Vol. 43 (2012) *ACTA PHYSICA POLONICA B* No 8

ON A TIME-CHANGED GEOMETRIC BROWNIAN MOTION AND ITS APPLICATION IN FINANCIAL MARKET

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(Received December 27, 2011; revised version received April 30, 2012)

In this paper, we introduce a time-changed geometric Brownian motion and investigate the corresponding martingale properties and fractional Fokker–Planck type equation. As an application, we prove that the market model considered is arbitrage-free and gives pricing formulae for the prices of European call options when the underlying asset price follows the time-changed geometric Brownian motion.

DOI:10.5506/APhysPolB.43.1667
PACS numbers: 02.50.–r, 02.50.Ey, 89.65.Gh, 05.40.Fb

1. Introduction

Recent developments in the area of statistical physics confirm that the classical diffusion models based on the Brownian motion fail to provide satisfactory description of many complex systems (*e.g.* see [1]). Therefore, systems exhibiting anomalous diffusive behavior, especially subdiffusive behavior, attract growing attention in many fields, including physics, finance, biophysics and so on (*e.g.* [2,3,4,5,6]). The common description of subdiffusive processes is in terms of the fractional Fokker–Planck equation (FFPE) which is first derived from the continuous-time random walk (CTRW) with heavy-tailed waiting times [2, 7, 8]. The CTRW model has been proved to be a useful tool for the description of systems out of equilibrium [2, 9], especially of anomalous diffusion phenomena. In the CTRW model without external force, the motion of a particle is completely determined by the two probability density functions (PDFs), namely, jump length PDF and waiting

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time PDF. Different types of CTRW models come out through specifying waiting time PDFs. If the waiting times between consecutive jumps follow a power law, then the CTRW results in a subordinate process (or anomalous diffusion) $X(T_\alpha(t))$, where $X(\tau)$ is a stable Lévy process and $T_\alpha(t)$ is the inverse α-stable subordinator independent of $X(\tau)$ [10,11]. In particular, if the jump lengths satisfy certain conditions, $X(\tau)$ becomes the Brownian motion $B(\tau)$. Recently, Magdziarz theoretically showed that this type of subdiffusions can be used to study option pricing [12] and subsequently Liang *et al.* [13] generalized Magdziarz's model to a composite-diffusive regime. Meanwhile, Janczura and Wyłomańska [14] presented two examples of economic data exhibiting subdiffusive behavior and modeled the market data using the subdiffusion with a constant force. In [12], Magdziarz introduced a subdiffusive geometric Brownian motion (SGBM) and gave the corresponding FFPE and the Black–Scholes formulae for the fair prices of European options when the underlying asset price is given by the SGBM $Z(T_\alpha(t))$, where $T_\alpha(t)$ is the inverse α-stable subordinator defined in the following way

$$T_\alpha(t) = \inf\{\tau > 0 : U_\alpha(\tau) > t\}\,, \qquad 0 < \alpha < 1\,, \tag{1}$$

$\{U_\alpha(\tau)\}_{\tau\geq 0}$ is the strictly increasing α-stable Lévy process with Laplace transform $\mathbb{E}(e^{-uU_\alpha(\tau)}) = e^{-\tau u^\alpha}$, $Z(\tau)$ follows a geometric Brownian motion

$$Z(\tau) = Z_0 \exp\{\mu\tau + \sigma B(\tau)\}\,, \qquad Z_0 > 0\,, \tag{2}$$

or is equivalently defined in the form of the Itô stochastic differential equation

$$dZ(\tau) = \left(\mu + \frac{\sigma^2}{2}\right) Z(\tau)dt + \sigma Z(\tau)dB(\tau)\,, \qquad Z(0) = Z_0 > 0\,, \tag{3}$$

with constant drift μ and volatility σ, and $B(\tau)$ is the standard Brownian motion independent of $T_\alpha(t)$.

The inverse α-stable subordinator $T_\alpha(t)$ is continuous and nondecreasing and hence it can be used as a time-change process. In this paper, we extend the notion of time-change process to the more general case. That is, by replacing the time-change process $T_\alpha(t)$ by the first-passage time process $T_\nu(t)$ of a mixture of stable subordinators w.r.t. (with respect to) a Borel probability measure ν on $(0,1)$, we introduce a new time-changed geometric Brownian motion

$$Z_\nu(t) = Z(T_\nu(t))\,, \tag{4}$$

where $Z(\tau)$ satisfies (3) and the process $B(\tau)$ is independent of $T_\nu(t)$ (see Section 2) which contains previously SGBM $Z(T_\alpha(t))$ as a special subclass. Inspired by the idea and method in [12], we show some properties of the processes related to $Z_\nu(t)$ and discuss the generalized Black–Scholes formula of the European options when the asset prices are described by the time-changed Brownian motion.

This paper is organized as follows. In Section 2, a time-changed geometric Brownian motion and the corresponding fractional Fokker–Planck type equation are discussed. In Section 3, martingale properties for the processes $B(T_\nu(t))$ and $Z_\nu(t)$ are obtained. In Section 4, as an application, the generalized Black–Scholes type formulae of the European options are given when the asset prices are described by the time-changed Brownian motion.

2. Time-changed geometric Brownian motion

2.1. Subordinator and its inverse

A Lévy process $\{U(\tau)\}_{\tau\geq 0}$ with nonnegative increments is called a *subordinator*. The Laplace transform of $U(\tau)$ has the form

$$\mathbb{E}\left[e^{-uU(\tau)}\right] = e^{-\tau\psi(u)}\,, \qquad u \geq 0\,,$$

with the Laplace exponent $\psi(u)$ given by

$$\psi(z) = \beta z + \int\limits_{(0,+\infty)} \left(1 - e^{-xz}\right)\, d\rho(x)$$

for any complex z with $\operatorname{Re} z \geq 0$, where $\beta \geq 0$ is a drift parameter and ρ is a measure satisfying $\int_{(0,+\infty)} \min\{1, x\}\, d\rho(x) < +\infty$ [15] which is called the *Lévy measure* of $\{U(\tau)\}_{\tau\geq 0}$. In particular, when $\psi(u) = u^\alpha$, $\alpha \in (0,1)$, $U(\tau) = U_\alpha(\tau)$ is a *α-stable subordinator*. Given a subordinator $\{U(\tau)\}_{\tau\geq 0}$, the first-passage time process defined by

$$T(t) := \inf\{\tau > 0 : U(\tau) > t\}$$

is called the *inverse subordinator* of $U(\tau)$.

Let ν be a Borel probability measure on $(0,1)$ and $\psi_\nu(u) = \int_{(0,1)} u^x\, d\nu(x)$. Consider a stochastic processes $\{U_\nu(\tau)\}_{\tau\geq 0}$ with Laplace transform $\mathbb{E}e^{-uU_\nu(\tau)} = e^{-\tau\psi_\nu(u)}$. It can be shown that the process $\{U_\nu(\tau)\}_{\tau\geq 0}$ has the same one-dimensional distributions with some subordinator (see Theorem 2.1).

Lemma 2.1 *([15], p. 216) For every $\alpha \in (0,1)$, the Lévy measure of $U_\alpha(t)$ is absolutely continuous w.r.t. the Lebesgue measure on $(0,+\infty)$ with the density function $h_\alpha(x) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-\alpha-1}$, $x > 0$.*

Theorem 2.1 *The stochastic process* $\{U_\nu(\tau)\}_{\tau\geq 0}$ *with the Laplace transform* $\mathbb{E}e^{-uU_\nu(\tau)} = e^{-\tau\psi_\nu(u)}$ *has the same one-dimensional distributions with a subordinator, where* $\psi_\nu(u) = \int_{(0,1)} u^x\, d\nu(x)$ *and* ν *is a Borel probability measure on* $(0,1)$.

Proof. It follows from Lemma 2.1 that, for $u \geq 0$,

$$\begin{aligned}\psi_\nu(u) &= \int\limits_{(0,1)} u^\alpha d\nu(\alpha) = \int\limits_{(0,1)} \int\limits_{(0,+\infty)} (1-e^{-ux})\frac{\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}dxd\nu(\alpha) \\ &= \int\limits_{(0,+\infty)} (1-e^{-ux}) \int\limits_{(0,1)} \frac{\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}d\nu(\alpha)dx < +\infty\,.\end{aligned}$$

If set $h_\nu(x) = \int_{(0,1)} \frac{\alpha}{\Gamma(1-\alpha)}x^{-\alpha-1}d\nu(\alpha)$, then $h_\nu(x) < +\infty$ for arbitrary $x > 0$ and

$$\int\limits_{(0,1]} xh_\nu(x)dx + \int\limits_{(1,+\infty)} h_\nu(x)dx < +\infty\,.$$

Let ρ_ν be the Borel measure on $(0,+\infty)$ satisfying $d\rho_\nu(x) = h_\nu(x)dx$. Taking advantage of Lévy–Itô decomposition, there exits a subordinator which has the Laplace exponent $\psi_\nu(u)$ and Lévy measure ρ_ν. That is, this subordinator has the same one-dimensional distributions with $\{U_\nu(\tau)\}_{\tau\geq 0}$ and the theorem holds.

Therefore, in this paper we always denote $\{U_\nu(\tau)\}_{\tau\geq 0}$ as the subordinator with the Laplace exponent $\psi_\nu(u)$.

2.2. Time-changed geometric Brownian motion

We introduce a time-changed geometric Brownian motion $Z_\nu(t) = Z(T_\nu(t))$, where ν and $\psi_\nu(u)$ are given in Subsection 2.1, $Z(\tau)$ satisfies (3), $T_\nu(t)$ is the inverse of the subordinator $U_\nu(t)$ with the Laplace exponent $\psi_\nu(u)$ and the processes $B(\tau)$ and $T_\nu(t)$ are independent. As a special case, when $\nu = \delta_\alpha$, the Dirac measure concentrated on a single point $\alpha \in (0,1)$, the processes $U_\nu(\tau)$ and $Z_\nu(t)$ become respectively the α-stable subordinator $U_\alpha(\tau)$ with Laplace transform $\mathbb{E}e^{-uU_\alpha(\tau)} = e^{-\tau u^\alpha}$ and the SGBM $Z(T_\alpha)$ introduced by Magdziarz [12]. Therefore, $Z_\nu(t) = Z(T_\nu(t))$ is a generalization of the SGBM. The process U_ν represents a mixture of independent stable subordinators w.r.t. the measure ν. For example, if $\nu = c_1\delta_{\alpha_1} + c_2\delta_{\alpha_2}$ with $\alpha_1, \alpha_2 \in (0,1)$, then $U_\nu(t) = c_1^{1/\alpha_1}U_{\alpha_1}(t) + c_2^{1/\alpha_2}U_{\alpha_2}(t)$ is the mixture of two independent stable subordinators $U_{\alpha_1}(t)$ and $U_{\alpha_2}(t)$, where $c_1, c_2 > 0$ are constants and $c_1 + c_2 = 1$. Moreover, if ν is a weighted sum of finite Dirac measures, then the following property holds. Fig. 1 shows simple realizations of the processes $Z_\nu(t)$ corresponding different parameters.

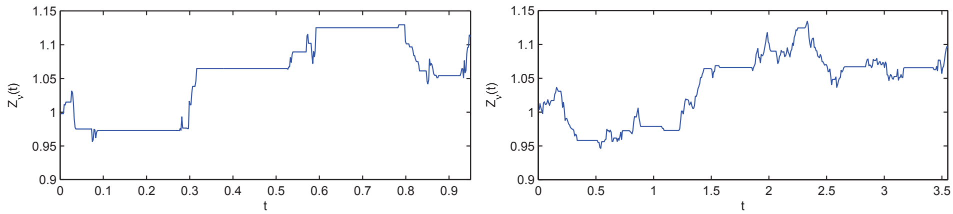

Fig. 1. Simple realizations of the processes $Z_\nu(t)$ corresponding different parameters $\mu = 0.05$, $\sigma = 0.1$, $Z_\nu(0) = 1$ and ν, where $\nu = \delta_{0.6}$ in the left figure, and $\nu = \frac{1}{2}\delta_{0.6} + \frac{1}{2}\delta_{0.9}$ in the right figure.

Property 2.1 *If* $\nu = \sum\limits_{i=1}^{n} c_i\delta_{\alpha_i}$, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n < 1$, $0 < c_i < 1$, $\sum\limits_{i=1}^{n} c_i = 1$, *then* $\psi_\nu(k) = \sum\limits_{i=1}^{n} c_i k^{\alpha_i}$ *and*

$$\begin{aligned}\mathbb{E}T_\nu(t) &= \mathscr{L}^{-1}_{k\to t}\left\{\frac{1}{k\psi_\nu(k)}\right\}(t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^{kt}}{\sum\limits_{i=1}^{n} c_i k^{\alpha_i+1}}dk \\ &= \frac{t^{\alpha_1}}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^{z}}{\sum\limits_{i=1}^{n} c_i t^{\alpha_1-\alpha_i} z^{\alpha_i+1}}dz\,.\end{aligned}$$

Note that

$$\begin{aligned}&\left|\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{e^{z}}{\sum\limits_{i=1}^{n} c_i t^{\alpha_1-\alpha_i} z^{\alpha_i+1}}dz\right| \le \frac{1}{\pi}\int_{0}^{+\infty}\frac{e}{\left|\sum\limits_{i=1}^{n} c_i t^{\alpha_1-\alpha_i}(1+ix)^{\alpha_i+1}\right|}dx \\ &\le \frac{1}{\pi}\int_{0}^{+\infty}\frac{e}{c_1(1+x^2)^{\frac{\alpha_1+1}{2}}}dx < +\infty\,.\end{aligned}$$

Then, there exists a positive number $M(\alpha_1) < +\infty$ *such that* $\mathbb{E}T_\nu(t) = \mathbb{E}B^2(T_\nu(t)) \le M(\alpha_1)t^{\alpha_1}$*.*

In general, the PDF of the time-changed geometric Brownian motion $Z_\nu(t) = Z(T_\nu(t))$ satisfies the following fractional Fokker–Planck equation, which first appeared in [16]. Here, we give a little different proof.

Theorem 2.2 *The PDF $w(x,t)$ of $Z_\nu(t)$ satisfies the fractional Fokker–Planck type equation*

$$\int\limits_{(0,1)} {}_0^C D_t^\alpha w(x,t) d\nu(\alpha) = -\left(\mu + \frac{\sigma^2}{2}\right) \frac{\partial}{\partial x} x w(x,t) + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} x^2 w(x,t)\,,$$

where ${}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-x)^{-\alpha} f'(x) dx$ is Caputo's fractional derivative.

Proof. The random variables (RV) and their PDFs are listed as follows

RV	$U_\nu(\tau)$	$T_\nu(t)$	$Z(t)$	$Z_\nu(t)$
PDF	$s(t,\tau)$	$l(\tau,t)$	$z(x,t)$	$w(x,t)$

Denote by $\hat{l}(\tau,k) = \mathscr{L}_{t\to k}\{l(\tau,t)\}(k)$ the Laplace transform of $l(\tau,t)$ w.r.t. t. Then

$$\begin{aligned}
\hat{l}(\tau,k) &= \mathscr{L}_{t\to k}\{l(\tau,t)\}(k) = \mathscr{L}_{t\to k}\left\{\frac{\partial}{\partial\tau}\Pr(T_\nu(t)\le\tau)\right\}(k) \\
&= -\mathscr{L}_{t\to k}\left\{\frac{\partial}{\partial\tau}\Pr(U_\nu(\tau)\le t)\right\}(k) = -\frac{\partial}{\partial\tau}\mathscr{L}_{t\to k}\left\{\int\limits_0^t s(x,\tau)dx\right\}(k) \\
&= -\frac{\partial}{\partial\tau}\left(\frac{1}{k}e^{-\tau\psi_\nu(k)}\right) = \frac{\psi_\nu(k)}{k}e^{-\tau\psi_\nu(k)}\,. \qquad (5)
\end{aligned}$$

By using the total probability formula and (5), we get

$$\begin{aligned}
w(x,t) &= \int\limits_0^{+\infty} z(x,\tau) l(\tau,t) d\tau\,, \qquad w(x,0) = z(x,0) \\
\hat{w}(x,k) &= \int\limits_0^{+\infty} z(x,\tau)\hat{l}(\tau,k)d\tau = \int\limits_0^{+\infty} z(x,\tau)\frac{\psi_\nu(k)}{k}e^{-\tau\psi_\nu(k)}d\tau \\
&= \frac{\psi_\nu(k)}{k}\hat{z}\left(x,\psi_\nu(k)\right)\,. \qquad (6)
\end{aligned}$$

Since $Z(t)$ is given by (3), $z(x,t)$ satisfies the standard Fokker–Planck equation

$$\frac{\partial z(x,t)}{\partial t} = -\left(\mu + \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} x z(x,t) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} x^2 z(x,t) \qquad (7)$$

with the boundary condition $z(x,0)=\delta_{Z_0}(x)$. The Laplace form of (7) is

$$k\hat{z}(x,k)-z(x,0)=-\left(\mu+\frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}x\hat{z}(x,k)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2\hat{z}(x,k)\,.$$

Replacing k by $\psi_\nu(k)$, we get

$$\begin{aligned}&\psi_\nu(k)\hat{z}(x,\psi_\nu(k))-z(x,0)\\&=-\left(\mu+\frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}x\hat{z}(x,\psi_\nu(k))+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2\hat{z}(x,\psi_\nu(k))\,.\end{aligned}$$

Noting that $w(x,0)=z(x,0)=\delta_{Z_0}(x)$ and (6), we have

$$\begin{aligned}&\psi_\nu(k)\hat{w}(x,k)-\frac{\psi_\nu(k)}{k}w(x,0)\\&=-\left(\mu+\frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}x\hat{w}(x,k)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2\hat{w}(x,k)\,.\end{aligned}\tag{8}$$

Taking the inverse Laplace transform of both sides of (8), from Theorem 2.1 and

$$\mathscr{L}_{t\to k}\left\{{}_0^CD_t^\alpha f(t)\right\}(k)=k^\alpha\mathscr{L}_{t\to k}\{f(t)\}(k)-k^{\alpha-1}f(0)\,,\quad [22]$$

we obtain the result of the theorem

$$\int\limits_{(0,1)}{}_0^CD_t^\alpha w(x,t)d\nu(\alpha)=-\left(\mu+\frac{\sigma^2}{2}\right)\frac{\partial}{\partial x}xw(x,t)+\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}x^2w(x,t)\,.$$

Remark: The equation in Theorem 2.2 can be seen as a diffusion equation with fractional derivative of distributed-order, which is widely used in the kinetic description of anomalous diffusion. In this field, there are some important papers (*e.g.* [17, 18, 19, 20]).

3. Martingale properties

For convenience, we assume that the Brownian motion $\{B(t)\}_{t\geq 0}$, the subordinator $\{U_\nu(t)\}_{t\geq 0}$ and its inverse $\{T_\nu(t)\}_{t\geq 0}$ are defined on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$. $\{B(t)\}_{t\geq 0}$ with the Brownian filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is continuous, U_ν is strictly increasing with *càdlàg* (right-continuous with left limits) trajectories and so $\{T_\nu(t)\}_{t\geq 0}$ is continuous. Let $\widetilde{\mathscr{F}}:=\sigma\left(T_\nu(t):t\geq 0\right)$ be independent of the sub-algebra $\sigma\left(\mathscr{F}_t:t\geq 0\right)$ of $\mathscr{F}$ and $\mathscr{G}_t=\mathscr{F}_t\vee\widetilde{\mathscr{F}}:=\sigma\left(\mathscr{F}_t,\widetilde{\mathscr{F}}\right)$. Then, we can obtain the following martingale properties.

Lemma 3.1 *$\mathbb{E}\left[e^{T_\nu(t)}\right] < +\infty$, $\mathbb{E}\left[T_\nu^n(t)\right] < +\infty$ and $\mathscr{L}_{t\to k}\left\{\mathbb{E}T_\nu^n(t)\right\}(k) = \frac{n!}{k\psi_\nu^n(k)}$, $n \in \mathbb{N}$.*

Proof. It follows from (5) that

$$\begin{aligned}\mathscr{L}_{t\to k}\left\{\mathbb{E}\left[e^{T_\nu(t)}\right]\right\}(k) &= \mathscr{L}_{t\to k}\left\{\int_0^{+\infty} e^x l(x,t)dx\right\}(k) = \int_0^{+\infty} e^x \hat{l}(x,k)dx \\ &= \int_0^{+\infty} \frac{\psi_\nu(k)}{k} e^{-x(\psi_\nu(k)-1)}dx\,. \end{aligned} \tag{9}$$

Note that if $k > 1$, then $\psi_\nu(k) > 1$. Thus it follows from (9) that $\mathscr{L}_{t\to k}\left\{\mathbb{E}\left[e^{T_\nu(t)}\right]\right\}(k) < +\infty$ which implies $\mathbb{E}\left[e^{T_\nu(t)}\right] < +\infty$. As a direct conclusion, for any $n \in \mathbb{N}, \mathbb{E}\left[T_\nu^n(t)\right] < +\infty$. Furthermore,

$$\begin{aligned}\mathscr{L}_{t\to k}\left\{\mathbb{E}\left[T_\nu^n(t)\right]\right\}(k) &= \mathscr{L}_{t\to k}\left\{\int_0^{+\infty} x^n l(x,t)dx\right\}(k) = \int_0^{+\infty} x^n \hat{l}(x,k)dx \\ &= \frac{1}{k\psi_\nu^n(k)}\int_0^{+\infty} x^n e^{-x}dx = \frac{n!}{k\psi_\nu^n(k)}\,. \end{aligned}$$

By using the similar argument as above, the following result can be obtained.

Corollary 3.1 *The Laplace transform of the inverse subordinator $T_\nu(t)$ can be represented as*

$$\mathbb{E}\left[e^{-uT_\nu(t)}\right] = \mathscr{L}^{-1}_{k\to t}\left\{\frac{\psi_\nu(k)}{k\left(\psi_\nu(k)+u\right)}\right\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{kt}\frac{\psi_\nu(k)}{k\left(\psi_\nu(k)+u\right)}dk\,, \tag{10}$$

where γ is a fixed positive number such that $\psi_\nu(\gamma) > |u|$.

Lemma 3.2 *$B(T_\nu(t))$ is a continuous martingale.*

The proof of this lemma is similar to that in [21].

Corollary 3.2 *For every $\lambda \in \mathbb{R}$, the process $\exp\left\{\lambda B(T_\nu(t)) - \frac{\lambda^2}{2}T_\nu(t)\right\}$ is a continuous $\{\mathscr{H}_t\}$-martingale.*

Proof. Note the fact that the quadratic variance of $B(T_\nu(t))$ satisfies $\langle B(T_\nu(t))\rangle = T_\nu(t)$ and the following result

$$
\begin{aligned}
\mathbb{E}\exp\left\{\lambda B(T_\nu(t)) - \frac{\lambda^2}{2}T_\nu(t)\right\} &= \int_0^{+\infty}\left(\mathbb{E}\exp\left\{\lambda B(x) - \frac{\lambda^2}{2}x\right\}\right) l(x,t)dx \\
&= \int_0^{+\infty}\left(\mathbb{E}\exp\left\{\lambda B(0) - 0\frac{\lambda^2}{2}\right\}\right) l(x,t)dx = 1 \\
&= \mathbb{E}\exp\left\{\lambda B(T_\nu(0)) - \frac{\lambda^2}{2}T_\nu(0)\right\}.
\end{aligned}
$$

From Lemma 3.1 the conclusion is obtained.

Theorem 3.1 *For $T > 0$, there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and $\{Z_\nu(t)\}_{t\in[0,T]}$ is a martingale w.r.t. $\mathbb{Q}$.*

Proof. Let $P(T) = \exp\left\{-\lambda B(T_\nu(T)) - \frac{\lambda^2}{2}T_\nu(T)\right\}$ with $\lambda = \frac{\sigma}{2} + \frac{\mu}{\sigma}$, we have $\mathbb{E}P(T) = \mathbb{E}P(0) = 1$. Define a probability measure $\mathbb{Q}$ satisfying $d\mathbb{Q} = P(T)d\mathbb{P}$, then the two measures $\mathbb{Q}$ and $\mathbb{P}$ are equivalent. For $t \in [0,T]$, denote $P(t) = \mathbb{E}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathscr{H}_t\right)$, we get that

$$
\begin{aligned}
dP(t) &= P(t)d\left(-\lambda B(T_\nu(t)) - \frac{\lambda^2}{2}T_\nu(t)\right) + \frac{1}{2}P(t)d\left(\lambda^2 T_\nu(t)\right) \\
&= -\lambda P(t)dB(T_\nu(t))
\end{aligned} \tag{11}
$$

with $P(0)$ =1. It follows from Girsanov's theorem, Lemma 3.1 and (11) that

$$
K_\nu(t) := B(T_\nu(t)) - \int_0^t \frac{1}{P(t)} d\langle P(s), \qquad B(T_\nu(s))\rangle = B(T_\nu(t)) + \lambda T_\nu(t)
$$

is a local martingale w.r.t. $\mathbb{Q}$, where $\langle\cdot,\cdot\rangle$ denotes the quadratic covariance of two processes. Therefore, the quadratic variation $\langle K_\nu(t)\rangle = T_\nu(t)$ and so $\exp\left\{\sigma K_\nu(t) - \frac{\sigma^2}{2}T_\nu(t)\right\} = Z_\nu(t)$ is a continuous local martingale and super-martingale w.r.t. $\mathbb{Q}$. On the other hand, it follows from Corollary 3.1

that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}} \exp\left\{\sigma K_\nu(T) - \frac{\sigma^2}{2} T_\nu(T)\right\} \\
&= \mathbb{E} \exp\left\{\sigma K_\nu(T) - \frac{\sigma^2}{2} T_\nu(T) - \lambda B(T_\nu(T)) - \frac{\lambda^2}{2} T_\nu(T)\right\} \\
&= \mathbb{E} \exp\left\{(\sigma - \lambda) B(T_\nu(T)) - \frac{(\sigma-\lambda)^2}{2} T_\nu(T)\right\} \\
&= \mathbb{E} \exp\left\{(\sigma - \lambda) B(T_\nu(0)) - \frac{(\sigma-\lambda)^2}{2} T_\nu(0)\right\} = 1 \\
&= \mathbb{E}^{\mathbb{Q}} \exp\left\{\sigma K_\nu(0) - \frac{\sigma^2}{2} T_\nu(0)\right\},
\end{aligned}
$$

where $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation w.r.t. the measure $\mathbb{Q}$. Thus, $\{Z_\nu(t)\}_{t\in[0,T]}$ is a martingale w.r.t. $\mathbb{Q}$. That ends the proof.

Theorem 3.2 *For every $\theta > 0$, denote by $\mathbb{Q}_\theta$ the probability measure on $(\Omega, \mathscr{F})$ such that $d\mathbb{Q}_\theta = ce^{-\theta T_\nu(T)} d\mathbb{Q} = ce^{-\theta T_\nu(T)} P(T) d\mathbb{P}$, where $c^{-1} = \mathbb{E}\left(e^{-\theta T_\nu(T)} P(T)\right)$ is a constant, then $\mathbb{Q}_\theta$ is equivalent to $\mathbb{P}$ and $\{Z_\nu(t)\}_{t\in[0,T]}$ is a martingale w.r.t. $\mathbb{Q}_\theta$.*

Proof. Let $X(t) = \exp\left\{-\lambda B(t) - \frac{\lambda^2}{2} t\right\}$ with $\lambda = \frac{\sigma}{2} + \frac{\mu}{\sigma}$, then $X(t)$ is a $\{\mathscr{G}_t\}$-martingale and $P(t) = X(T_\nu(t))$. Moreover, the process $(XZ)(t) := X(t)Z(t) = \exp\left\{(\sigma - \lambda)B(t) - \frac{1}{2}(\sigma-\lambda)^2 t\right\}$ is also a $\{\mathscr{G}_t\}$-martingale and so $(XZ)(t \wedge T_\nu(T))$ is a $\{\mathscr{G}_t\}$-martingale.

For every $t > s \geq 0$ and $A \in \mathscr{G}_s$

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}_\theta}\left[1_A Z(t \wedge T_\nu(T))\right] &= \mathbb{E}\left[1_A Z(t \wedge T_\nu(T)) c\, e^{-\theta T_\nu(T)} P(T)\right] \\
&= \mathbb{E}\left\{\mathbb{E}\left[1_A Z(t \wedge T_\nu(T)) c\, e^{-\theta T_\nu(T)} X(T_\nu(T)) \middle| \mathscr{G}_t\right]\right\} \\
&= \mathbb{E}\left\{1_A c\, e^{-\theta T_\nu(T)} Z(t \wedge T_\nu(T)) X(t \wedge T_\nu(T))\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[1_A c\, e^{-\theta T_\nu(T)} Z(t \wedge T_\nu(T)) X(t \wedge T_\nu(T)) \middle| \mathscr{G}_s\right]\right\} \\
&= \mathbb{E}\left[1_A c\, e^{-\theta T_\nu(T)} Z(s \wedge T_\nu(T)) X(s \wedge T_\nu(T))\right] \\
&= \mathbb{E}\left\{\mathbb{E}\left[1_A c\, e^{-\theta T_\nu(T)} Z(s \wedge T_\nu(T)) P(T) \middle| \mathscr{G}_s\right]\right\} \\
&= \mathbb{E}\left[1_A c\, e^{-\theta T_\nu(T)} P(T) Z(s \wedge T_\nu(T))\right] \\
&= \mathbb{E}^{\mathbb{Q}_\theta}\left[1_A Z(s \wedge T_\nu(T))\right],
\end{aligned}
$$

that is $\mathbb{E}^{\mathbb{Q}_\theta}\left[Z(t\wedge T_\nu(T))\big|\mathscr{G}_s\right] = Z(s\wedge T_\nu(T))$. Thus $Z(t\wedge T_\nu(T))$ is a $\{\mathscr{G}_t\}$-martingale w.r.t. $\mathbb{Q}_\theta$.

Next, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_\theta}\left(\sup_{t\geq 0} Z(t\wedge T_\nu(T))\right) &= \mathbb{E}^{\mathbb{Q}_\theta}\left(\sup_{t\in[0,T]} Z(T_\nu(t))\right)\\
&= \mathbb{E}\left(c\, e^{-\theta T_\nu(T)}P(T)\sup_{t\in[0,T]} Z(T_\nu(t))\right)\\
&\leq cZ_0\mathbb{E}\left(e^{-\lambda B(T_\nu(T))}e^{|\mu|T_\nu(T)}\sup_{t\in[0,T]} e^{\sigma B(T_\nu(t))}\right).
\end{aligned} \tag{12}$$

Let k be an arbitrary constant, similar to Lemma 3.1, we have that $\mathbb{E}e^{kT_\nu(T)} < +\infty$. It follows that

$$\begin{aligned}
\mathbb{E}\left(\left(\sup_{t\in[0,T]} e^{kB(T_\nu(t))}\right)^2\right) &= \int_0^{+\infty}\mathbb{E}\left(\left(\sup_{t\in[0,x]} e^{kB(t)}\right)^2\right) l(x,T)dx\\
&\leq \int_0^{+\infty}\mathbb{E}\left(\left(\sup_{t\in[0,x]} e^{kB(t)-\frac{k^2}{2}t}\right)^2\right) e^{k^2x}l(x,T)dx\\
&\leq \int_0^{+\infty} 4\mathbb{E}\left(e^{2kB(x)-k^2x}\right)e^{k^2x}l(x,T)dx\\
&= 4\mathbb{E}e^{2k^2T_\nu(T)} < +\infty\,.
\end{aligned}$$

The formula obtained above implies that $\mathbb{E}e^{kB(T_\nu(T))} < +\infty$. By Hölder inequality and (12), we have $\mathbb{E}^{\mathbb{Q}_\theta}\left(\sup_{t\geq 0} Z(t\wedge T_\nu(T))\right) < +\infty$ and so $Z(t\wedge T_\nu(T))$ is a uniformly integrable martingale w.r.t. $\mathbb{Q}_\theta$. It follows that there exists a random variable Y_θ such that $Z(t\wedge T_\nu(T)) = \mathbb{E}^{\mathbb{Q}_\theta}\left(Y_\theta|\mathscr{G}_t\right)$ and $Z_\nu(t) = Z(T_\nu(T)\wedge T_\nu(T)) = \mathbb{E}^{\mathbb{Q}_\theta}\left(Y_\theta|\mathscr{H}_t\right)$. So $Z_\nu(t)$ is a martingale w.r.t. the probability measure $\mathbb{Q}_\theta$.

4. Application: option pricing

In this section, we give the corresponding price formulae of the European call options when the time-changed geometric Brownian motion $Z_\nu(t)$ defined in (4) represents the price of the underlying asset.

First, suppose that the risk-free interest rate is 0, then the discounted stock price at time t equals $Z_\nu(t)$. By using the first and second fundamental theorems of asset pricing, the following result can be immediately obtained from Theorem 3.1 and Theorem 3.2.

Theorem 4.1 *The market model whose underlying asset price follows the time-changed GBM $\{Z_\nu(t)\}_{t\in[0,T]}$ is arbitrage-free but incomplete.*

Denote by $C_\nu(Z_0,T,K,r)$ (or $C(Z_0,T,K,r)$) the price of the European call option under the asset price model $Z_\nu(t)$ (or $Z(t)$) defined in (4) (or (2)) with the initial value of underlying asset Z_0, time to expiration date T, strike price K and the risk-free interest rate r. Especially, we drop the parameter r when $r = 0$. It is well known that the fair price $C(Z_0,T,K,r)$ of the European call option is given as follows

$$C(Z_0,T,K,r) = Z_0\boldsymbol{N}(d_1) - K\,e^{-rT}\boldsymbol{N}(d_2)\,,$$

where

$$d_1 = \frac{\ln\frac{Z_0}{K} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\,, \qquad d_2 = d_1 - \sigma\sqrt{T}$$

and $\boldsymbol{N}(\cdot)$ is cumulative normal density function.

Theorem 4.2 *Assume that the price of the underlying asset is given by (4), the risk-free interest rate is 0 and the equivalent martingale measure is Q, then the fair price of the European call option satisfies*

$$C_\nu(Z_0,T,K) = \int_0^{+\infty} C(Z_0,t,K)l(t,T)dt\,.$$

Proof. The fair price of the European option satisfies

$$\begin{aligned}
C_\nu(Z_0,T,K) &= \mathbb{E}^{\mathbb{Q}}\left((Z_\nu(T)-K)^+\right)\\
&= \mathbb{E}\left(\exp\left\{-\lambda B(T_\nu(T)) - \frac{\lambda^2}{2}T_\nu(T)\right\}(Z_\nu(T)-K)^+\right)\\
&= \int_0^{+\infty} l(t,T)\mathbb{E}\left(\exp\left\{-\lambda B(t) - \frac{\lambda^2}{2}t\right\}(Z(t)-K)^+\right)dt\\
&= \int_0^{+\infty} C(Z_0,t,K)l(t,T)dt\,.
\end{aligned}$$

Additionally, the Laplace transform $\mathscr{L}_{\tau\to u}\{l(\tau,t)\}(u)$ of $l(\tau,t)$ is given by (10).

Remark: When $\nu=\delta_\alpha$ with $\alpha\in(0,1)$, $\psi_\nu(k)=k^\alpha$, $T_\nu(t)$ degenerates to the inverse α-stable subordinator with the Laplace transform

$$\mathbb{E}e^{-uT_\nu(t)}=\mathscr{L}^{-1}_{k\to t}\left\{\frac{k^\alpha}{k\left(k^\alpha+u\right)}\right\}(t)=E_\alpha(-ut^\alpha)\,,$$

where $E_\alpha(z)=\sum\limits_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}$ is the Mittag–Leffler function [22]. This model is consistent with Magdziarz's [12].

Next, assume that the risk-free interest rate $r>0$, we apply the actuarial approach (Theorem 2.1 in [23]) to derive the corresponding option pricing formula.

Theorem 4.3 *If the price of the underlying asset is given by (4), the price of the corresponding European call option satisfies* $C_\nu(Z_0,T,K,r)=\int_0^{+\infty}C\left(Z_0,t,\frac{\mathbb{E}Z_\nu(T)}{\mathbb{E}Z(t)}K,r\frac{T}{t}\right)\frac{\mathbb{E}Z(t)}{\mathbb{E}Z_\nu(T)}l(t,T)dt.$

Proof. Computing directly the expectations of $Z(t)$ and $Z_\nu(T)$ educes

$$\mathbb{E}Z(t)=Z_0e^{\mu t+\frac{\sigma^2}{2}t}\,,\qquad \mathbb{E}Z_\nu(T)=\int\limits_0^{+\infty}l(x,T)\mathbb{E}Z(x)dx\,.$$

Let $g_\nu(t,T)=\frac{\mathbb{E}Z(t)}{\mathbb{E}Z_\nu(T)}$. By the actuarial approach (Theorem 2.1 in [23]), we have

$$\begin{aligned}
C_\nu(Z_0,T,K,r)&=\mathbb{E}\left(Z_\nu(T)\frac{Z_0}{\mathbb{E}Z_\nu(T)}-K\,e^{-rT}\right)^+\\
&=\int\limits_0^{+\infty}\left[\mathbb{E}\left(Z(t)\frac{Z_0}{\mathbb{E}Z_\nu(T)}-K\,e^{-rT}\right)^+\right]l(t,T)dt\\
&=\int\limits_0^{+\infty}g_\nu(t,T)\left[\mathbb{E}\left(Z(t)\frac{Z_0}{\mathbb{E}Z(t)}-\frac{K}{g_\nu(t,T)}e^{-(r\frac{T}{t})t}\right)^+\right]l(t,T)dt\\
&=\int\limits_0^{+\infty}C\left(Z_0,t,\frac{K}{g_\nu(t,T)},r\frac{T}{t}\right)g_\nu(t,T)l(t,T)dt\,.
\end{aligned}$$

Remark: From the anonymous referee we know the latest results in [24], in which authors introduced a subdiffusive arithmetic Brownian motion as a model of stock prices and investigated the corresponding option pricing. Moreover, the proofs in [24] are fairly beautiful.

The authors would like to thank the Referees for their helpful comments and suggestions. This work was supported by the Science and Technology Commission of Shanghai Municipality (No 11ZR1410300) and by Shanghai Leading Academic Discipline Project (No B407).

REFERENCES

[1] M. Magdziarz, *J. Stat. Phys.* **135**, 763 (2009).

[2] R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).

[3] R. Metzler, J. Klafter, *J. Phys. A* **37**, 161 (2004).

[4] S.C. Kou, *Ann. Appl. Stat.* **2**, 501 (2008).

[5] S.C. Kou, X.S. Xie, *Phys. Rev. Lett.* **93**, 180603 (2004).

[6] E. Scalas, in: *The Complex Networks of Economic Interactions*, Lecture Notes in Economics and Mathematical Systems, Vol. 567, Springer, Berlin 2006.

[7] E. Barkai, R. Metzler, J. Klafter, *Phys. Rev.* **E61**, 132 (2000).

[8] R. Metzler, E. Barkai, J. Klafter, *Phys. Rev. Lett.* **82**, 3563 (1999).

[9] K.S. Fa, *Phys. Rev.* **E82**, 012101 (2010).

[10] M. Magdziarz, K. Weron, *Physica A* **367**, 1 (2006).

[11] A. Piryatinska, A.I. Saichev, W.A. Woyczynski, *Physica A* **349**, 375 (2005).

[12] M. Magdziarz, *J. Stat. Phys.* **136**, 553 (2009).

[13] J.-R. Liang *et al.*, *J. Stat. Phys.* **146**, 205 (2012).

[14] J. Janczura, A. Wyłomańska, *Acta Phys. Pol. B* **40**, 1341 (2009).

[15] K.I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

[16] M.G. Hanhn, K. Kobayashi, S. Umarov, *Proc. Amer. Math. Soc.* **139**, 691 (2011).

[17] A.V. Chechkin, R. Gorenfo, I.M. Sokolov, *Phys. Rev.* **E66**, 046129 (2002).

[18] M.M. Meerschaert, H.P. Scheffler, *Stoch. Proc. Appl.* **116**, 1215 (2006).

[19] M. Naber, *Fractals* **12**, 23 (2004).

[20] M.M. Meerschaert, E. Nane, P. Vellaisamy, *J. Math. Anal. Appl.* **379**, 216 (2011).

[21] M. Magdziarz, *Stoch. Models* **26**, 256 (2010).

[22] I. Podlubny, *Fractional Differential Equations*, Mathematics in Sciences and Engineering Vol. 198, Academic Press, San Diego 1999.

[23] M. Bladt, T.H. Rydberg, *Insur. Math. Econ.* **22**, 65 (1998).

[24] M. Magdziarz, S. Orzeł, A. Weron, *J. Stat. Phys.* **145**, 187 (2011).