COULOMB-GAS APPROACH FOR PERCOLATION THEORY

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The aim of this work is to present a non-trivial confirmation of the powerful Coulomb gas-techniques for Boundary Conformal Field Theory (BCFT). We show that we can re-derive the known Cardy's result of percolation problem via the techniques developed by S. Kawai in the Coulomb gas formalism.

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1. Critical percolation and conformal invariance

In BCFTs, Coulomb gas formalism presents a strong tool to obtain correlation functions without having to solve differential equations not only in the case of conformal minimal models but also in different models such as percolation model. In his papers [1, 2], Kawai presents a general formalism to compute correlation functions in the half plane using the free-field construction of boundary states and applying the Coulomb gas formalism. This formalism was applied for the critical Ising model with free and fixed boundary conditions obtained from Cardy's boundary states. In this work, we will use Kawai's techniques to provide the percolation crossing probability which is a two-point correlation function on the upper half plane.

In the thermodynamic limit and, of course, at the critical point, percolation is believed to be described by a conformal field theory $M(p' = 2, p = 3)$ with vanishing central charge [3]. Crossing probability is of great interest in studies of percolation. In two dimensions and in geometries with edges

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(a rectangle for example), a crossing event is a configuration of bonds or sites on the lattice covering this geometry such that there exists at least one cluster connecting two disjoint segments of the boundary. See figure 1 for the case of the rectangle.

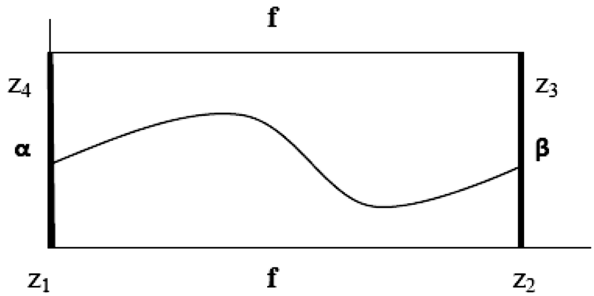

Fig. 1. Crossing cluster between the two segments of the rectangle.

Let π be the crossing probability associated with this event, and p be the probability for each bond to be open (and $1-p$ to be closed). Then, there is a critical value of p called the percolation threshold such as

$$\pi = \left\{ \begin{array}{ccc} 0 & \text{if} & p < p_c \\ \pi(p,r) & \text{if} & p = p_c \\ 1 & \text{if} & p > p_c \end{array} \right\} , \tag{1}$$

where r represents the aspect ratio of the rectangle (height/width). The most familiar way to think about percolation as a critical phenomenon is through the $Q \to 1$ limit of the Q-state Potts model. Let $Z_{\alpha\beta}$ be the partition function of the Q-state Potts model with the constraint that all spins at lattice sites on S_1 are fixed in the state α and all the spins on S_2 are fixed in the state β. The boundary spins are free on the horizontal sides of the rectangle. The existence of a cluster would force spins on the two segments (S_1, S_2) to be in the same state. In this way, a formula for $\pi(p,r)$ on the lattice is obtained as [4, 5]

$$\pi(p,r) = \lim_{Q \to 1} (Z_{\alpha\alpha} - Z_{\alpha\beta}) , \tag{2}$$

where the partition functions we need are given in terms of correlators by

$$\begin{aligned} Z_{\alpha\alpha} &= Z_{\mathrm{f}} \left\langle \phi_{(f\alpha)}(z_1) \phi_{(\alpha f)}(z_2) \phi_{(f\alpha)}(z_3) \phi_{(\alpha f)}(z_4) \right\rangle , \\ Z_{\alpha\beta} &= Z_{\mathrm{f}} \left\langle \phi_{(f\alpha)}(z_1) \phi_{(\alpha f)}(z_2) \phi_{(f\beta)}(z_3) \phi_{(\beta f)}(z_4) \right\rangle , \end{aligned} \tag{3}$$

where Z_{f} is the partition function with free boundary conditions and $\phi_{(ij)}$ denote the boundary operator corresponding to a switch from boundary condition (i) to (j) at the point x. In our case, the relevant boundary changing operator is identified as being the $\phi_{(12)}$ boundary primary field in the $M(2,3)$ theory ($Q \to 1$ limit) [6,7]. For this particular minimal model we see, from the Kac table formula

$$\begin{aligned} h_{r,s} &= \frac{[r(m+1)-sm]^2-1}{4m(m+1)}\,, \\ r,s &> 0\,, \qquad m=2 \end{aligned} \tag{4}$$

that $h_{1,2}=0$ and we have, in addition, $\lim_{Q\to 1} Z_{\mathrm{f}} = 1$. Then, to obtain the crossing probability formula one has to find the form of the four-point correlation functions [8]

$$G(\eta) = \left\langle \phi_{(12)}(z_1)\phi_{(12)}(z_2)\phi_{(12)}(z_3)\phi_{(12)}(z_4)\right\rangle \tag{5}$$

which will depend only on the cross ratio

$$\eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}\,.$$

2. Correlation functions from the Coulomb-gas approach

The correlator (5) can be written in the upper half plane — by application of the method of images — (see [9]) as

$$\begin{aligned} G(\eta) &= \left\langle \phi_{(12)}(z_1)\phi_{(12)}(z_2)\phi_{(12)}(z_3)\phi_{(12)}(z_4)\right\rangle \\ &= \left\langle \phi_{(12)}(z_1,\overline{z}_1)\phi_{(12)}(z_2,\overline{z}_2)\right\rangle_{\mathrm{UHP}}\,, \end{aligned} \tag{6}$$

where $z_3=\overline{z}_1$ and $z_4=\overline{z}_2$.

In the Coulomb gas approach, the boundary 2-point correlation function for physical boundary conditions can be obtained by introducing the screened vertex operators [1,2]. In this case, the correlator (6) becomes

$$G(\eta) = \left\langle B(\alpha) \middle| V_{(12)}^{m_1,n_1}(z_1) V_{(12)}^{\overline{m}_1,\overline{n}_1}(\overline{z}_1) V_{(12)}^{m_2,n_2}(z_2) V_{(12)}^{\overline{m}_2,\overline{n}_2}(\overline{z}_2) \middle| 0,0;\alpha_0 \right\rangle\,,$$

where $|B(\alpha)\rangle$ is called the boundary coherent state, $|0,0;\alpha_0\rangle$ represents the vacuum state and V and $\overline{V}$ are the screened Vertex operators.

In order that the correlator be non-vanishing, we must satisfy the charge neutrality conditions

$$m+\overline{m}=0\,, \qquad n+\overline{n}=1 \tag{7}$$

then we have the boundary charge $\alpha = \alpha_{1,1} = 0$ for the first condition and $\alpha = \alpha_{1,3} = -\alpha_-$ for the second one, where

$$\begin{aligned} \alpha_{r,s} &= \tfrac{1}{2}(1-r)\alpha_+ + \tfrac{1}{2}(1-s)\alpha_- \,, \\ \alpha_+ &= \sqrt{\tfrac{p}{p'}} = \sqrt{\tfrac{3}{2}} \,, \\ \alpha_- &= -\sqrt{\tfrac{p'}{p}} = -\sqrt{\tfrac{2}{3}} \,. \end{aligned}$$

The first condition (7) corresponds to the conformal block

$$\begin{aligned} I_1 &= N_1 \left\{ \frac{(z_1 - \overline{z}_1)(\overline{z}_2 - z_2)}{(z_1 - z_2)(\overline{z}_1 - \overline{z}_2)(z_1 - \overline{z}_2)(\overline{z}_1 - z_2)} \right\}^{2h_{1,2}} \frac{\Gamma\left(1-\alpha_-^2\right)^2}{\Gamma\left(2-2\alpha_-^2\right)} \\ &\quad \times F\left(2b, 1-\alpha_-^2, 2-\alpha_-^2; \eta\right) = N_1 \frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)} F\left(0, \tfrac{1}{3}, \tfrac{4}{3}; \eta\right) \,, \end{aligned} \tag{8}$$

where N_1 is a constant, $b = 2\alpha_{1,2}(2\alpha_0 - \alpha_{1,2})$ with $\alpha_0 = \frac{1}{2\sqrt{6}}$ and F is the hypergeometric function of the Gaussian type. Whereas the second condition corresponds to the conformal block

$$\begin{aligned} I_2 &= N_2 \left\{ \frac{(z_1 - \overline{z}_1)(\overline{z}_2 - z_2)}{(z_1 - z_2)(\overline{z}_1 - \overline{z}_2)(z_1 - \overline{z}_2)(\overline{z}_1 - z_2)} \right\}^{2h_{1,2}} \\ &\quad \times \frac{\Gamma\left(1-\alpha_-^2\right)\Gamma\left(3\alpha_-^2 - 1\right)}{\Gamma\left(2\alpha_-^2\right)} \times (-\eta)^{2h_{1,2}+\frac{\alpha_-^2}{2}} F\left(\alpha_-^2, 1-\alpha_-^2, 2\alpha_-^2; \eta\right) \\ &= N_2 \frac{\Gamma\left(\frac{1}{3}\right)\Gamma(1)}{\Gamma\left(\frac{4}{3}\right)} (-\eta)^{\frac{1}{3}} F\left(\tfrac{2}{3}, \tfrac{1}{3}, \tfrac{4}{3}; \eta\right) \,, \end{aligned} \tag{9}$$

where N_2 is a constant. Using the properties of the Γ and the hypergeometric functions [10] we can write the conformal blocks of equations (8) and (9) as

$$I_1 = N_1 \frac{\Gamma\left(\frac{1}{3}\right)^2}{\Gamma\left(\frac{2}{3}\right)} \,, \tag{10}$$

$$I_2 = -3N_2(\eta)^{\frac{1}{3}} F\left(\tfrac{1}{3}, \tfrac{2}{3}, \tfrac{4}{3}; \eta\right) \,. \tag{11}$$

To find the appropriate combination (*i.e.* the values of N_1 and N_2) describing $\pi(p,r)$, we give the precise correspondence between the aspect ratio r of the rectangle and the cross ratio η by the two expressions

$$r = \frac{-iz_4}{z_2} \quad \text{and} \quad \eta = \frac{z_2^2}{|z_3|^2} \,. \tag{12}$$

For infinitely wide lattice ($r \to 0$ and $\eta \to 1$), the vertical crossing probability $\pi_{\rm v}(p,r)$ should be 1. But for infinitely narrow lattice ($r \to \infty$ and $\eta \to 0$) it should be zero. Thus, we find $N_1 = 0$ and $N_2 = -\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2$. Then the vertical crossing probability takes the form

$$\pi_{\rm v}(p,r) = \pi((z_1,z_2);(z_3,z_4)) = 3\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}(\eta)^{\frac{1}{3}}F\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};\eta\right) . \tag{13}$$

The horizontal crossing probability $\pi_{\rm h}(p,r)$ should be 1 for infinitely narrow lattice and zero for infinitely wide lattice. Then, in this case we have $N_1 = N_2 = \Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})^2$ and we can write

$$\pi_{\rm h}(p,r) = \pi((z_1,z_4);(z_2,z_3)) = 1 - 3\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}(\eta)^{\frac{1}{3}}F\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};\eta\right) . \tag{14}$$

Of course, $\pi_{\rm h} + \pi_{\rm v} = 1$ which means that whenever there is a horizontal cluster, it cannot exist a vertical one. The two events are incompatible.

If we change the labelling of the rectangle corner's from (z_1,z_2,z_3,z_4) to (z_2,z_3,z_4,z_1), we retrieve Cardy's results [3]

$$\pi_{\rm v}(p,r) = \pi((z_1,z_4);(z_2,z_3)) = 1 - 3\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}(1-\eta)^{\frac{1}{3}}F\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};1-\eta\right) \tag{15}$$

and

$$\pi_{\rm h}(p,r) = \pi((z_1,z_2);(z_3,z_4)) = 3\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)^2}(1-\eta)^{\frac{1}{3}}F\left(\tfrac{1}{3},\tfrac{2}{3},\tfrac{4}{3};1-\eta\right) \tag{16}$$

and from equations (13), (14), (15) and (16) we retrieve also that

$$\pi(\eta) = 1 - \pi(1-\eta) . \tag{17}$$

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