of the energy operator. To comply with the common knowledge about the spectral properties of the operator $-\Delta + x^2$, we find that the (hitherto neglected) presence of the special boundary data has been enforced. Then only the self-adjoint extensions of the corresponding non-negative operator may be introduced, and $\rho_*^{1/2}(x)$ may receive a consistent interpretation as an asymptotic (terminal) target in the semigroup evolution.

This particular example indicates, in turn, the main jeopardy hidden in the presented reconstruction problem. Once an invariant PDF has been settled, it is necessary to identify a consistent self-adjointness domain for the involved operator $\hat{H}$. Then only an invariant PDF may be interpreted as an asymptotic (terminal) one for a well defined semigroup $\exp(-t\hat{H})$.

A problem of reconstructing a self-adjoint operator from the knowledge of its ground state is not trivial at all, see *e.g.* Ref. [23] for an investigation of the diffusion-type framework. In the harmonic oscillator problem of Comment 2 it is known that a corresponding self-adjoint Hamiltonian shares odd-parity (label $n = 1, 3, 5, ...$) part of the spectrum with a "normal" harmonic oscillator, while an even-labeled part is substantially different, that extends to the associated eigenfunctions as well.

We are indebted to Professor Witold Karwowski for enlightening discussions and an explanation of the role of boundary data in discriminating between the "normal" and "abnormal" (zero bottom eigenvalue) harmonic oscillator spectral problems.

Appendix A

Continuity conditions at $k = 0$

From (40), the functions at $k = 0$ read

$$C_{12}\left[\sqrt{k}K_\nu(u)\right]_{k=0} = C_{22}\left[\sqrt{k}N_\nu(u)\right]_{k=0} . \tag{A.1}$$

The derivatives at $k = 0$

$$C_{21}\left[\sqrt{k}J_\nu(u)\right]'_{k=0} + C_{22}\left[\sqrt{k}N_\nu(u)\right]'_{k=0} = C_{12}\left[\sqrt{k}K_\nu(u)\right]'_{k=0} . \tag{A.2}$$

Such forms of (A.1) and (A.2) are dictated by the following asymptotic expansions of Bessel functions near $k = 0$ in variables (42)

$$
\begin{aligned}
K_\nu(u) &\approx \frac{\Gamma(-\nu)}{2}\sqrt{k}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu + \frac{\Gamma(\nu)}{2\sqrt{k}}\left[\frac{\mu+2}{\sqrt{2}}\right]^\nu , \\
N_\nu(u) &\approx -\frac{\cos\pi\nu\ \Gamma(-\nu)}{\pi}\sqrt{k}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu - \frac{\Gamma(\nu)}{\pi\sqrt{k}}\left[\frac{\mu+2}{\sqrt{2}}\right]^\nu , \\
J_\nu(u) &\approx \frac{\sqrt{k}}{\Gamma(1+\nu)}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu .
\end{aligned} \tag{A.3}
$$

Eq. (A.3) means that $[\sqrt{k}J_\nu(u)]_{k=0} = 0$. For reference purposes, the derivatives like $[\sqrt{k}N_\nu(u)]'_{k=0}$ (we first multiply by $\sqrt{k}$ and then differentiate) read

$$
\begin{aligned}
\left[\sqrt{k}N_\nu(u)\right]'_{k=0} &= -\frac{\cos\pi\nu\ \Gamma(-\nu)}{\pi}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu , \\
\left[\sqrt{k}K_\nu(u)\right]'_{k=0} &= \frac{\Gamma(-\nu)}{2}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu , \\
\left[\sqrt{k}J_\nu(u)\right]'_{k=0} &= \frac{1}{\Gamma(1+\nu)}\left[\frac{\sqrt{2}}{\mu+2}\right]^\nu .
\end{aligned} \tag{A.4}
$$

Substitution of values of functions and derivatives into (A.1) and (A.2) yields

$$
C_{22} = -\frac{\pi}{2}C_{12}\,, \qquad C_{21} = \frac{\pi}{2}C_{12}\cot\frac{\pi\nu}{2}\,, \tag{A.5}
$$

which, after employing the identity $\Gamma(1+\mu)\Gamma(-\mu) = -\pi/\sin(\pi\mu)$ gives rise to Eq. (41).

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