

PARTICLE SPECTRA IN STATISTICAL MODELS  
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Single particle momentum spectra are calculated within three micro-canonical statistical ensembles, namely, with conserved system energy, system momentum, as well as system energy and momentum. Deviations from the exponential spectrum of the grand canonical ensemble are quantified and discussed. For mean particle multiplicity and temperature, typical for  $p + p$  interactions at the LHC energies, the effect of the conservation laws extends to transverse momenta as low as about  $3 \text{ GeV}/c$ . The presented results should be considered as the next step in development of statistical models for particle production in high energy collisions. They can be useful to interpret spectra measured in nuclear collisions at high energies, in particular, their system size dependence.

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## 1. Introduction

Questions concerning possible phases of strongly interacting matter and transitions between them have been motivating experimental and theoretical study of relativistic nuclear collisions for many years now [1]. Results on collision energy dependence of hadron production properties in central lead-lead (Pb+Pb) collisions indicate that a high density phase of strongly interacting matter, a quark-gluon plasma (QGP), is produced at an early stage of collisions at energies higher than about 8 GeV (center-of-mass energy per nucleon-nucleon pair) [2]. Signals of the onset of deconfinement are not

observed in proton–proton ( $p+p$ ) interactions. This is probably because of a small volume of the created system. Also, in other cases, when interpreting signatures of the onset of deconfinement and/or QGP in nucleus–nucleus ( $A+A$ ) collisions it is popular to refer to a comparison with properly normalized data on  $p+p$  interactions at the same collision energy per nucleon.

Statistical models in thermodynamical approximation are, in general, sufficient to describe mean particle multiplicities in central Pb+Pb collisions at high energies. This is because the volume of the created matter is large and, therefore, an influence of the material and motional conservation laws can be neglected. On the other hand, data on  $p+p$  interactions are notoriously difficult to interpret within statistical approaches. This should be attributed to an importance of the conservation laws and thus invalidity of thermodynamic models. Instead of the grand canonical ensemble (GCE), the canonical (CE) or micro-canonical (MCE) ones should be used. Of course, this may also impact conclusions from a comparison of results on Pb+Pb and  $p+p$  interactions as well as the study of system size dependence in  $A+A$  collisions.

An influence of material conservation laws on particle yields has been studied within the CE since a long time [3]. In particular, strangeness [4], baryon number [5], and charm [6] conservation laws were considered separately in detail. A complete treatment of the exact material conservation laws within the CE and MCE formulations was developed and applied to analyze hadron yields in elementary collisions in Refs. [7,8]. The main result is that the density of conserved charge carriers decreases with decreasing system volume. This so-called CE suppression becomes significant for a mean multiplicity of conserved charges of the order of one.

Similar to the CE suppression of particle yields, one may expect that a shape of single particle momentum spectra changes when energy and momentum conservation laws are imposed. This conjecture is addressed quantitatively in our paper in which three micro-canonical statistical ensembles, namely, with conserved system energy, system momentum as well as system energy and momentum, are considered.

The paper is organized as follows. Partition functions for the three ensembles are defined and calculated in Section 2. The corresponding single particle momentum spectra are shown and discussed in Section 3. Summary presented in Section 4 closes the paper.

## 2. Partition functions with conserved momentum and energy

For simplicity, a non-interacting gas of mass-less particles (without conserved charges) will be studied. Moreover, the classical Boltzmann approximation, which neglects (small) quantum effects, will be used. This allows to

derive analytical formulas for single particle spectra in three micro-canonical statistical ensembles. These are ensembles with the fixed volume,  $V$ , and conserved system energy  $E$  only, system momentum  $\vec{P}$  only as well as  $E$  and  $\vec{P}$  being conserved together. They are referred to as  $(E, V)$ ,  $(T, \vec{P}, V)$ , and  $(E, \vec{P}, V)$  ensembles, respectively, where  $T$  denotes the system temperature. For comparison, spectra obtained within the GCE, *i.e.* the  $(T, V)$  ensemble, will be used. The corresponding partition functions read

$$Z(T, V) = \sum_{N=0}^{\infty} Z_N(T, V) = \sum_{N=0}^{\infty} \left[ \frac{V}{(2\pi)^3} \right]^N \frac{W_N(T)}{N!}, \quad (1)$$

$$Z(E, V) = \sum_{N=1}^{\infty} Z_N(E, V) = \sum_{N=1}^{\infty} \left[ \frac{V}{(2\pi)^3} \right]^N \frac{W_N(E)}{N!}, \quad (2)$$

$$Z(E, \vec{P}, V) = \sum_{N=2}^{\infty} Z_N(E, \vec{P}, V) = \sum_{N=2}^{\infty} \left[ \frac{V}{(2\pi)^3} \right]^N \frac{W_N(E, \vec{P})}{N!}, \quad (3)$$

$$Z(T, \vec{P}, V) = \sum_{N=2}^{\infty} Z_N(T, \vec{P}, V) = \sum_{N=2}^{\infty} \left[ \frac{V}{(2\pi)^3} \right]^N \frac{W_N(T, \vec{P})}{N!}, \quad (4)$$

where

$$W_N(T) = \int d\vec{p}_1 \dots d\vec{p}_N \exp \left( -\frac{\sum_{r=1}^N p_r}{T} \right) = (8\pi T^3)^N, \quad N \geq 0, \quad (5)$$

$$W_N(E) = \int d\vec{p}_1 \dots d\vec{p}_N \delta \left( E - \sum_{r=1}^N p_r \right), \quad N \geq 1, \quad (6)$$

$$W_N(E, \vec{P}) = \int d\vec{p}_1 \dots d\vec{p}_N \delta \left( E - \sum_{r=1}^N p_r \right) \delta \left( \vec{P} - \sum_{r=1}^N \vec{p}_r \right), \quad N \geq 2, \quad (7)$$

$$W_N(T, \vec{P}) = \int d\vec{p}_1 \dots d\vec{p}_N \exp \left( -\frac{\sum_{r=1}^N p_r}{T} \right) \delta \left( \vec{P} - \sum_{r=1}^N \vec{p}_r \right), \quad N \geq 2. \quad (8)$$

Note, that the minimal possible number of particles in the ensembles with conserved momentum, Eqs. (3)–(4) and (7)–(8), is  $N = 2$ .

Using the integral representation of the  $\delta$ -functions,

$$\delta \left( E - \sum_{r=1}^N p_r \right) = \frac{1}{2\pi} \int d\alpha \exp \left( i\alpha E - i\alpha \sum_{r=1}^N p_r \right), \quad (9)$$

$$\delta \left( \vec{P} - \sum_{r=1}^N \vec{p}_r \right) = \frac{1}{(2\pi)^3} \int d^3 \vec{\lambda} \exp \left( i \vec{\lambda} \vec{P} - i \vec{\lambda} \sum_{r=1}^N \vec{p}_r \right), \quad (10)$$

one finds

$$W_N(E) = \frac{2^{3N} \pi^N}{(3N-1)!} E^{3N-1}, \quad (11)$$

$$W_N(E, \vec{P}) = \frac{(8\pi i)^N}{(2\pi)^4} \int d\alpha d^3 \vec{\lambda} \exp \left[ i \left( \alpha E + \vec{\lambda} \vec{P} \right) \right] \left[ \frac{\alpha}{(\alpha^2 - \lambda^2)^2} \right]^N, \quad (12)$$

$$W_N(T, \vec{P}) = \frac{(8\pi T^3)^N}{(2\pi)^3} \int d^3 \vec{\lambda} \frac{\exp \left( i \vec{\lambda} \vec{P} \right)}{(1 + T^2 \lambda^2)^{2N}}. \quad (13)$$

Calculating the  $\alpha$ - and  $\lambda$ -integrals [10] in Eq. (12), one obtains

$$\begin{aligned} W_N(E, \vec{P}) &= \left( \frac{\pi}{2} \right)^{N-1} \frac{(E^2 - P^2)^{N-2}}{2P} \\ &\times \sum_{r=0}^N C_N^r \frac{(E+P)^{N-r} (E-P)^r}{(2N-r-2)! (N+r-2)!} \\ &\times \left[ \frac{E+P}{2N-r-1} - \frac{E-P}{N+r-1} \right]. \end{aligned} \quad (14)$$

Performing the summation and rewriting (14) using the hypergeometric functions, one gets

$$\begin{aligned} W_N(E, \vec{P}) &= \left( \frac{\pi}{2} \right)^{N-1} \frac{(E^2 - P^2)^{N-2}}{2P} (E+P)^N \\ &\times \left[ \frac{(E+P)}{(N-2)!(2N-1)!_2} F_1 \left( 1-2n, -n; n-1; \frac{E-P}{E+P} \right) \right. \\ &\left. - \frac{(E-P)}{(N-1)!(2N-2)!_2} F_1 \left( 2-2n, -n; n; \frac{E-P}{E+P} \right) \right]. \end{aligned} \quad (15)$$

Equation (13), after taking the integral, reads

$$W_N(T, \vec{P}) = \frac{1}{(2\pi)^2} \left( \frac{8\pi}{T} \right)^N \sqrt{\pi} \left( \frac{PT}{2} \right)^{2N-3/2} \frac{K_{2N-3/2}(P/T)}{(2N-1)!}, \quad (16)$$

where  $K_{2N-3/2}(P/T)$  are the modified Bessel functions. From Eqs. (15) and (16) follow that  $W_N(E, \vec{P}) = W_N(E, P)$  and  $W_N(T, \vec{P}) = W_N(T, P)$ . Therefore, as it is intuitively expected, the partition functions (3) and (4)

depend on the absolute value  $P$  of the 3-vector  $\vec{P}$ , and they are independent of the  $\vec{P}$  direction. For  $\vec{P} = 0$ , from Eqs. (15) and (16), one gets

$$W_N(E, \vec{P} = 0) = \left(\frac{\pi}{2}\right)^{N-1} \frac{(2N-1)(4N-4)!}{((2N-1)!)^2(3N-4)!} E^{3N-4}, \quad (17)$$

$$W_N(T, \vec{P} = 0) = \left(\frac{\pi}{2}\right)^{N-1} \frac{(4N-4)!}{(2N-1)!(2N-2)!} T^{3N-3}. \quad (18)$$

Using Eqs. (5), (11), and (15)–(16), one obtains the partition functions in the GCE (1), and the micro-canonical ensembles (2)–(4). Then, the corresponding mean multiplicity is calculated as

$$\langle N \rangle = \frac{\sum_N N Z_N}{Z}. \quad (19)$$

### 3. Single particle momentum spectra

The single particle momentum spectrum in the GCE (5) reads [9]

$$\begin{aligned} F(p; T) &\equiv \frac{1}{\langle N \rangle} \frac{dN}{p^2 dp} = \frac{V}{2\pi^2 \langle N \rangle} \exp\left(-\frac{p}{T}\right) \\ &= \frac{1}{2T^3} \exp\left(-\frac{p}{T}\right) \equiv F_{\text{Boltz}}(p). \end{aligned} \quad (20)$$

The single particle momentum spectra in the micro-canonical ensembles (2)–(4) are [9]

$$F(p; E) = \frac{V}{2\pi^2 \langle N \rangle} \frac{1}{Z(E, V)} \sum_{N=1}^{\infty} Z_N(E - p, V), \quad (21)$$

$$F(p; E, \vec{P} = 0) = \frac{V}{2\pi^2 \langle N \rangle} \frac{1}{Z(E, \vec{P} = 0, V)} \sum_{N=2}^{\infty} Z_N(E - p, p, V), \quad (22)$$

$$F(p; T, \vec{P} = 0) = \frac{V}{2\pi^2 \langle N \rangle} \frac{\exp(-p/T)}{Z(T, \vec{P} = 0, V)} \sum_{N=2}^{\infty} Z_N(T, p, V). \quad (23)$$

Relations  $Z_N(E - p, \vec{p}, V) = Z_N(E - p, p, V)$  and  $Z_N(T, \vec{p}, V) = Z_N(T, p, V)$  in the right-hand side of Eqs. (22) and (23), respectively, were used. Note also that the spectra (20)–(23) satisfy the same normalization condition

$$\int_0^{\infty} p^2 dp F(p) = 1. \quad (24)$$

The single particle momentum spectra obtained within the  $(T, V)$ ,  $(E, V)$ ,  $(E, \vec{P}, V)$ , and  $(T, \vec{P}, V)$  ensembles are shown in Fig. 1. They are calculated for  $T = 160$  MeV and  $E = 48$  GeV in the  $(T, V)$ ,  $(T, \vec{P}, V)$  and  $(E, V)$ ,  $(E, \vec{P}, V)$  ensembles, respectively. The average multiplicity in the GCE is selected to be  $\langle N \rangle = 100$ . Note, that the selected mean multiplicity and temperature are close to those measured in  $p + p$  interactions at the LHC energies. The energy in the  $(E, V)$  and  $(E, \vec{P}, V)$  ensembles was set to be equal to the mean energy in the  $(T, V)$  ensemble, *i.e.*  $E = 3T\langle N \rangle$ . Finally, the GCE relation  $\langle N \rangle = VT^3/\pi^2$  was used to obtain the volume  $V$ , which is used in all ensembles. For these values of  $E$ ,  $V$ , and  $T$  the average multiplicities (19) in  $(E, V)$ ,  $(E, \vec{P}, V)$ , and  $(T, \vec{P}, V)$  ensembles are then approximately equal to that in the GCE.

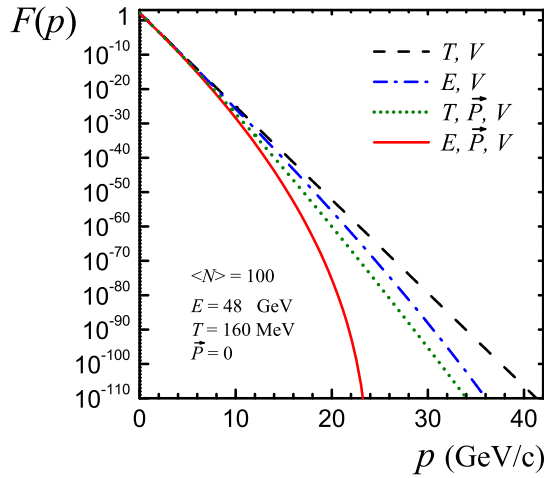


Fig. 1. Single particle momentum spectra  $F(p)$  in the  $(T, V)$  (dashed line),  $(E, V)$  (dash-dotted line),  $(E, \vec{P} = 0, V)$  (solid line), and  $(T, \vec{P} = 0, V)$  (dotted line) ensembles.

As seen in Fig. 1, at high momenta the spectra calculated imposing energy and/or momentum conservation are significantly below the GCE exponential spectrum. This is expected because at the threshold momentum the particle yield has to equal zero, namely,  $F(p; E) \rightarrow 0$  at  $p \rightarrow E$ , and  $F(p; E, \vec{P} = 0) \rightarrow 0$  at  $p \rightarrow E/2$ .

In order to quantify impact of the energy and momentum conservation at momenta significantly below the threshold one, the ratio of the spectra in the  $(T, \vec{P}, V)$ ,  $(E, V)$ , and  $(E, \vec{P}, V)$  ensembles to the spectrum in the  $(T, V)$  ensemble is shown in Fig. 2. The suppression of the spectra due to the energy and momentum conservation is already significant (a factor of

about 2) at momentum  $p \cong 3 \text{ GeV}/c$  which is as low as about 12.5% of the threshold one. Note, that the ratio is lower than the one by  $(1 \div 2)\%$  at low momenta of several hundred  $\text{MeV}/c$  as a result of the suppression of the spectra (21)–(23) at higher momenta and their normalization to unity, Eq. (24).

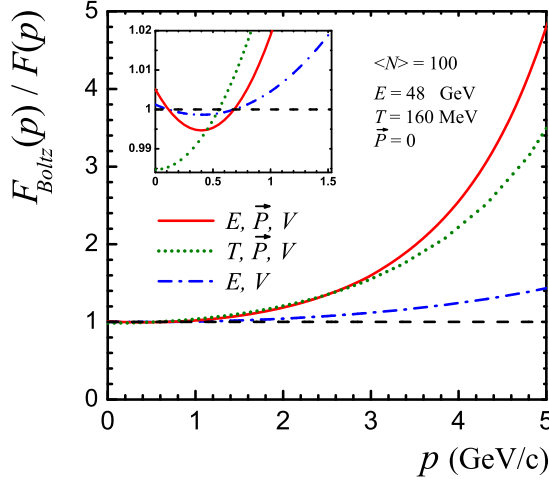


Fig. 2. The ratio of  $F_{\text{Boltz}}(p)/F(p)$ , where  $F(p)$  equal to  $F(p; E)$  (21),  $F(p; E, \vec{P} = 0)$  (22) and  $F(p; T, \vec{P} = 0)$  (23), is shown as a function of  $p$  by the dash-dotted, solid and dotted lines, respectively. The small momentum region,  $p < 1.5 \text{ GeV}/c$ , is presented in the inset. Further details are given in the text.

Dependence of the ratio  $F_{\text{Boltz}}(p)/F(p)$  at  $p = 3 \text{ GeV}/c$  on mean particle multiplicity is shown in Fig. 3. The temperature is fixed as  $T = 160 \text{ MeV}$ . The energy and volume are calculated as  $E = 3T\langle N \rangle$  and  $V = \pi^2\langle N \rangle/T^3$ , and they have the same values in all statistical ensembles. For small statistical systems (at low  $\langle N \rangle$ ) the effect of the energy-momentum conservation is strong and the ratio  $F_{\text{Boltz}}(p)/F(p)$  is large. The ratio decreases to unity with increasing  $\langle N \rangle$  for all three ensembles with conserved  $E$  and/or  $\vec{P}$ . This is expected, because in the thermodynamical limit an influence of the energy-momentum conservation on single particle momentum spectra at any fixed particle momentum should disappear. Note, that in the  $(E, V)$  and  $(E, \vec{P}, V)$  ensembles,  $F_{\text{Boltz}}(p)/F(p) \rightarrow \infty$  at the threshold values of  $\langle N \rangle$  equal to  $p/3T$  and  $2p/3T$ , for  $p = E$  and  $p = E/2$ , respectively.

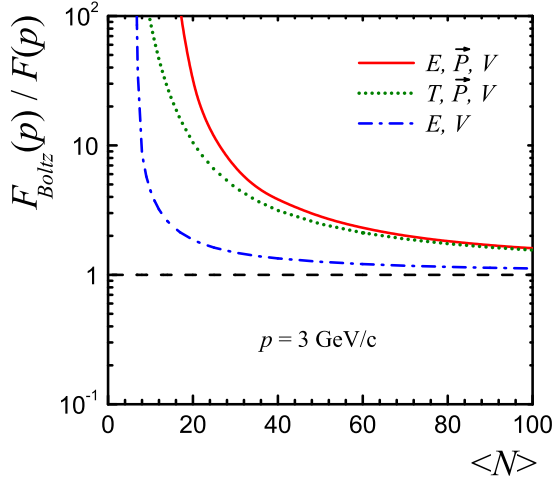


Fig. 3. The ratio of  $F_{\text{Boltz}}(p)/F(p)$ , where  $F(p)$  equal to  $F(p; E)$  (21),  $F(p; E, \vec{P} = 0)$  (22) and  $F(p; T, \vec{P} = 0)$  (23) is shown as a function of  $\langle N \rangle$  at  $p = 3 \text{ GeV}/c$  by the dash-dotted, solid and dotted lines, respectively. Further details are given in the text.

#### 4. Summary

Single particle momentum spectra are calculated in the system of massless non-interacting Boltzmann particles within three micro-canonical statistical ensembles, namely, with the conserved system energy, system momentum, as well as system energy and momentum. We find a strong suppression of the spectra at large momenta in comparison to the exponential spectrum of the grand canonical  $(T, V)$  ensemble. In the  $(E, V)$  and  $(E, \vec{P} = 0, V)$  ensembles, the spectra approach zero for momentum approaching its threshold value,  $p \rightarrow E$  and  $p \rightarrow E/2$ , respectively. There is no threshold in the  $(T, \vec{P} = 0, V)$  ensemble, nevertheless the spectrum is also strongly suppressed. For the mean particle multiplicity and temperature typical for  $p+p$  interactions at the LHC energy, the suppression of the spectra due to the energy and momentum conservation is already significant (a factor of about 2) at  $p = 3 \text{ GeV}/c$ , *i.e.* at momenta as low as about 12.5% of the threshold momentum for the  $(E, \vec{P} = 0, V)$  ensemble.

The presented results should be considered as the next step in development of statistical models for particle production in high energy collisions. They can be useful to interpret spectra measured at high energy collisions. In particular, the energy-momentum conservation effects are important to describe the system size dependence in nuclear collisions. In the present study, we consider only gas of massless Boltzmann particles. This allows to



present all results in analytical form. An interesting region of particle momenta is found to be  $p = (2 \div 4)$  GeV/ $c$ . Effects of Bose or Fermi statistics can be safely neglected in this region, because  $\exp(p/T) \gg 1$ . Moreover, in this region  $m_\pi/p \ll 1$  and, consequently, the pion mass can be safely neglected in calculations. On the other hand, an essential part of pions in this kinematic region comes from decays of heavy resonances. Therefore, a detailed comparison with data on pion transverse momentum spectra as well as an analysis of kaon and proton spectra would require further development of the model.

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