ALGEBRAIC PROPERTIES OF TRANSLATIONAL SHAPE INVARIANT POTENTIALS IN ARBITRARY STEPS

Wang-Chang Su

Department of Physics, National Chung-Cheng University
Chia-Yi 621, Taiwan
physuw@ccu.edu.tw

(Received February 9, 2012)

Within the framework of supersymmetric quantum mechanics, we investigate the general algebraic properties of translational shape invariant potentials in arbitrary k steps, in which the k remainders $R_s(a_m)$ are analytic functions of the parameter a_m that is related to others by translation: $a_m = a_{m-1} + \delta$. The present study is based on the fact that the simplified potential algebra of shape invariance condition in k steps is equivalent to that of generalized deformed oscillators with a built-in Z_k-grading structure. We shall show that, despite the complexity in the study, the general algebraic properties still can be systematically determined.

DOI:10.5506/APhysPolB.43.1683
PACS numbers: 03.65.–w, 03.65.Fd, 11.30.Pb

1. Introduction

Supersymmetric quantum mechanics (SUSY QM) was initially introduced as a toy model to understand dynamical supersymmetry (SUSY) breaking in quantum field theories [1,2]. It was soon realized that SUSY QM by itself could be a very fascinating research topic. It is because, by the method of factorization [3], SUSY QM enables us to construct the so-called SUSY partner of a given one-dimensional nonrelativistic Hamiltonian. The process can be successively used to generate an entire hierarchy of isospectral SUSY partner Hamiltonians. For complete reviews on SUSY QM, refer to [4,5,6,7] and references therein.

Let us be more specific. Two potentials $V^{(-)}(x, a_0)$ and $V^{(+)}(x, a_0)$ are said to be SUSY partners, if they are related to each other by the superpotential $W(x, a_0)$ as

$$V^{(\pm)}(x, a_0) = W^2(x, a_0) \pm W'(x, a_0)\,, \tag{1}$$

where $W'(x,a_0) \equiv \frac{d}{dx}W(x,a_0)$ and a_0 is a set of parameters. In units of $\hbar = 2m = 1$, the corresponding SUSY partner Hamiltonians $H^{(-)}(x,a_0)$ and $H^{(+)}(x,a_0)$ take the standard form: $H^{(\pm)}(x,a_0) = -\frac{d^2}{dx^2} + V^{(\pm)}(x,a_0)$, respectively. The SUSY partner Hamiltonians $H^{(\pm)}(x,a_0)$ can then be shown to be exactly isospectral, except for a zero-energy ground-state eigenfunction. The existence of such a zero-energy eigenstate is completely determined by the asymptotic behavior of the superpotential $W(x \to \pm\infty, a_0) \equiv w_\pm$. In this way, the Witten index acquires the topological expression: $\Delta = \frac{1}{2}[\mathrm{sgn}(w_+) - \mathrm{sgn}(w_-)]$; hence $\Delta \neq 0$ indicates good SUSY, whereas $\Delta = 0$ signifies the breaking of SUSY [5]. Here, $\mathrm{sgn}(w_\pm)$ is the sign of $w_\pm$.

The concept of shape invariance [8] can be incorporated with the formulation of SUSY QM, which effectively gives rise to an integrability condition to the solvable potentials of nonrelativistic Schrödinger equation. Many classes of solvable shape invariant potentials that retain SUSY are subsequently constructed, including *(i)* the translational class [9, 10], where the parameters a_0 and a_1 are related by $a_1 = a_0 + \delta$, *(ii)* the scaling class [11,12], where $a_1 = qa_0$, for $0 < q < 1$, *(iii)* the cyclic class [13], where $a_0 = a_p$, $a_1 = a_{p+1}$, and so on, for $p = 2, 3, ...$, and *(iv)* the "exotic" class [11], where $a_1 = qa_0^p$ and its generalization $a_1 = qa_0/(1 + ra_0)$, for $0 < q < 1$, $ra_0 \ll 1$, and $p = 2, 3, \ldots$[1].

By the term "shape invariance", it is suggested that the pair of partner potentials $V^{(\pm)}(x,a_0)$ defined in equation (1) is similar in shape but differs only up to a change of parameters and additive constants. Written in terms of the superpotential $W(x,a_0)$, the shape invariance condition in "one step" reads

$$W^2(x,a_0) + W'(x,a_0) =, W^2(x,a_1) - W'(x,a_1) + R(a_0)\,, \tag{2}$$

where $a_1 = f(a_0)$ is a function of a_0 and the remainder $R(a_0)$ is independent of x. By equation (2), the entire energy spectrum of the initial Hamiltonian $H^{(-)}(x,a_0)$ can be determined algebraically: $E_0^{(-)} = 0$ and $E_n^{(-)} = \sum_{i=0}^{n-1} R(a_i)$ (for $n = 1, 2, 3, \ldots$) [8, 14]. Here, we assume that the superpotential $W(x,a_0)$ is constructed in such a way that the Hamiltonian $H^{(-)}(x,a_0)$ possesses the unique zero-energy ground state. It should be remarked that the simple expression of energy eigenvalues is actually a direct consequence of what is generally referred to as the potential algebra [15,16]. That is, all shape invariant potentials described by equation (2) admit an underlying algebraic structure, thus can be studied by group theoretical methods [17, 18].

[1] Strictly speaking, the four classes are not entirely independent, since they can be transformed into one another by suitable reparameterizations.

The usage of shape invariance condition can be readily extended to two and even multi-steps [11]. Based on this method, some solvable shape invariant potentials in two or multi-steps are therefore established [13,14,19,20,21]. The extension is straightforward. For this, we consider the general case of shape invariance condition in k steps, where k is an arbitrary positive integer. In order to maintain unbroken SUSY, the k superpotentials $W_s(x, a_0)$ (for $s = 0, 1, \ldots, k-1$) are chosen to fulfill the asymptotic behaviors: $\mathrm{sgn}(w_{0+}) = \mathrm{sgn}(w_{1+}) \ldots = \mathrm{sgn}(w_{(k-1)+})$ as well as the Witten index requirements: $\Delta_s \neq 0$. Otherwise, they are quite arbitrary functions of x and a_0. In terms of these k superpotentials, the shape invariance condition in k steps is given by

$$\begin{aligned} W_0^2(x,a_0) + W_0'(x,a_0) &= W_1^2(x,a_0) - W_1'(x,a_0) + R_0(a_0)\,, \\ W_1^2(x,a_0) + W_1'(x,a_0) &= W_2^2(x,a_0) - W_2'(x,a_0) + R_1(a_0)\,, \\ \cdots &= \cdots \\ W_{k-1}^2(x,a_0) + W_{k-1}'(x,a_0) &= W_0^2(x,a_1) - W_0'(x,a_1) + R_{k-1}(a_0)\,, \end{aligned} \tag{3}$$

where the k remainders $R_s(a_0)$ are arbitrary and independent of x. Much in the same way, from equation (3), the energy eigenvalues for the initial Hamiltonian $H_0^{(-)}(x, a_0)$ can be algebraically determined by

$$E_{nk+s}^{(-)} = \sum_{m=0}^{n-1} \sum_{t=0}^{k-1} R_t(a_m) + \sum_{t=0}^{s-1} R_t(a_n)\,, \tag{4}$$

where the convention $\sum_{t=0}^{-1} = 0$ is used. Here, $n = 0, 1, 2, \ldots$ and $s = 0, 1, \ldots, k-1$.

It is interesting to note that, under certain circumstances, a simplified version of potential algebra of shape invariance condition in k steps, as described in equations (3) and (4), can be built. For the simplest case in $k = 2$ steps, it is found that the corresponding simplified potential algebra is similar to that of shape invariance in one-step (2), and is based only on three angular-momentum-like generators [20, 22]. As for the simplified algebraic properties in arbitrary k steps, they can be realized and subsequently established based on the algebra of generalized deformed oscillators with a built-in Z_k-grading structure [23]. We mention here that, as a special case, the simplified potential algebra in k steps developed in [23] includes those cyclic shape invariant potentials of period k [13, 24, 25].

In the present article, the investigation on the simplified potential algebra of shape invariance condition in k steps is continued. The results of [23] will be extended. There, the algebraic structures of translational shape invariant potentials in k steps were step-by-step carried out, in which the remainder

functions $R_s(a_m)$ depend only linearly on the parameter a_m which is related to other parameters by translation $a_m = a_{m-1} + \delta$. Here, we shall generalize the study to the case that the remainder functions $R_s(a_m)$ are analytic functions of the parameter a_m. In other words, the remainder functions $R_s(a_m)$ are assumed to admit series expansions in the parameter a_m[2]. We will show that, despite tedious computations that are needed to handle this problem, the general algebraic properties of translational shape invariant potentials in arbitrary steps still can be systematically determined.

The article is organized as follows. In Sec. 2, for the purpose of completeness, we briefly review on how the simplified potential algebra of shape invariance in k steps is realized by the algebra of Z_k-graded generalized deformed oscillators. In Sec. 3, based on the Z_k-graded deformed oscillator algebra, we explicitly work out the detailed algebraic properties of translational shape invariant potentials in k steps, in which remainders $R_s(a_m)$ are analytic in the parameter a_m. A closed-form example of analytic remainders is then given to illustrate the corresponding algebraic structure. Finally, Sec. 4 contains a discussion of the present article.

2. Z_k-graded shape invariant potentials

In this section, we review the equivalence between the simplified potential algebra of shape invariance in k steps and the Z_k-graded generalized deformed oscillator algebra, which was first established in [23]. Deformed oscillators have been proposed and studied in many different deformation schemes in the literature [27,28,29,30]. The so-called Z_k-graded generalized deformed oscillators are the ordinary deformed oscillators that have built-in an extra Z_k-grading symmetry [31,32].

To begin with, let us consider in equation (3) the identification of parameters $a_0 \to \boldsymbol{\alpha}(N_0)$ and more generally $a_m \to \boldsymbol{\alpha}(N_0 - m)$, where N_0 is an arbitrary integer and $m = 0, 1, 2, \ldots$ The precise form of the function $\boldsymbol{\alpha}(N_0)$ is determined by requiring that the change $\boldsymbol{\alpha}(N_0) \to \boldsymbol{\alpha}(N_0 - 1)$ corresponds to the change of parameters $a_0 \to a_1$. Next, to go further beyond equation (3), we restrict ourselves to the simplified version of potential algebra of shape invariance condition in k steps. This is achieved by introducing extra constraints on both originally arbitrary k superpotentials and k remainders. To be more explicit, we demand that

$$W_s(x, \boldsymbol{\alpha}(N_0)) \equiv \mathcal{W}\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right), \quad R_s(\boldsymbol{\alpha}(N_0)) \equiv \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right), \tag{5}$$

[2] In some cases, the generalization of the remainders $R_s(a_m)$ to be analytic functions of the parameter a_m might be purely mathematical. It is because we are aware of the fact that for bound state problems the nth energy eigenvalue E_n obeys $E_n \leq \text{const} \times n^2$, for large enough value of n [26].

where the identification $a_{\frac{s}{k}} \to \boldsymbol{\alpha}(N_0 - \frac{s}{k})$ is implied. As a result, the k relations of equation (3) are cast into a compact one, in terms of the unified superpotential $\mathcal{W}(x, \boldsymbol{\alpha}(N_0))$ and unified remainder $\mathcal{R}(\boldsymbol{\alpha}(N_0))$, as

$$\mathcal{W}^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) + \mathcal{W}'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right)$$
$$= \mathcal{W}^2\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) - \mathcal{W}'\left(x, \boldsymbol{\alpha}\left(N_0 - \frac{s+1}{k}\right)\right) + \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{s}{k}\right)\right) . \quad (6)$$

Note that the k relations in equation (3) are easily reproduced from equation (6) by letting, one at a time, $s = 0, 1, \ldots, k-1$.

At this stage, equation (6) clearly represents a constraint on the unified superpotential $\mathcal{W}(x, \boldsymbol{\alpha}(N_0))$, when the parameter N_0 is changed by $-\frac{1}{k}$ as we go from the first equality to the second one of equation (3). In quantum mechanics, this change of parameter $N_0 \to N_0 - \frac{1}{k}$ is very common and is usually formulated by the action of raising and lowering operators of the simple harmonic oscillator. With this in mind, we first define the analogous number operator $\mathcal{N}$ of equation (6) by

$$\mathcal{N} \equiv \frac{1}{i}\frac{\partial}{\partial \phi} , \quad (7)$$

and designate the parameter N_0 as the eigenvalue of the number operator $\mathcal{N}$ acting on the particular eigenstate $|N_0\rangle$. Thus, the entire Fock space of number eigenstates can be denoted by the direct sum $\mathcal{H} = \sum_{s=0}^{k-1} \oplus \mathcal{H}_s$ that consists of k distinct Fock subspaces

$$\mathcal{H}_s \equiv \left\{ \left| N_0 - \frac{nk+s}{k} \right\rangle \right| \; n = 0, 1, 2, \ldots \right\} . \quad (8)$$

Second, the Z_k-grading structure of the Fock space $\mathcal{H}$ is easily realized by projection operators that by definition project onto the k distinct subspaces $\mathcal{H}_s$. Written out explicitly, the k projection operators Π_s (for $s = 0, 1, \ldots, k-1$) are expressible in terms of the number operator $\mathcal{N}$ as [31, 32]

$$\Pi_s \equiv \frac{1}{k}\sum_{t=0}^{k-1} e^{2\pi i t(\mathcal{N}+s/k)} , \qquad \sum_{s=0}^{k-1} \Pi_s = I . \quad (9)$$

In this way, the number eigenstates are simultaneous eigenstates of both operators $\mathcal{N}$ and Π_s, in which the eigenvalue equations are, respectively,

$$\mathcal{N}\left| N_0 - \frac{nk+s}{k} \right\rangle = \left(N_0 - \frac{nk+s}{k}\right)\left| N_0 - \frac{nk+s}{k} \right\rangle , \quad (10)$$

$$\Pi_t \left| N_0 - \frac{nk+s}{k} \right\rangle = \delta_{t,s} \left| N_0 - \frac{nk+s}{k} \right\rangle , \quad (11)$$

where $n = 0, 1, 2, \ldots$ and $s, t = 0, 1, \ldots, k-1$.

Third, we build the analogous ladder operators, that is, $\mathcal{A}$ and $\mathcal{A}^\dagger = (\mathcal{A})^\dagger$, by using the unified superpotential $\mathcal{W}(x, \alpha(\mathcal{N}))$

$$\mathcal{A} = e^{-i\phi/k}\left[\frac{\partial}{\partial x} + \mathcal{W}(x, \boldsymbol{\alpha}(\mathcal{N}))\right], \qquad \mathcal{A}^\dagger = \left[-\frac{\partial}{\partial x} + \mathcal{W}(x, \boldsymbol{\alpha}(\mathcal{N}))\right] e^{i\phi/k}. \tag{12}$$

Since the operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ in equation (12) remain invariant under the transformation $\phi \to \phi + 2\pi k$, we shall take the configuration space for the variable ϕ to be in the interval $[0, 2\pi k]$.

Finally, we are at a position to explain the associated potential algebra that is implicitly given in equation (6). Direct computations reveal that the simplified potential algebra of shape invariance in k steps is, in fact, described by the set of generators I, $\mathcal{A}$, $\mathcal{A}^\dagger$, $\mathcal{N}$, and Π_s, which fulfill the Hermiticity conditions $(\mathcal{A})^\dagger = \mathcal{A}^\dagger$, $\mathcal{N}^\dagger = \mathcal{N}$, $\Pi_s^\dagger = \Pi_s$, and the following relations

$$\left[\mathcal{N}, \mathcal{A}^\dagger\right] = \frac{1}{k}\mathcal{A}^\dagger, \qquad [\mathcal{N}, \mathcal{A}] = -\frac{1}{k}\mathcal{A}, \tag{13}$$

$$\mathcal{A}^\dagger\mathcal{A} = \mathcal{F}(\boldsymbol{\alpha}(\mathcal{N})), \qquad \mathcal{A}\mathcal{A}^\dagger = \mathcal{F}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right), \tag{14}$$

$$[\mathcal{N}, \Pi_s] = 0, \quad \Pi_s\Pi_t = \delta_{s,t}, \quad \mathcal{A}^\dagger\Pi_s = \Pi_{s+1}\mathcal{A}^\dagger, \quad \mathcal{A}\,\Pi_s = \Pi_{s-1}\mathcal{A}, \tag{15}$$

where the convention is used for the projection operators $\Pi_t = \Pi_s$, if $t - s = 0 \bmod k$. We realize immediately that the simplified potential algebra described by equations (13), (14), and (15) is familiar and similar to that of generalized deformed oscillators with a Z_k-grading structure [31, 32]. In other words, the identification (5) simplifies the would-be complex potential algebra of shape invariance condition in k steps to the relatively simplified version (6), which in turn is found to be identical to the well established Z_k-graded deformed oscillator algebra.

The Hermitian positive function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ appearing in equation (14) is called the structure function as is suggested in the generalized deformed oscillator algebra. According to equation (6), the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ is related to the unified remainder function $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ through this remainder-structure-function relation

$$\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N})) - \mathcal{F}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right) = -\mathcal{R}\left(\boldsymbol{\alpha}\left(\mathcal{N} + \frac{1}{k}\right)\right). \tag{16}$$

Furthermore, when acting on the number eigenstates, the ladder operators $\mathcal{A}$ and $\mathcal{A}^\dagger$ constructed in equation (12) are as usual the lowering and raising operators of the simplified potential algebra of shape invariance in k steps,

which, as desired, change the eigenvalues of the number operator $\mathcal{N}$ by $-\frac{1}{k}$ and $+\frac{1}{k}$, respectively,

$$\mathcal{A}\left|N_0-\frac{n}{k}\right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right)}\left|N_0-\frac{n+1}{k}\right\rangle, \tag{17}$$

$$\mathcal{A}^{\dagger}\left|N_0-\frac{n}{k}\right\rangle = \sqrt{\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{n-1}{k}\right)\right)}\left|N_0-\frac{n-1}{k}\right\rangle, \tag{18}$$

where $n = 0, 1, 2, \ldots$ We mention here that if the number eigenvalue spectrum exhibits a lowest-weight eigenstate $\mathcal{A}|N_0 - \frac{n_0}{k}\rangle = 0$, for a given integer n_0, then the condition $\mathcal{F}(\boldsymbol{\alpha}(N_0 - \frac{n_0}{k})) = 0$ will be satisfied. Otherwise, if it exhibits a highest-weight eigenstate $\mathcal{A}^{\dagger}|N_0 - \frac{n_0}{k}\rangle = 0$, then we will have $\mathcal{F}(\boldsymbol{\alpha}(N_0 - \frac{n_0-1}{k})) = 0$.

As a byproduct, the use of simplified potential algebra of shape invariance in k steps enables us to express the energy eigenvalues of initial Hamiltonian $H_0^{(-)}(x, a_0) = H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ completely in terms of the structure function $\mathcal{F}(\boldsymbol{\alpha}(N_0 - \frac{m}{k}))$. We can show this relation by projecting the operator equation (16) on the eigenstate $|N_0 - \frac{m+1}{k}\rangle$. Then, by applying the resultant relation recursively, we obtain via equation (4) the eigenenergies of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in the form

$$E_n^{(-)} = \sum_{m=0}^{n-1} \mathcal{R}\left(\boldsymbol{\alpha}\left(N_0-\frac{m}{k}\right)\right) = \mathcal{F}\left(\boldsymbol{\alpha}(N_0)\right) - \mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right), \tag{19}$$

where $n = 0, 1, 2, \ldots$ It is worth pointing that equation (19) represents the energy spectrum of the simplified shape invariance condition in k steps, which is to be compared with the initially more complicated energy spectrum in equation (4).

Three remarks are in order.

1. For the simplest case of shape invariance in $k = 2$ steps, the Z_2-grading structure is characterized by the usual Klein operator. In this sense, the Z_2-graded deformed oscillator algebra is similar to the Calogero–Vasiliev oscillator algebra [33]. Moreover, it is also related to the so-called R-deformed Heisenberg algebra [28] that has found many interesting applications, in the context of noncommutative geometry, supersymmetry, and anyons [34].

2. By inspecting the energy spectrum in equation (19), we learn that the remainder $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k}))$ is nothing but the energy gap between two adjacent eigenstates. Therefore, we must have $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k})) > 0$, for

any positive integer m, in order to prevent energy levels from crossing. If $\mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{m}{k})) \leq 0$ happens, it simply means that the associated shape invariant potential is of finite depth, thus contains only a finite number of bound states.

3. As is presented in equations (13), (14), and (15), the simplified potential algebra of shape invariance in k steps will not be complete, without knowing the details of the structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$, which is afterwards related to the unified remainder $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ via equation (16). It, therefore, implies that different remainder functions will give different structure functions, thus resulting in different shape invariant potentials in k steps. That is to say, in the terminology of deformed oscillator algebra, different structure functions correspond to different deformation schemes, resulting in different Z_k-graded deformed oscillators.

3. Translational shape invariant potentials in k steps

The detailed algebraic structures of translational shape invariant potentials in k steps will be explicitly constructed, in which the parameters of partner potentials are related to each other by translation: $a_m = a_{m-1} + \delta$. Here, δ is a constant. As mentioned earlier, the same but relatively simpler problem has been discussed in [23], in which the k remainders $R_s(a_m)$ in equation (3) are chosen to be only linear functions of the parameter a_m. In the present section, we are interested in the more general situation, in which the remainders $R_s(a_m)$ are analytic functions of the parameter a_m.

We hence consider translational shape invariant potentials in k steps, such that the relations $a_{m+\frac{s}{k}} = a_m + \frac{s}{k}\delta = a_0 + (m + \frac{s}{k})\delta$ are automatically fulfilled, where $m = 0, 1, 2, \ldots$ and $s = 0, 1, \ldots, k-1$. In addition, without loss of generality, we demand that, in the identification procedure mentioned in the paragraph before equation (5), the function $\boldsymbol{\alpha}(N_0 - \frac{n}{k})$ be identical to the parameter $a_{\frac{n}{k}}$. In other words, for $n = 0, 1, 2, \ldots,$

$$\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right) \equiv a_{\frac{n}{k}} = (a_0 + N_0) - \left(N_0 - \frac{n}{k}\right)\delta\,. \tag{20}$$

The algebraic properties of translational shape invariant potentials in k steps can be established as follows. Let us first choose the k unrelated remainders $R_s(a_m)$ in equation (3) to admit the following series expansions $(m = 0, 1, 2, \ldots)$

$$R_s(a_m) = \sum_{i=0}^{I} \alpha_{s,i} (a_m)^i\,, \tag{21}$$

where the expansion coefficients $\alpha_{s,i}$ $(s = 0, 1, \ldots, k-1)$ are arbitrary constants, and the parameter I is any positive integer, which denotes the highest powered order in the expansions. Of course, the value of I can go to infinity so that equation (21) becomes an infinite series.

Next, according to equation (5), we can identify the k remainders $R_s(a_m)$ to be the unified remainder $\mathcal{R}(a_{m+\frac{s}{k}})$ at different values of $a_{m+\frac{s}{k}}$. At the same time, we memorize that under such identification the simplified potential algebra of shape invariance in k steps will turn out to acquire a built-in Z_k-grading structure. In mathematical formalism, we are thus led to write

$$R_s(a_m) \equiv \mathcal{R}\Big(a_{m+\frac{s}{k}}\Big) = \sum_{i=0}^{I} \beta_{s,i} \Big(a_m + \frac{s}{k}\,\delta\Big)^i , \tag{22}$$

where $\beta_{s,i}$ are another expansion coefficients that are introduced to characterize the inherited Z_k-grading structure of the remainder $\mathcal{R}(a_{m+\frac{s}{k}})$. We assume here that $\mathcal{R}(a_{m+\frac{s}{k}})$ admits the power series expansion, too. Then, comparing the power series expansions in a_m for both equations (21) and (22), we immediately arrive at the relationship between the coefficients

$$\alpha_{s,i} = \sum_{j=i}^{I} \binom{j}{i} \Big(\frac{s}{k}\,\delta\Big)^{j-i} \beta_{s,j} , \tag{23}$$

where $\binom{j}{i} = j!/i!(j-i)!$ is the binomial coefficient.

In the same vein, if we denote $n = mk + s$ in equation (22), a totally different expansion, in the power series of $\frac{n}{k}$, of the unified remainder $\mathcal{R}(a_{\frac{n}{k}}) = \mathcal{R}(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ can be deduced. With the help of (20), equation (22) becomes

$$\mathcal{R}\left(a_{\frac{n}{k}}\right) = \sum_{s=0}^{k-1} \sum_{i=0}^{I} \beta_{s,i} \left(a_0 + \frac{n}{k}\,\delta\right)^i \Delta_{n,s} = \sum_{s=0}^{k-1} \sum_{i=0}^{I} \omega_{s,i} \left(\frac{n}{k}\right)^i \Delta_{n,s} , \tag{24}$$

where the expansion coefficients $\omega_{s,i}$ are given by

$$\omega_{s,i} = \delta^i \sum_{j=i}^{I} \binom{j}{i} \beta_{s,j} (a_0)^{j-i} . \tag{25}$$

In equation (24), the symbol $\Delta_{n,s}$ is presented to single out the specific contribution, when $n = mk + s$. It is defined as the analogous Kronecker delta for the cyclic group of order k

$$\Delta_{n,s} = \begin{cases} 1 , & \text{for } \; n = s \mod k , \\ 0 , & \text{for } \; n \neq s \mod k . \end{cases} \tag{26}$$

Besides, an equivalent operator expression of equation (24) can be established, in favor of the number operator $\mathcal{N}$ (7) and projection operators Π_s (9), in the compact form as

$$\mathcal{R}\left(\boldsymbol{\alpha}\left(\mathcal{N}\right)\right) \equiv \sum_{i=0}^{I} \mathcal{R}_i\left(\boldsymbol{\alpha}\left(\mathcal{N}\right)\right) = \sum_{s=0}^{k-1} \sum_{i=0}^{I} \omega_{s,i}\ \left(N_0 - \mathcal{N}\right)^i\ \Pi_s\,. \tag{27}$$

From this, equation (24) can be readily recovered by acting the remainder operator $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ directly on the number eigenstate $|N_0 - \frac{n}{k}\rangle$. In the following subsections, by using equation (27), the relevant algebraic quantities of translational shape invariant potentials in arbitrary k steps will be completely determined. Because the remainder $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ shown in equation (27) is expanded in the power series of $(N_0 - \mathcal{N})$, we shall choose to present our results in accord with the increasing power of $(N_0 - \mathcal{N})$. That is, we will analyze the remainder $\mathcal{R}_i(\boldsymbol{\alpha}(\mathcal{N}))$, step-by-step, in each order of $(N_0 - \mathcal{N})$.

Before going into the details, let us introduce the short-hand notation $f_0(n,k)$ that will be useful in the later presentations

$$f_0(n,k) \equiv \sum_{s=0}^{k-1} \left[\sum_{t=0}^{k-1} \omega_{s+t} \right] \Delta_{n,s}\,, \tag{28}$$

where the cyclic convention is used: $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ (for instance, $\omega_k = \omega_0$ and $\omega_{k+1} = \omega_1$). Due to the presence of the Kronecker delta $\Delta_{n,s}$, only one term in the index s summation is singled out, which fulfills the condition: $n - s = 0 \bmod k$. Furthermore, despite the $\Delta_{n,s}$ term appearing in the summation, the function $f_0(n,k)$ is actually a constant independent of n. To show this, we take as an example $k = 4$ in equation (28) and immediately obtain $f_0(n,4) = (\omega_0 + \omega_1 + \omega_2 + \omega_3)$, for all integral n. Note that the second subscript "i" of all $\omega_{s+t,i}$ has been suppressed.

3.1. The zeroth order remainder

Let us begin with the simplest class of translational shape invariant potentials in k steps, in which the unified remainder takes the simple form $\mathcal{R}_0(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1} \omega_{s,0}\,\Pi_s$, which clearly is a Z_k-graded constant in $(N_0$–$\mathcal{N})$. The Z_k-graded Fock space of shape invariance condition in k steps is denoted by $\mathcal{H} = \sum_{s=0}^{k-1} \oplus \mathcal{H}_s$, in which the k Fock subspaces are $\mathcal{H}_s = \{|N_0 - m - \frac{s}{k}\rangle \ | m = 0, 1, 2, \dots\}$, for $s = 0, 1, \dots, k-1$. When acting the remainder $\mathcal{R}_0(\boldsymbol{\alpha}(\mathcal{N}))$ on the number eigenstates $|N_0 - \frac{n}{k}\rangle$ (for $n = 0, 1, 2, \dots$), it yields the result: $\sum_{s=0}^{k-1} \omega_{s,0} \Delta_{n,s}$, where $\Delta_{n,s}$ is the analogous Kronecker delta defined in equation (26). The first few members of the remainder are listed as $\mathcal{R}_0(\boldsymbol{\alpha}(N_0 - \frac{n}{k})) = (\omega_0, \omega_1, \omega_2, \dots, \omega_{k-1}, \omega_0, \omega_1, \dots)$ corresponding to the choice of $n = (0, 1, 2, \dots, k-1, k, k+1, \dots)$. For the purpose of simplicity, we suppress the second subscript "0" of the quantity $\omega_{s,0}$.

In this class, the structure function $\mathcal{F}_0(\boldsymbol{\alpha}(\mathcal{N}))$ for translational shape invariant potentials in k steps can be constructed, via the remainder-structure-function relation (16). A direct computation shows that

$$\mathcal{F}_0\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right)=C_0-\left[\,f_0(n,k)\,\frac{n}{k}+f_1(n,k)\,\right]\,, \tag{29}$$

where $n=0,1,2,\ldots$ and C_0 is an arbitrary constant to render $\mathcal{F}_0(\boldsymbol{\alpha}(N_0-\frac{n}{k}))$ positive definite. The function $f_0(n,k)$ has been defined in equation (28), whereas the function $f_1(n,k)$ will be called the first grading function, since it is presented to characterize the inherited Z_k-grading symmetry of the structure function $\mathcal{F}_0(\boldsymbol{\alpha}(\mathcal{N}))$ in k steps[3]. To be more specific, it is found to be

$$f_1(n,k)=\sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\frac{t}{k}-\frac{1}{2}\right)\omega_{s+t}\right]\Delta_{n,s}\,, \tag{30}$$

where we use the convention $\omega_{s+t}\equiv\omega_{s+t \bmod k}$. Because of the Kronecker delta $\Delta_{n,s}$, only one term in the index s summation is singled out that satisfies the condition: $n-s=0 \bmod k$. As an illustration, we take $k=4$ in equation (30) and obtain $f_1(4i,4)=\frac{1}{4}(-2\omega_0-\omega_1+\omega_3)$, $f_1(4i+1,4)=\frac{1}{4}(-2\omega_1-\omega_2+\omega_0)$, $f_1(4i+2,4)=\frac{1}{4}(-2\omega_2-\omega_3+\omega_1)$, and $f_1(4i+3,4)=\frac{1}{4}(-2\omega_3-\omega_0+\omega_2)$, where $i=0,1,2,\ldots$

Interestingly, the two terms inside the square bracket on the right-hand side of equation (30) can be further sorted neatly. To the purpose, we define the modified version of the first grading function by

$$\tilde{f}_1(n,k)=\sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\frac{t+n}{k}-\frac{1}{2}\right)\omega_{s+t}\right]\Delta_{n,s}\,. \tag{31}$$

Equation (29) consequently reduces to the simple expression

$$\mathcal{F}_0\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right)=C_0-\tilde{f}_1(n,k)\,. \tag{32}$$

Note again that the second subscript "0" of all $\omega_{s,0}$ is suppressed.

The energy spectrum of the initial Hamiltonian $H_0^{(-)}(x,\boldsymbol{\alpha}(N_0))$ can thus be obtained, according to equation (19), as

$$E_n^{(-)}=\tilde{f}_1(n,k)-\tilde{f}_1(0,k)\,. \tag{33}$$

In the literature, the potentials that possess such energy spectrum, which is linear in the number eigenvalue $(\frac{n}{k})$, are known as the cyclic shape invariant potentials of period k. The analytical properties for those potentials can be found in the study of cyclic shape invariance condition [13, 24, 25]. See also the special case of cyclic shape invariant potentials of period two [19].

[3] By the same reasoning, we shall call $f_0(n,k)$ the zeroth grading function.

3.2. The first order remainder

We next consider translational shape invariant potentials in k steps, in which the unified remainder is linear in $(N_0-\mathcal{N})$ as $\mathcal{R}_1(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1}\omega_{s,1}$ $(N_0-\mathcal{N})\,\Pi_s$. When acting the remainder $\mathcal{R}_1(\boldsymbol{\alpha}(\mathcal{N}))$ on the associated number eigenstates $|N_0-\frac{n}{k}\rangle$ (for $n=0,1,2,\ldots$), we find that $\sum_{s=0}^{k-1}\omega_{s,1}(\frac{n}{k})\Delta_{n,s}$, where $\Delta_{n,s}$ is in equation (26). The first few members of the remainder are listed as $\mathcal{R}_1(\boldsymbol{\alpha}(N_0-\frac{n}{k})) = (0, \frac{1}{k}\omega_1, \frac{2}{k}\omega_2, \ldots, \frac{k-1}{k}\omega_{k-1}, \omega_0, \frac{k+1}{k}\omega_1, \ldots)$ corresponding to the choices of $n = (0,1,2,\ldots,k-1,k,k+1,\ldots)$. Similarly, we suppress the second subscript "1" of the quantity $\omega_{s,1}$.

It is worth pointing that when $n=0$, the remainder $\mathcal{R}_1(\boldsymbol{\alpha}(N_0)) = 0$. It looks odd at the first sight, since the remainder $\mathcal{R}_1(\boldsymbol{\alpha}(N_0))$ by construction is the energy gap between the ground and first excited number eigenstates, that is, it has to be greater than zero, when the first excited eigenstate does exist. Nevertheless, it should cause no problem because we are only discussing the first order term of the remainder in the expansion of $(N_0-\mathcal{N})$, there presumably is a zeroth order term, which results in $\mathcal{R}_0(\boldsymbol{\alpha}(N_0)) \neq 0$, when $n=0$.

The corresponding structure function $\mathcal{F}_1(\boldsymbol{\alpha}(\mathcal{N}))$ can be determined by using equation (16). After some calculations, we obtain $(n=0,1,2,\ldots)$

$$\mathcal{F}_1\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right) = C_1 - \frac{1}{2}\left[f_0(n,k)\left(\frac{n}{k}\right)^2 + 2f_1(n,k)\,\frac{n}{k} + f_2(n,k)\right], \tag{34}$$

where the constant C_1 is introduced to keep the associated structure function positive definite, if necessary. Further, in equation (34) the grading functions $f_0(n,k)$ and $f_1(n,k)$ are defined in equations (28) and (30), respectively, while the second grading function $f_2(n,k)$ is found to be

$$f_2(n,k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\frac{t}{k}\left(\frac{t}{k}-1\right)+\frac{1}{6}\right)\omega_{s+t}\right]\Delta_{n,s}\,, \tag{35}$$

where the convention $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ is understood. Likewise, only one term is singled out in the index s summation that satisfies the condition: $n-s = 0 \bmod k$. Setting $k=4$ in equation (35), we find that $f_2(4i,4) = \frac{1}{48}(8\omega_0-\omega_1-4\omega_2-\omega_3)$, $f_2(4i+1,4) = \frac{1}{48}(8\omega_1-\omega_2-4\omega_3-\omega_0)$, $f_2(4i+2,4) = \frac{1}{48}(8\omega_2-\omega_3-4\omega_0-\omega_1)$, and $f_2(4i+3,4) = \frac{1}{48}(8\omega_3-\omega_0-4\omega_1-\omega_2)$, where $i=0,1,2,\ldots$ Here, we suppress in all grading functions the second subscript "1" of $\omega_{s,1}$.

Much the same as in the preceding subsection, if we introduce the modified version of the second grading function by

$$\tilde{f}_2(n,k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\frac{t+n}{k}\left(\frac{t+n}{k}-1\right)+\frac{1}{6}\right)\omega_{s+t}\right]\Delta_{n,s}\,, \tag{36}$$

the structure function $\mathcal{F}_1(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ in equation (34) will reduce to

$$\mathcal{F}_1\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_1 - \frac{1}{2}\tilde{f}_2(n,k)\,. \tag{37}$$

Consequently, the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ can be algebraically determined: $E_n^{(-)} = \frac{1}{2}(\tilde{f}_2(n,k) - \tilde{f}_2(0,k))$, by using equation (19).

We note that, in this class, a similar algebraic result for shape invariant potentials in two steps has been discussed in [20, 22], using different approaches. Similarly, based on different Z_k-grading functions, the algebraic properties of shape invariant potentials in arbitrary k steps are also established [23]. As for the analytical properties of translational shape invariant potentials in k steps, we do not know much about them, except for the singular Pöschl–Teller I & II potentials in two steps [20].

3.3. The second order remainder

We continue to discuss translational shape invariant potentials in k steps, in which the unified remainder is second order in $(N_0 - \mathcal{N})$, which is given by $\mathcal{R}_2(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1} \omega_{s,2}(N_0 - \mathcal{N})^2\, \Pi_s$. When acting the remainder $\mathcal{R}_2(\boldsymbol{\alpha}(\mathcal{N}))$ on the associated number eigenstates $|N_0 - \frac{n}{k}\rangle$ (for $n = 0, 1, 2, \ldots$), we find that $\sum_{s=0}^{k-1} \omega_{s,2}(\frac{n}{k})^2 \Delta_{n,s}$, where $\Delta_{n,s}$ is in equation (26). The first few members of the remainder are listed as $\mathcal{R}_2(\boldsymbol{\alpha}(N_0 - \frac{n}{k})) = (0, (\frac{1}{k})^2\omega_1, (\frac{2}{k})^2\omega_2, \ldots, (\frac{k-1}{k})^2\omega_{k-1}, \omega_0, (\frac{k+1}{k})^2\omega_1, \ldots)$ corresponding to the choices of $n = (0, 1, 2, \ldots, k-1, k, k+1, \ldots)$. The second subscript "2" of all $\omega_{s,2}$ is suppressed.

In the same vein, the structure function $\mathcal{F}_2(\boldsymbol{\alpha}(\mathcal{N}))$ can be determined via equation (16). We afterwards find (for $n = 0, 1, 2, \ldots$)

$$\begin{aligned}\mathcal{F}_2\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) \;=\; & C_2 - \frac{1}{3}\Bigg[f_0(n,k)\left(\frac{n}{k}\right)^3 + 3f_1(n,k)\left(\frac{n}{k}\right)^2 \\ & + 3f_2(n,k)\,\frac{n}{k} + f_3(n,k)\Bigg]\,,\end{aligned} \tag{38}$$

where the constant C_2 is introduced to yield the structure function positive definite, if necessary. In equation (38), the grading functions $f_0(n,k)$, $f_1(n,k)$ and $f_2(n,k)$ have been defined. The third grading function $f_3(n,k)$ is found to take the form

$$f_3(n,k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\frac{t}{k}\left(\frac{t}{k} - \frac{1}{2}\right)\left(\frac{t}{k} - 1\right)\omega_{s+t}\right]\Delta_{n,s}\,, \tag{39}$$

where the convention $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ is known. The term that is singled out in the index s summation satisfies the condition: $n - s = 0 \bmod k$. As an example, if we let $k = 4$ in equation (39), then $f_3(4i, 4) = \frac{1}{64}(3\omega_1 - 3\omega_3)$, $f_3(4i+1, 4) = \frac{1}{64}(3\omega_2 - 3\omega_0)$, $f_3(4i+2, 4) = \frac{1}{64}(3\omega_3 - 3\omega_1)$, and $f_3(4i+3, 4) = \frac{1}{64}(3\omega_0 - 3\omega_2)$, where $i = 0, 1, 2, \ldots$ The second subscript "2" of $\omega_{s,2}$ in the four grading functions is once more omitted.

Now, the structure function $\mathcal{F}_2(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ can be shown to be expressible into the following more compact form

$$\mathcal{F}_2\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_2 - \frac{1}{3}\tilde{f}_3(n, k)\,, \tag{40}$$

if we introduce the modified third grading function by

$$\tilde{f}_3(n, k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\frac{t+n}{k}\left(\frac{t+n}{k} - \frac{1}{2}\right)\left(\frac{t+n}{k} - 1\right)\omega_{s+t}\right]\Delta_{n,s}\,. \tag{41}$$

In the same way, the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ is algebraically determined: $E_n^{(-)} = \frac{1}{3}(\tilde{f}_3(n, k) - \tilde{f}_3(0, k))$, by equation (19).

To the author's knowledge, we experience little in analytical properties of translational shape invariant potentials in k steps, in which the remainder is quadratic in the number eigenvalue $(\frac{n}{k})$. Perhaps, the only exception is the restricted shape invariant potentials in two steps studied in [21], in which the gap parameters $\omega_{0,2}$ and $\omega_{1,2}$ of the unified remainder obey the relation $\omega_{1,2} = -\omega_{0,2}$. It thus indicates that $f_0(n, k) = 0$ in equation (38). That is to say, the energy spectrum of the restricted type of potentials is still quadratic in $(\frac{n}{k})$, similar to that in the preceding class. The partial algebraic properties for those restricted shape invariant potentials in two steps have been discussed in [22].

3.4. The third order remainder

Before discussing the general properties of translational shape invariant potentials in k steps, in which the remainder can be arbitrary order in $(N_0 - \mathcal{N})$, let us present one more class. After this class, we shall know enough to deduce the common algebraic structures that share with all the translational shape invariant potentials in k steps.

We hence analyze translational shape invariant potentials in k steps, in which the unified remainder is expressed as $\mathcal{R}_3(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1}\omega_{s,3}$ $(N_0 - \mathcal{N})^3\,\Pi_s$. When acting the remainder $\mathcal{R}_3(\boldsymbol{\alpha}(\mathcal{N}))$ on the associated number eigenstates $|N_0 - \frac{n}{k}\rangle$ (for $n = 0, 1, 2, \ldots$), we have $\sum_{s=0}^{k-1}\omega_{s,3}(\frac{n}{k})^3\Delta_{n,s}$, where $\Delta_{n,s}$ is in equation (26). The first few members of the remainder

are listed as $\mathcal{R}_3(\boldsymbol{\alpha}(N_0 - \frac{n}{k})) = (0, (\frac{1}{k})^3\omega_1, (\frac{2}{k})^3\omega_2, \ldots, \omega_0, (\frac{k+1}{k})^3\omega_1, \ldots)$ corresponding to the choices of $n = (0, 1, 2, \ldots, k, k+1, \ldots)$. Similarly, the second subscript "3" of all $\omega_{s,3}$ is suppressed.

Much in the same way as in the previous classes, the structure function $\mathcal{F}_3(\boldsymbol{\alpha}(\mathcal{N}))$ can be determined via equation (16). Written out explicitly, it is (for $n = 0, 1, 2, \ldots$)

$$\mathcal{F}_3\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_3 - \frac{1}{4}\left[f_0(n,k)\left(\frac{n}{k}\right)^4 + 4f_1(n,k)\left(\frac{n}{k}\right)^3 + 6f_2(n,k)\left(\frac{n}{k}\right)^2 + 4f_3(n,k)\,\frac{n}{k} + f_4(n,k)\right], \quad (42)$$

where the constant C_3 is chosen to make the structure function positive, if necessary. In equation (42), the grading functions $f_i(n,k)$ (for $i = 0, 1, 2, 3$) have been defined earlier, while the fourth grading function $f_4(n,k)$ is given by

$$f_4(n,k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\left(\frac{t}{k}\right)^2\left(\frac{t}{k} - 1\right)^2 - \frac{1}{30}\right)\omega_{s+t}\right]\Delta_{n,s}\,, \quad (43)$$

where the convention $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ is understood. Likewise, only the term, satisfying the condition $n = s \bmod k$, survives in the index s summation. Taking $k = 4$ in equation (43), we obtain the awkward results: $f_4(4i,4) = \frac{1}{3840}(-128\omega_0 + 7\omega_1 + 112\omega_2 + 7\omega_3)$, $f_4(4i+1,4) = \frac{1}{3840}(-128\omega_1 + 7\omega_2 + 112\omega_3 + 7\omega_0)$, $f_4(4i+2,4) = \frac{1}{3840}(-128\omega_2 + 7\omega_3 + 112\omega_0 + 7\omega_1)$, and $f_4(4i+3,4) = \frac{1}{3840}(-128\omega_3 + 7\omega_0 + 112\omega_1 + 7\omega_2)$, where $i = 0, 1, 2, \ldots$ Again, the second subscript "3" of $\omega_{s,3}$ in all the grading functions is omitted.

Furthermore, the complicated expression of the structure function $\mathcal{F}_3(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ in equation (42) can be recast into the relatively simple form

$$\mathcal{F}_3\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_3 - \frac{1}{4}\tilde{f}_4(n,k)\,, \quad (44)$$

after we denote the modified fourth grading function by

$$\tilde{f}_4(n,k) = \sum_{s=0}^{k-1}\left[\sum_{t=0}^{k-1}\left(\left(\frac{t+n}{k}\right)^2\left(\frac{t+n}{k} - 1\right)^2 - \frac{1}{30}\right)\omega_{s+t}\right]\Delta_{n,s}\,. \quad (45)$$

The energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ can be similarly determined via equation (19) as $E_n^{(-)} = \frac{1}{4}(\tilde{f}_4(n,k) - \tilde{f}_4(0,k))$.

Up to now, let us give a brief remark on what we have learned concerning the structure functions and grading functions, constructed from the order-by-order remainders $\mathcal{R}_i(\boldsymbol{\alpha}(\mathcal{N}))$ in $(N_0 - \mathcal{N})$. As for the structure functions

$\mathcal{F}_i(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ presented in equation (29), (34), (38), and (42), they seem to follow some sort of binomial-like expansions of power $(i+1)$, for $i = 0, 1, 2, 3$, respectively. With regard to the grading functions $f_i(n,k)$ shown in equations (30), (35), (39), and (43) that are used to characterize the Z_k-grading symmetry, they seem to relate to one another by certain integral relations. The similar pattern can also be detected among the modified grading functions $\tilde{f}_i(n,k)$. In the next subsection, we will show that the naive observation is indeed correct and that the general expressions for the structure functions and grading functions can be formally written out.

3.5. The arbitrary order remainder

We are now at a position to talk over the general algebraic structures of translational shape invariant potentials in k steps, in which the remainder can be arbitrary order in $(N_0 - \mathcal{N})$. To be more specific, the unified remainder in the present class takes the form $\mathcal{R}_I(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1} \omega_{s,I} (N_0 - \mathcal{N})^I \, \Pi_s$, where I is an arbitrary integer. When acting the remainder $\mathcal{R}_I(\boldsymbol{\alpha}(\mathcal{N}))$ on the associated number eigenstates $|N_0 - \frac{n}{k}\rangle$ (for $n = 0, 1, 2, \ldots$), we obtain the result $\sum_{s=0}^{k-1} \omega_{s,I} (\frac{n}{k})^I \Delta_{n,s}$, where $\Delta_{n,s}$ is defined in equation (26).

The structure function $\mathcal{F}_I(\boldsymbol{\alpha}(\mathcal{N}))$ can be similarly determined via equation (16). After some algebras, we find that it is as anticipated given by the following binomial-like expansion as (for $n = 0, 1, 2, \ldots$)

$$\mathcal{F}_I\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_I - \frac{1}{I+1}\left[\sum_{i=0}^{I+1} \binom{I+1}{i} f_i(n,k) \left(\frac{n}{k}\right)^{I+1-i}\right], \quad (46)$$

where $\binom{I+1}{i}$ is the binomial coefficient and the constant C_I is chosen to render the associated structure function positive, if necessary.

In addition, all the grading functions $f_i(n,k)$ appearing in equation (46) can be systematically determined through certain integral equations. In Appendix A, we depict the determination of the grading functions $f_i(n,k)$ by the method of recurrence relations, which in turn shows that the constructed grading functions are surprisingly expressible in terms of the Bernoulli functions $B_i(x)$ as

$$f_i(n,k) = \sum_{s=0}^{k-1} \left[\sum_{t=0}^{k-1} B_i\left(\frac{t}{k}\right) \omega_{s+t}\right] \Delta_{n,s}\,, \quad (47)$$

where the convention $\omega_{s+t} \equiv \omega_{s+t \bmod k}$ is understood. In equation (47), the symbol $\Delta_{n,s}$ singles out in the s summation the term that satisfies the condition $n = s \bmod k$, and the second subscript "I" of $\omega_{s,I}$ is suppressed as before.

In the same manner, by introducing the modified notion of grading functions

$$\tilde{f}_i(n,k) = \sum_{s=0}^{k-1} \left[\sum_{t=0}^{k-1} B_i \left(\frac{t+n}{k} \right) \omega_{s+t} \right] \Delta_{n,s} \,, \tag{48}$$

we readily obtain the structure function $\mathcal{F}_I(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ in the compact expression

$$\mathcal{F}_I\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C_I - \frac{1}{I+1}\tilde{f}_{I+1}(n,k) \,. \tag{49}$$

As a closing note, the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ in the present class can be obtained by equation (19), as before.

3.6. A closed-form example

An instructive example is given here to demonstrate the closed-form algebraic properties of translational shape invariant potentials in arbitrary k steps. To show this, the unified remainder is taken to be an analytic function of $(N_0 - \mathcal{N})$ and admits the Taylor series: $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N})) = \sum_{s=0}^{k-1} \sum_{I=0}^{\infty} \omega_{s,I} (N_0 - \mathcal{N})^I \, \Pi_s$. When the remainder $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ is acted on the number eigenstate $|N_0 - \frac{n}{k}\rangle$ (for $n = 0, 1, 2, \ldots$), we find that

$$\mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = \sum_{s=0}^{k-1} \left[\sum_{I=0}^{\infty} \omega_{s,I} \left(\frac{n}{k}\right)^I \right] \Delta_{n,s} \,. \tag{50}$$

The corresponding structure function $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ can be readily constructed from equation (49). We therefore have

$$\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = C - \sum_{s,t=0}^{k-1} \left[\sum_{I=0}^{\infty} \frac{1}{I+1} B_{I+1}\left(\frac{t+n}{k}\right) \omega_{s+t,I} \right] \Delta_{n,s} \,, \tag{51}$$

where C is a constant to yield the above structure function positive definite, if necessary. Here, the modified grading function $\tilde{f}_{I+1}(n,k)$ in equation (51) has been replaced by using equation (48), and $B_{I+1}(x)$ is the $(I+1)$th Bernoulli function. Note that the second subscript "I" of $\omega_{s+t,I}$ is explicitly written out.

To proceed, we consider the example of quantum mechanical system, in which the remainder function, *i.e.*, the energy gap between two adjacent eigenstates, is of the form

$$\mathcal{R}\left(\boldsymbol{\alpha}\left(N_0 - \frac{n}{k}\right)\right) = \sum_{s=0}^{k-1} \gamma_s \, e^{-\frac{n}{k}\varepsilon_s} \, \Delta_{n,s} = \sum_{s=0}^{k-1} \gamma_s \left[\sum_{I=0}^{\infty} \frac{(-\varepsilon_s)^I}{I!} \left(\frac{n}{k}\right)^I \right] \Delta_{n,s} \,, \tag{52}$$

where γ_s and ε_s are positive constants. When comparing (52) with (50), we immediately have $\omega_{s,I} = \gamma_s(-\varepsilon_s)^I/I!$, which results in the following exact structure function of the system

$$\begin{aligned}\mathcal{F}\left(\boldsymbol{\alpha}\left(N_0-\frac{n}{k}\right)\right) &= C-\sum_{s,t=0}^{k-1}\gamma_{s+t}\left[\sum_{I=0}^{\infty}B_{I+1}\left(\frac{t+n}{k}\right)\frac{(-\varepsilon_{s+t})^I}{(I+1)!}\right]\Delta_{n,s}\,,\\ &= C-\sum_{s,t=0}^{k-1}\gamma_{s+t}\left[\frac{e^{-\left(\frac{t+n}{k}\right)\varepsilon_{s+t}}}{e^{-\varepsilon_{s+t}}-1}+\frac{1}{\varepsilon_{s+t}}\right]\Delta_{n,s}\,, \qquad (53)\end{aligned}$$

where, to go from the first line to the second one, we use $B_0(x) = 1$ and the property of generating function that defines Bernoulli functions [35]. Finally, according to equation (19), the energy spectrum of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ is given by $(n = 0, 1, 2, \ldots)$

$$E_n^{(-)} = \sum_{t=0}^{k-1}\left\{\sum_{s=0}^{k-1}\gamma_{s+t}\left[\frac{e^{-\left(\frac{t+n}{k}\right)\varepsilon_{s+t}}}{e^{-\varepsilon_{s+t}}-1}+\frac{1}{\varepsilon_{s+t}}\right]\Delta_{n,s}-\gamma_t\left[\frac{e^{-\frac{t}{k}\varepsilon_t}}{e^{-\varepsilon_t}-1}+\frac{1}{\varepsilon_t}\right]\right\}. \qquad (54)$$

Correctness of the exact expression of energy spectrum (54) can be easily checked for the ordinary shape invariant potentials in SUSY QM, that is, for the special case $k = 1$. When $k = 1$, there is no grading structure in the associated potential algebra, so that the grading indices are taken as $s = t = 0$. If we now denote $\gamma_0 = \gamma$ and $\varepsilon_0 = \varepsilon$, equation (54) is reduced to the simple form

$$E_n^{(-)} = \gamma\,\frac{1-e^{-n\varepsilon}}{1-e^{-\varepsilon}}\,, \qquad (55)$$

which yields $E_0^{(-)} = 0$ and $E_1^{(-)} = \gamma > 0$ as is necessary by the requirement of unbroken SUSY. Furthermore, let us compute the energy difference between two adjacent eigenstates from equation (55). It is found to be

$$\mathcal{R}(\boldsymbol{\alpha}(N_0-n)) = E_{n+1}^{(-)} - E_n^{(-)} = \gamma\, e^{-n\varepsilon}\,. \qquad (56)$$

Consistency of equations (56) and (52) (when letting $k = 1$ and $s = 0$ in the latter one) is therefore manifest. Finally, it is pointed out that in the limiting case $\varepsilon \to 0$, equation (55) becomes $E_n^{(-)} = n\gamma$, which is nothing but the eigenenergy spectrum of simple harmonic oscillator.

4. Conclusions

In this article, the algebra of translational shape invariant potentials in k steps is investigated, within the framework of SUSY QM. We start with the k arbitrary relations (3) of shape invariance condition in k steps and then impose the identification (5) among the k superpotentials and k remainders. The simplified potential algebra of shape invariance in k steps (6) is accordingly deduced, which is found to be equivalent to be the generalized deformed oscillator algebra with a built-in Z_k-grading structure. Resembling the latter one, the simplified potential algebra of shape invariance in k steps is, therefore, described by the similar set of operators $\{I, \mathcal{A}, \mathcal{A}^\dagger, \mathcal{N}, \Pi_s\}$ (for $s = 0, 1, \ldots, k-1$), which fulfill equations (13), (14), and (15).

The detailed algebraic properties of translational shape invariant potentials in k steps are then determined. The remainders $R_s(a_m)$ are taken to be analytic functions of the parameter a_m, in which the parameters of partner potentials are related to each other by translation: $a_m = a_{m-1} + \delta$. Without loss of generality, we further set $a_{\frac{n}{k}} = \boldsymbol{\alpha}(N_0 - \frac{n}{k})$ in equation (20) and expand the unified remainder $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ in the power series of $(N_0 - \mathcal{N})$ (27). For the purpose of illustration, we explicitly work out four classes of translational shape invariant potentials in k steps, in which the unified remainder $\mathcal{R}_i(\boldsymbol{\alpha}(\mathcal{N}))$ (for $i = 0, 1, 2, 3$) is zeroth, first, second, and third order in $(N_0 - \mathcal{N})$, respectively. In each class, we determine the corresponding structure function $\mathcal{F}_i(\boldsymbol{\alpha}(\mathcal{N}))$ that defines the algebra of Z_k-graded deformed oscillators. In addition, the grading functions $f_i(n, k)$ and the modified versions $\tilde{f}_i(n, k)$ that characterize the inherited Z_k-grading symmetry are also constructed.

From the constructed structure functions $\mathcal{F}_i(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ (for $i = 0, 1, 2, 3$), we observe that, when the unified remainder $\mathcal{R}_I(\boldsymbol{\alpha}(\mathcal{N}))$ is Ith order in $(N_0 - \mathcal{N})$, the general form of structure function $\mathcal{F}_I(\boldsymbol{\alpha}(N_0 - \frac{n}{k}))$ can be written in the binomial-like expansion (46). Up to a constant, it is also identical to the $(I+1)$th modified grading function $\tilde{f}_{I+1}(n, k)$ (49). In addition, all the grading functions $f_i(n, k)$ are found to be expressible in terms of the Bernoulli functions (47), which are shown in Appendix A to relate to one another by recurrence relations. Finally, we discuss a typical example of remainder function (52) to demonstrate how the infinite series of structure function (51) can be summed up to render the closed-form result (53). Consistency of the obtained energy eigenvalues (54) is afterwards verified, for the special case $k = 1$. To conclude the paper, we emphasize that the energy eigenvalues of the initial Hamiltonian $H_0^{(-)}(x, \boldsymbol{\alpha}(N_0))$ of shape invariance in k steps can be completely determined by purely algebraic means, when the unified remainder $\mathcal{R}(\boldsymbol{\alpha}(\mathcal{N}))$ is any analytic function of $(N_0 - \mathcal{N})$.

This work is funded in part by the National Science Council in Taiwan — grant no. NSC 100-2112-M-194-001.

Appendix A

In this appendix, the grading functions $f_i(n,k)$ that are used to construct the Z_k-graded structure functions $\mathcal{F}(\boldsymbol{\alpha}(\mathcal{N}))$ of shape invariant potentials in k-steps, presented in Sec. 3, will be systematically determined by the recurrence relations developed below. It turns out that the general form of grading functions is expressible in terms of the Bernoulli functions.

From equations (30), (35), (39), and (43), we conclude that the general expression for the ith grading function $f_i(n,k)$ is given by

$$f_i(n,k) = \sum_{s=0}^{k-1} \left[\sum_{t=0}^{k-1} B_i\left(\frac{t}{k}\right) \omega_{s+t} \right] \Delta_{n,s} \,, \tag{57}$$

in which the ith coefficient function $B_i(x)$ (for $x \equiv \frac{t}{k}$ and $i = 0, 1, 2, \ldots$) is formally defined as a certain polynomial of x of order i. Note that $B_i(x)$ is not the Bernoulli functions, yet. When integrating the coefficient function $B_i(x)$ over x, we obtain the resultant $(i+1)$th function $D_{i+1}(x)$ of x of order $i+1$ in the form

$$D_{i+1}(x) \equiv (i+1) \int_0^x B_i\left(x'\right) dx' \,, \tag{58}$$

in such a way that the boundary conditions are to be satisfied: $D_{i+1}(0) = D_{i+1}(1) = 0$. Further, the next, $(i+1)$th coefficient function $B_{i+1}(x)$ is given by $D_{i+1}(x)$ through the definition

$$B_{i+1}(x) \equiv D_{i+1}(x) + c_{i+1} \,, \tag{59}$$

where c_{i+1} is a constant that is to be determined by a relation similar to equation (58), with i replaced by $i+1$, such that $D_{i+2}(0) = D_{i+2}(1) = 0$. Based on the two defining recurrence relations (58) and (59), the entire sequence of the coefficient functions $B_i(x)$ can thus be determined, term-by-term.

To be more explicitly, let us work out some of the functions $B_i(x)$ and $D_i(x)$. We start with the simplest case $D_1(x) = x$, then $B_1(x) = x + c_1$, according to equation (59). Substituting the expression of $B_1(x)$ into equation (58), we arrive at the function $D_2(x) = x(x + 2c_1)$. To fulfill the boundary condition $D_2(1) = 0$, we have to choose $c_1 = -\frac{1}{2}$ and therefore

$$B_1(x) = x - \tfrac{1}{2} \,. \tag{60}$$

Secondly, from $D_2(x) = x(x-1)$, we write $B_2(x) = D_2(x) + c_2$, which is then substituted to equation (58) to render $D_3(x) = x(x^2 - \frac{3}{2}x + 3c_2)$. The boundary condition thus implies $c_2 = \frac{1}{6}$ and in addition

$$B_2(x) = x(x-1) + \tfrac{1}{6}\,. \tag{61}$$

Thirdly, from $D_3(x) = x(x-\frac{1}{2})(x-1)$, we define $B_3(x) = D_3(x) + c_3$. The use of equation (58) shows that $D_4(x) = x^2(x-1)^2 + 4c_3x$. As a result, we have $c_3 = 0$ and

$$B_3(x) = x(x-1)\left(x - \tfrac{1}{2}\right)\,. \tag{62}$$

Fourthly, from $D_4(x) = x^2(x-1)^2$, we let $B_4(x) = D_4(x) + c_4$. Then equation (58) renders us that $D_5(x) = x(x-\frac{1}{2})(x-1)(x^2 - x - \frac{1}{3})$, with $c_4 = -\frac{1}{30}$. Therefore,

$$B_4(x) = x^2(x-1)^2 - \tfrac{1}{30}\,. \tag{63}$$

The search for the higher coefficient functions $B_i(x)$ can be continued, though the resultant coefficient functions become complicated and complicated. For the purpose of completeness, we list some of the calculated functions $B_i(x)$, for i up to eight, when the recurrence relations (58) and (59) are repeatedly used. They are

$$B_5(x) = x(x-1)\left(x-\tfrac{1}{2}\right)\left(x^2 - x - \tfrac{1}{3}\right)\,, \tag{64}$$

$$B_6(x) = x^2(x-1)^2\left(x^2 - x - \tfrac{1}{2}\right) + \tfrac{1}{42}\,, \tag{65}$$

$$B_7(x) = x(x-1)\left(x-\tfrac{1}{2}\right)\left(x^4 - 2x^3 + x + \tfrac{1}{3}\right)\,, \tag{66}$$

$$B_8(x) = x^2(x-1)^2\left(x^4 - 2x^3 - \tfrac{1}{3}x^2 + \tfrac{4}{3}x + \tfrac{2}{3}\right) - \tfrac{1}{30}\,. \tag{67}$$

Incidentally, the above constructed coefficient functions $B_i(x)$ ($i = 1$ to 8) that are used to define the structure functions $f_i(k,s)$ in equation (57), are nothing but the Bernoulli functions, which satisfy $B_i'(x) = iB_{i-1}(x)$ and reduce to the usual Bernoulli numbers when setting $B_i(x=0) = B_i$ [35].

A final remark is given. All the odd indexed constants c_{2m+1}, except for c_1, in the above computations are found identically to be zero, so that we have $B_{2m+1}(x) = D_{2m+1}(x)$, for any $m = 1, 2, 3, \ldots$ The reason for this is that the function $D_{2m+1}(x)$ is actually an antisymmetric function about the point $x = \frac{1}{2}$ and vanishes at the boundaries $D_{2m+1}(0) = D_{2m+1}(1)$. Therefore, the integration of $D_{2m+1}(x)$ over $x \in [0,1]$ solely give vanishing result, even without the help of the constant c_{2m+1}. Nevertheless, this is not the case for the even indexed functions $D_{2m}(x)$, where they are symmetric functions about $x = \frac{1}{2}$. To obtain vanishing result after integration over $x \in [0,1]$, we need to add to every $D_{2m}(x)$ the associated constant c_{2m}, thus resulting in the relation $B_{2m}(x) = D_{2m}(x) + c_{2m}$.

REFERENCES

[1] E. Witten, *Nucl. Phys.* **B188**, 513 (1981).

[2] F. Cooper, B. Freedman, *Ann. Phys.* **146**, 262 (1983).

[3] L. Infeld, T.E. Hull, *Rev. Mod. Phys.* **23**, 21 (1952).

[4] F. Cooper, A. Khare, U. Sukhatme, *Phys. Rep.* **251**, 267 (1995).

[5] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, Springer, Berlin 1996.

[6] B.K. Bagchi, *Supersymmetry in Quantum and Classical Mechanics*, Chapman and Hall, London 2000.

[7] F. Cooper, A. Khare, U. Sukhatme, *Supersymmetry in Quantum Mechanics*, World Scientific, Singapore 2001.

[8] L.E. Gendenshtein, *JETP Lett.* **38**, 356 (1983).

[9] F. Cooper, J.N. Ginocchio, A. Khare, *Phys. Rev.* **D36**, 2458 (1987).

[10] C. Chuan, *J. Phys. A: Math. Gen.* **24**, L1165 (1991); R. Dutt, A. Khare, U. Sukhatme, *Am. J. Phys.* **56**, 163 (1988); G. Levai, *J. Phys. A: Math. Gen.* **22**, 689 (1989).

[11] A. Khare, U. Sukhatme, *J. Phys. A: Math. Gen.* **26**, L901 (1993); D.T. Barclay *et al.*, *Phys. Rev.* **A48**, 2786 (1993).

[12] V. Shabat, *Inv. Prob.* **8**, 303 (1992); V. Spiridonov, *Phys. Rev. Lett.* **69**, 398 (1992).

[13] U.P. Sukhatme, C. Rasinariu, A. Khare, *Phys. Lett.* **A234**, 401 (1997).

[14] R. Dutt, A. Khare, U. Sukhatme, *Phys. Lett.* **B181**, 295 (1986); J. Dabrowska, A. Khare, U. Sukhatme, *J. Phys. A: Math. Gen.* **21**, L195 (1988); F. Cooper, J.N. Ginocchio, A. Wipf, *Phys. Lett.* **A129**, 145 (1988); A. Khare, U. Sukhatme, *J. Phys. A: Math. Gen.* **21**, L501 (1988).

[15] A.O. Barut, A. Inomata, R. Wilson, *J. Phys. A: Math. Gen.* **20**, 4075 (1987); *J. Phys. A: Math. Gen.* **20**, 4083 (1987); M.J. Englefield, *J. Math. Phys.* **28**, 827 (1987); M.J. Englefield, C. Quesne, *J. Phys. A: Math. Gen.* **24**, 3557 (1987); R.D. Tangerman, J.A. Tjon, *Phys. Rev.* **A48**, 1089 (1993).

[16] H.Y. Cheung, *Nuovo Cim.* **B101**, 193 (1988); J. Wu, Y. Alhassid, F. Gürsey, *Ann. Phys.* **196**, 163 (1989); J. Wu, Y. Alhassid, *J. Math. Phys.* **31**, 557 (1990).

[17] A. Gangopadhyaya, J.V. Mallow, U.P. Sukhatme, *Phys. Rev.* **A58**, 4287 (1998); S. Chaturvedi *et al.*, *Phys. Lett.* **A248**, 109 (1998); R. Dutt, A. Gangopadhyaya, C. Rasinariu, U. Sukhatme, *Phys. Rev.* **A60**, 3482 (1999); A. Gangopadhyaya, J.V. Mallow, U.P. Sukhatme, *Phys. Lett.* **A283**, 279 (2001).

[18] A.B. Balantekin, *Phys. Rev.* **A57**, 4188 (1998); A.B. Balantekin, M.A. Cândido Ribeiro, A.N.F. Aleixo, *J. Phys. A: Math. Gen.* **32**, 2785 (1999); A.N.F. Aleixo, A.B. Balantekin, *J. Phys. A: Math. Gen.* **37**, 8513 (2004).

[19] A. Gangopadhyaya, U.P. Sukhatme, *Phys. Lett.* **A224**, 5 (1996).

[20] R. Dutt, A. Gangopadhyaya, C. Rasinariu, U. Sukhatme, *J. Phys. A: Math. Gen.* **34**, 4129 (2001).

[21] W.-C. Su, *J. Phys. A: Math. Theor.* **41**, 255307 (2008).

[22] W.-C. Su, *J. Phys. A: Math. Theor.* **41**, 435301 (2008).

[23] W.-C. Su, *J. Phys. A: Math. Theor.* **42**, 385202 (2009).

[24] C. Quesne, N. Vansteenkiste, *Helv. Phys. Acta* **72**, 71 (1999).

[25] M. Daoud, M.R. Kibler, *J. Math. Phys.* **47**, 122108 (2006).

[26] M.M. Nieto, L.M. Simmons Jr, *Am. J. Phys.* **47**, 634 (1979).

[27] C. Daskaloyannis, *J. Phys. A: Math. Gen.* **24**, L789 (1991); *J. Phys. A: Math. Gen.* **25**, 2261 (1992); C. Daskaloyannis, K. Ypsilantis, *J. Phys. A: Math. Gen.* **25**, 4157 (1992); C. Daskaloyannis, *J. Phys. A: Math. Gen.* **26**, 1589 (1993).

[28] M.S. Plyushchay, *Ann. Phys.* **245**, 339 (1996); *Nucl. Phys.* **B491**, 619 (1997); S. Klishevich, M.S. Plyushchay, *Mod. Phys. Lett* **A14**, 2739 (1999); M.S. Plyushchay, *Int. J. Phys.* **A15**, 3679 (2000).

[29] J. Katriel, C. Quesne, *J. Math. Phys.* **37**, 1650 (1996); C. Quesne, N. Vansteenkiste, *J. Phys. A: Math. Gen.* **28**, 7019 (1995); *Helv. Phys. Acta* **69**, 141 (1996).

[30] G. Brodimas *et al.*, *Lett. Nuovo Cim.* **31**, 177 (1981); A. Jannussis, D. Vavougios, *Hadron. J.* **7**, 947 (1984); *Hadron. J.* **10**, 75 (1987); K. Odaka, T. Kishi, S. Kamefuchi, *J. Phys. A: Math. Gen.* **24**, L591 (1991); J. Beckers, N. Debergh, *J. Phys. A: Math. Gen.* **24**, L1277 (1991).

[31] C. Quesne, N. Vansteenkiste, *Phys. Lett.* **A240**, 21 (1998); *Int. J. Theor. Phys.* **39**, 1175 (2000).

[32] M. Daoud, M. Kibler, *Phys. Part. Nucl.* **33**, S43 (2002); *Phys. Lett.* **A321**, 147 (2004).

[33] M.A. Vasiliev, *Int. J. Mod. Phys.* **A6**, 1115 (1991).

[34] P.A. Horváthy, M.S. Plyushchay, *Phys. Lett.* **B595**, 547 (2004); F. Correa, M.S. Plyushchay, *Ann. Phys.* **322**, 2493 (2007); P.A. Horváthy, M.S. Plyushchay, M. Valenzuela, *Nucl. Phys.* **B768**, 247 (2007); F. Correa, V. Jakubsky, L.-M. Nieto, M.S. Plyushchay, *Phys. Rev. Lett.* **101**, 030403 (2008).

[35] G.B. Arfken, H.J. Weber, *Mathematical Methods for Physicists*, Elsevier Academic Press, 6th Ed., 2005.