

METAMORPHOSIS VERSUS DECOUPLING IN NON-ABELIAN GAUGE THEORIES AT VERY HIGH ENERGIES

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In the present paper, we study the limit of zero mass in non-Abelian gauge theories both with Higgs mechanism and in the nonlinear realization of the gauge group (Stückelberg mass). We argue that in the first case the longitudinal modes undergo a metamorphosis process to the Goldstone scalar modes, while in the second, we guess, a decoupling process associated to a phase transformation. The two scenarios yield strikingly different behaviors at high energy, mainly ascribed to the presence of a massless Higgs doublet among the physical modes in the case of Higgs mechanism (*i.e.* not only the Higgs boson). The aim of this work is to show that the problem of unitarity at high energy in non-Abelian gauge theory with no Higgs boson can open new perspectives in quantum field theory.

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1. Introduction

The fate of the longitudinal mode of a vector boson at high energy is entangled with the problem of unitarity. On the basis of some well known papers in the late 70s and early 80s [1, 2, 3, 4], people have acquired the conviction that the Higgs boson is necessary in order to ensure physical unitarity in non-Abelian gauge theories. The heart of the argument is based on the behavior of the elastic scattering amplitude of the longitudinal modes $W_L W_L$ at high energy [1, 2, 3, 4, 5, 6].

Thus, the construction of Higgsless Electroweak Models faces tremendously difficult theoretical problems dealing with basic principles as renormalization, unitarity, foundation of bound state quantum field theory, predictivity of the model (finite number of free parameters), *etc.* Ref. [7] updates a recent proposal for a Higgsless scenario and provides a nice overview of most models.

The problem comes from the fact that the longitudinal polarization of a vector meson is a physical mode for any finite value of the mass (M). On the other side, for zero mass a vector meson has only two (transverse) polarizations. Thus, either the mode decouples from the physical states in the massless limit (like in massive QED) or we face a singular behavior at zero vector boson mass.

In non-Abelian gauge theories (we deal with SU(2), with or without the U(1) factor), the longitudinal polarization does not decouple in physical S-matrix elements at zero mass. A conundrum that shows up in really practical items as in the proof of physical unitarity and in phenomenology [8,9]. The number of physical modes changes in the limit of zero mass, thus the cancellation of the unphysical modes in the proof of S-matrix unitarity must proceed with different patterns in the two regimes. On the other side, in phenomenology one must introduce a cut-off to mark the events region, where the longitudinal polarization of the vector meson can be established within the errors (at very high momentum one cannot distinguish between the longitudinal polarization and the spin zero wave function). Therefore, the distinction between longitudinal mode for the spin one and a spin zero mode, in the limit of zero mass, has no operative meaning and, as said, conflicts with unitarity, due to the non-decoupling.

A naïve analysis, based on BRST transformations, indicates that in the massless limit the *unphysical* (at $M \neq 0$) components of the Higgs field eventually describe *physical* massless scalars for $M = 0$. Thus, we suggest that the longitudinal polarization mode, in default of the decoupling, undergoes a metamorphosis to the mode of a massless scalar. Physical unitarity of the S-matrix is preserved with this assignment of the fields to physical and unphysical modes. The Equivalence Theorem (CLTCG) [1,4,10,11,12,13,14,15,16] supports this setting¹.

¹ After the work of J.M. Cornwall, D.N. Levin, G. Tiktopoulos, M.S. Chanowitz and M.K. Gaillard, the relation between the S-matrix elements for longitudinal modes of the gauge fields and those of the Goldstone bosons has developed to a somewhat more complex result than the one implied by the theorem on the point transformations of fields in scattering theory [17]. The present work adds more consequences to the discovery of the above mentioned physicists. Thus, we choose to denote the theorem by their names.

Of course, the massless theory is plagued by infrared divergences; but we are going to ignore this difficulty, hoping that the two problems are not dangerously entangled. If instead infrared divergences are an insurmountable obstacle, as a last resource one can reverse the view point and consider the mass as the infrared regulator of an otherwise ill-defined field theory.

This possibility of a metamorphosis of states is suggested for non-Abelian gauge theories, where the mass M is generated by the Higgs mechanism. The reason being that the limit of symmetry restoration (zero vacuum expectation value (v.e.v.) of the Higgs boson field) seems doable on the effective action in a loop-wise perturbative expansion.

After the metamorphosis, the theory consists of a massless gauge field and a complex doublets of scalar fields (the fields used to induce the Higgs mechanism for $M \neq 0$). According to the standard analysis, the theory is not asymptotically free [18, 19, 20] due to the presence of scalars.

Thus, one gets a consistent setting for studying the physics of the intermediate vector mesons at energies $E \gg M_W, M_Z$. If needed, one can use the $M \neq 0$ theory as the infrared regulated theory.

In the present paper, we address the same question in the case, where M enters via a Stückelberg term. In this case, the local gauge group is realized *nonlinearly* and hence there is no need of a Higgs boson in the perturbative spectrum.

Both theories obey the same set of equations used in the present work: Slavnov–Taylor Identity (STI), gauge fixing equation and anti-ghost equation². Moreover, the CLTCG theorem takes the same form. However, they are strikingly different in the zero mass limit. In the linear case, the limit of v.e.v. to zero in the Feynman amplitudes seems to be manageable. While in the nonlinear case the situation is fuzzier. In recent works [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32], we proposed a divergences subtraction scheme for the nonlinear sigma model for the massive Yang–Mills theory and for the Electroweak Model $SU(2) \otimes U(1)$. Locality and perturbative unitarity are obeyed. However, the perturbative solution has a bad M^{-1} behavior for $M = 0$, essentially due to the nonlinear sigma model couplings.

Thus, the scenario of a metamorphosis of the longitudinal modes for $M = 0$ cannot be envisaged in the case of nonlinear realization of the gauge group.

From some considerations, based on the matching of the number of degrees of freedom and on the strong coupling limit of the lattice-regulated theory, we guess a $M \rightarrow 0$ behavior, where both the longitudinal polarizations and the Goldstone bosons decouple. In order to support this scenario,

² The Local Functional Equation [21], employed in the subtraction strategy of the ultraviolet divergences in nonlinear theories, is not used here.

we envisage the existence of two or more phases in the parameter space separated by some discontinuity. In particular, we assume that the loop expansion cannot be continued to $M = 0$. This would mark the difference with the linear case, where the Goldstone bosons survive as physical modes. For instance, the nonlinear theory would be an asymptotically free theory in the limit.

The conjecture on the limit $M = 0$ for the nonlinear case could be studied by lattice simulations. In particular, one should make a survey of the phase diagram in the parameter space (g^{-2}, M^2) and look for possible singularities responsible for the bad behavior of the loop expansion for low mass³.

A further comparison of the two scenarios could come from high energy processes. However, a quick analysis shows that this is pretty hard to achieve, as a simple example will show.

We work in the 't Hooft gauge.

2. The classical actions

In this section, we fix some notations. Matter fields are omitted in most part of the paper.

The work focuses on the gauge and scalar sectors of the following Yang–Mills classical actions written for the $SU(2)$ gauge group. We consider both cases of linear (Higgs) and nonlinear (Stückelberg) representation of the gauge group.

2.1. Yang–Mills with Higgs mechanism

We consider an $SU(2)$ Yang–Mills theory, where the mass is generated through the Higgs mechanism

$$S_H = \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \left(-\frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu} + [(\partial_\mu - iA_\mu)\Phi]^\dagger [(\partial^\mu - iA^\mu)\Phi] - \frac{\lambda}{4g^2} (\Phi^\dagger \Phi - 2v^2 g^2)^2 \right). \quad (1)$$

Λ is a mass scale for the analytic continuation in D dimensions. We use the short notation

$$\Lambda_g \equiv \frac{\Lambda^{(D-4)}}{g^2}. \quad (2)$$

We use the matrix notation

$$A_\mu = \frac{\tau_a}{2} A_{a\mu},$$

³ Recent lattice simulations [33] support this scenario.

$$G_{\mu\nu}[A] = G_{a\mu\nu} \frac{\tau_a}{2} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \quad (3)$$

and Φ is parametrized by

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} i\phi_1 + \phi_2 \\ \phi_0 - i\phi_3 \end{pmatrix}. \quad (4)$$

The action (1) is invariant under local $SU(2)_L$ left transformations

$$\begin{aligned} A'_\mu &= U_L A_\mu U_L^\dagger + iU_L \partial_\mu U_L^\dagger, \\ \Phi' &= U_L \Phi, \end{aligned} \quad (5)$$

and under global $SU(2)_R$ right transformations

$$\begin{aligned} A'_\mu &= A_\mu, \\ \Omega' &= \Omega U_R, \end{aligned} \quad (6)$$

where

$$\begin{aligned} \Omega_{\alpha\beta} &\simeq \sqrt{2}\Phi_\alpha \tilde{\Phi}_\beta, \\ \tilde{\Phi} &= i\tau_2 \Phi^*. \end{aligned} \quad (7)$$

Notice that, in general, $\Omega \notin SU(2)$.

The spontaneous breakdown of the global $SU(2)_L \otimes SU(2)_R$ symmetry proceeds via the nonzero vacuum expectation value

$$\langle 0|\phi_0|0\rangle = \langle 0|(h + 2vg)|0\rangle = 2vg \quad (8)$$

and the global $SU(2)$ invariance is left over on the vector indexes.

The spontaneous breakdown induces a mass for the vector bosons ($M \equiv gv$), for the Higgs boson ($M_H^2 \equiv \lambda v^2$) and a mixing $A_\mu - \phi$

$$\begin{aligned} S_{\text{H Bilinear}} &= \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \left(-\frac{1}{2}(\partial_\mu A_{a\nu} \partial^\mu A_a^\nu - \partial_\mu A_{a\nu} \partial^\nu A_a^\mu) \right. \\ &\quad \left. + \frac{M^2}{2} A_{a\nu} A_a^\nu + \frac{1}{2} \partial_\mu h \partial^\mu h + \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - M^2 A_{a\mu} \partial^\mu \phi_a - \frac{1}{2} M_H^2 h^2 \right). \end{aligned} \quad (9)$$

We use the 't Hooft gauge in order to remove the mixing. In doing this, we get a mass for the Goldstone bosons $\vec{\phi}$

$$S_{\text{H gf}} = \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \left(\frac{b^2}{2\xi} + \frac{M}{\xi} b_a \phi_a + b_a \partial_\mu A_a^\mu \right). \quad (10)$$

The Goldstone mass at the tree level is then $M_G^2 \equiv M^2 \xi^{-1}$. In Appendix B, we give the complete effective action at zero loop with the necessary Faddeev-Popov ghosts.

The perturbation expansion is in the number of loops and the amplitudes are made finite by using naïve dimensional renormalization. Finite renormalization is not a relevant item for the content of the paper.

2.2. Yang–Mills with Stückelberg mass

The nonlinear sigma model field Ω is an element of the $SU(2)$ group, which is parametrized in terms of the coordinate fields ϕ_a as follows (compare with Eq. (7))⁴

$$\Omega = \phi_0 + i\tau_a\phi_a, \quad \Omega^\dagger\Omega = 1, \quad \det\Omega = 1, \quad \phi_0^2 + \phi_a^2 = 1. \quad (11)$$

The $SU(2)$ flat connection is

$$\begin{aligned} F_\mu &= i\Omega\partial_\mu\Omega^\dagger = F_{a\mu}\frac{\tau_a}{2}, \\ F_{a\mu} &= 2(\phi_0\partial_\mu\phi_a - \partial_\mu\phi_0\phi_a + \epsilon_{abc}\partial_\mu\phi_b\phi_c). \end{aligned} \quad (12)$$

The field strength of F_μ vanishes

$$G_{\mu\nu}[F] = 0. \quad (13)$$

Under a local $SU(2)$ left transformation $U_L = \exp(i\alpha_a^L\frac{\tau_a}{2})$, one gets

$$\begin{aligned} \Omega' &= U_L\Omega, \\ F'_\mu &= U_L F_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger, \\ A'_\mu &= U_L A_\mu U_L^\dagger + iU_L\partial_\mu U_L^\dagger. \end{aligned} \quad (14)$$

The constraint in Eq. (11) implies that the gauge symmetry is nonlinearly realized on the fields ϕ_a , whose infinitesimal transformations are

$$\delta\phi_a = \tfrac{1}{2}\phi_0\alpha_a^L + \tfrac{1}{2}\epsilon_{abc}\phi_b\alpha_c^L, \quad \phi_0 = \sqrt{1 - \phi_a^2}, \quad \delta\phi_0 = -\tfrac{1}{2}\alpha_a^L\phi_a. \quad (15)$$

Under local $SU(2)_L$ symmetry the combination $A_\mu - F_\mu$ transforms in the adjoint representation of $SU(2)$. Hence one can construct out of $A_\mu - F_\mu$ and Ω invariants under $SU(2)_L$ local transformations. The Yang–Mills action, in the presence of a Stückelberg mass term [27] and in the 't Hooft gauge, is

$$\begin{aligned} S_S &= \frac{\Lambda^{(D-4)}}{g^2} \int d^Dx \left(-\frac{1}{4}G_{a\mu\nu}[A]G_a^{\mu\nu}[A] + \frac{M^2}{2}(A_{a\mu} - F_{a\mu})^2 \right), \\ S_{S\text{ gf}} &= \frac{\Lambda^{(D-4)}}{g^2} \int d^Dx \left(\frac{b^2}{2\xi} + 2\frac{M^2}{\xi}b_a\phi_a + b_a\partial_\mu A_a^\mu \right). \end{aligned} \quad (16)$$

S_S is invariant under local $SU(2)_L$ symmetry and also global $SU(2)_R$ symmetry. The choice of independent fields made in Eq. (15) fixed the direction of the spontaneous breakdown of the symmetry. The bilinear part of the action S_S is essentially the same as in the Higgs mechanism (9) apart from the absence of the Higgs boson terms.

⁴ In the nonlinear case, we use dimensionless fields ϕ_a .

3. Properties of the two-point functions (Higgs)

In Appendix D, we derive the two-point connected functions of the unphysical bosonic sector. The solutions are given in terms of the 1PI two-point functions⁵

$$\Gamma_{\phi\phi}, \quad ip^\nu \Gamma_{\phi A^\nu}, \quad \Gamma_L \quad (18)$$

which are related by the Eq. (106)

$$(p^\nu \Gamma_{A^\nu \phi})^2 + p^2 \Gamma_L \Gamma_{\phi\phi} = 0. \quad (19)$$

The connected two-point functions involving the Lagrangian multiplier are (see Eq. (108))

$$\begin{aligned} W_{A^\mu b} &= -\frac{i}{\Lambda_g} \frac{p^\mu}{p^2 - i \frac{M}{\xi} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}, \\ W_{\phi b} &= \frac{i}{\Lambda_g} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \frac{1}{p^2 - \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}. \end{aligned} \quad (20)$$

The two-point functions have the pole in the same position, *i.e.* the solution of

$$p^2 = \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}. \quad (21)$$

The connected two-point functions, for unphysical modes involving ϕ and A^μ , are given in Eqs. (123), (122) and (124)

$$W_{\phi\phi} = -\frac{p^2}{\Gamma_{\phi\phi}} \left(p^2 - \frac{1}{\Lambda_g \xi} \Gamma_L \right) \frac{1}{\left(p^2 - \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2}, \quad (22)$$

$$W_{A^\mu \phi} = i \frac{1}{\xi \Gamma_{\phi\phi}} \left(\frac{i}{\Lambda_g} p^\nu \Gamma_{A^\nu \phi} + M p^2 \right) \frac{p^\mu}{\left(p^2 - \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2}, \quad (23)$$

and

$$W_L = \frac{p^2}{\xi \Gamma_{\phi\phi}} \frac{\frac{\Gamma_{\phi\phi}}{\Lambda_g} - \frac{M^2}{\xi}}{\left(p^2 - \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2}. \quad (24)$$

⁵ We drop internal indexes whenever there is no ambiguity. Moreover, we use the notation

$$\Gamma_{A^\mu A^\nu} = \Gamma_T(p^2) \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + \Gamma_L(p^2) \frac{p_\mu p_\nu}{p^2}. \quad (17)$$

By direct computation, one can derive the identity

$$\Gamma_L = \frac{p^2 W_{b\phi}^2}{\xi W_{b\phi}^2 - \frac{1}{\Lambda_g} W_{\phi\phi}}. \quad (25)$$

It is interesting to see the properties of the numerators in Eqs. (23), (23) and (24). They form a matrix (variables: $\phi, M^{-1}\partial^\mu A_\mu$)

$$\mathcal{G} \equiv \frac{p^2}{\xi \Gamma_{\phi\phi}} \begin{pmatrix} -\xi p^2 + \frac{\Gamma_L}{\Lambda_g} & M^{-1} \left(\frac{i}{\Lambda_g} p^\nu \Gamma_{A^\nu \phi} + M p^2 \right) \\ M^{-1} \left(\frac{i}{\Lambda_g} p^\nu \Gamma_{A^\nu \phi} + M p^2 \right) & M^{-2} \left(p^2 \frac{\Gamma_{\phi\phi}}{\Lambda_g} - p^2 \frac{M^2}{\xi} \right) \end{pmatrix} \quad (26)$$

whose determinant is

$$\det(\mathcal{G}) = -\frac{1}{\Lambda_g} \frac{p^4}{M^2 \xi \Gamma_{\phi\phi}} \left(p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2. \quad (27)$$

Thus, the numerator-matrix has one vanishing eigenvalue on the double-poles. In fact, the field given by the linear combination [34]

$$X_1 = \frac{\phi}{\xi} + \frac{\partial^\mu A_\mu}{M} \quad (28)$$

is shown (Appendix E, Eq. (131)) to be the eigenvector of the vanishing eigenvalue of the matrix in Eq. (26).

4. The naïve $M = 0$ limit (Higgs)

The paper is focused on extreme processes, where one can neglect the mass parameters. In the Higgs case, this is equivalent to the limit $v = 0$, *i.e.* the limit of unbroken symmetry.

We do not consider skew limits as $M = 0$ and $M_H \neq 0$, which, although interesting in phenomenology [16], requires a series resummation as $v \rightarrow 0$.

In the limit $v = 0$, the symmetry $\Phi \rightarrow -\Phi$ is unbroken; as a consequence one has

$$\Gamma_{\phi A^\nu} = 0 \quad (29)$$

and therefore

$$\begin{aligned} \Gamma_L &= 0, \\ W_{A^\mu b} &= -i \frac{p^\mu}{\Lambda_g p^2}, \\ W_{\phi b} &= 0. \end{aligned} \quad (30)$$

Similarly, the limit in the Eqs. (23), (23) and (24) yields

$$\begin{aligned} W_{A^\mu\phi} &= 0, \\ W_{\phi\phi} &= -\frac{1}{\Gamma_{\phi\phi}}, \\ W_L &= \frac{1}{\Lambda_g\xi} \frac{1}{p^2}. \end{aligned} \quad (31)$$

The vanishing of the two-point functions $W_{A^\mu\phi}$ and $W_{b\phi}$ shows that in the limit ϕ describes modes orthogonal to the unphysical modes. This fact is a preliminary condition for the realization of the metamorphosis of the vector meson longitudinal polarizations into the Goldstone bosons, as described in the next sections. Moreover, this suggests that the limit $v = 0$ can be performed order-by-order in perturbation theory. This limit is possible on the amplitudes for generic external momenta, while for most S-matrix elements the limit cannot be performed due to infrared divergences. The amplitudes are those of massless non-Abelian field coupled to a massless fields Φ belonging to the spinorial representation of the $SU(2)$ group of local left transformations.

5. The longitudinal polarization and its fate for $M \rightarrow 0$

The longitudinal polarization⁶

$$\epsilon_L = \frac{1}{\tilde{M}} \left(|\vec{p}|, \frac{\vec{p}}{|\vec{p}|} E \right), \quad E = \sqrt{\tilde{M}^2 + \vec{p}^2} \quad (32)$$

can be written ($E_G \equiv \sqrt{\vec{p}^2 + \tilde{M}_G^2}$)

$$\epsilon_L = \frac{1}{\tilde{M}} \left(E_G, \vec{p} \right) + \left(-\frac{\tilde{M}_G^2}{\tilde{M}} \frac{1}{(p + E_G)}, \frac{\tilde{M}}{p(p + E)} \vec{p} \right). \quad (33)$$

Thus, for large value of the energy ($E \gg \tilde{M}$) we have

$$\epsilon_L = \frac{1}{\tilde{M}} (E_G, \vec{p}) + \mathcal{O} \left(\frac{\tilde{M}}{E} \right). \quad (34)$$

Equation (34) has attracted the attention of many physicists. We briefly add our comments.

⁶ In this section and in the sequel, $\tilde{M}$ and $\tilde{M}_G$ are fixed by the poles of the transverse and longitudinal tensors of the vector mesons after radiative corrections.

5.1. The need of a cut-off

Equation (34) shows that at very high energy one cannot experimentally distinguish the longitudinal mode of a vector field from a spin zero state described by a field like $\partial_\mu \phi$. Therefore, a cut-off energy E_C should be quoted in the experimental data, saying for what energy $E \ll E_C$ it is possible to distinguish the two states. For $E \geq E_C$ a statement about the spin content (spin one longitudinal *versus* spin zero) of the mode is void. The necessity of such cut-off energy is very relevant for our problem. In fact, if one is interested in the dynamics of the model at $M = 0$, whose S-matrix elements are plagued by infrared divergences, the cut-off can be used in order to evaluate the physically relevant observables. Thus, the model at $M = 0$ can be traded with a massive non-Abelian gauge theory with Higgs mechanism, provided that the mass M is small enough for the given kinematic setup.

5.2. Default of decoupling of longitudinal mode for non-Abelian gauge theories

It might be tempting to neglect the $\mathcal{O}(\frac{\tilde{M}}{E})$ parts in Eq. (34) and to perform the replacement

$$\epsilon_L \rightarrow \frac{1}{\tilde{M}} (E_G, \vec{p}) \quad (35)$$

in the Feynman amplitudes. Were it possible without ambiguity, then the problem of the decoupling of the longitudinal mode would be much easier. Unfortunately, this procedure is not allowed since mixed terms in quadratic or higher-order forms give finite contributions, which cannot be neglected without further scrutiny. The reason is connected to the validity of the condition

$$E_G \mathfrak{M}_0(E_G, \vec{p}) - p_i \mathfrak{M}_i(E_G, \vec{p}) = 0, \quad (36)$$

as it will be illustrated here to some extent. For instance, let us consider the situation, where ϵ_L^μ multiplies some amplitude $\mathfrak{M}_\mu$ which depends on the momentum p_ν . The limit $\tilde{M} = 0$ might be performed by evaluating the difference

$$\epsilon_L^\mu \mathfrak{M}_\mu(E, \vec{p}) - \frac{1}{\tilde{M}} (E_G \mathfrak{M}_0(E_G, \vec{p}) - p_i \mathfrak{M}_i(E_G, \vec{p})) \quad (37)$$

which is of order $\mathcal{O}(\frac{\tilde{M}}{E})$ as a standalone expression. Let us expand around E_G , where it is much easier to use the STI. We get (the common dependence from $\vec{p}$ being suppressed)

$$\begin{aligned} \epsilon_L^\mu \mathfrak{M}_\mu(E) - \frac{1}{\tilde{M}} p^\mu \mathfrak{M}_\mu(E_G) &= \frac{1}{\tilde{M}} \left([p - E_G] \mathfrak{M}_0(E_G) - p_i \left(\frac{E}{p} - 1 \right) \mathfrak{M}_i(E_G) \right. \\ &\left. + (E - E_G) \left\{ p \frac{\partial}{\partial E_G} \mathfrak{M}_0(E_G) - p_i \frac{E}{p} \frac{\partial}{\partial E_G} \mathfrak{M}_i(E_G) \right\} \right) + \mathcal{O}(\tilde{M}^3). \end{aligned} \quad (38)$$

Now we use the relations

$$\begin{aligned} \frac{E_G}{p} &= 1 + \mathcal{O}(\tilde{M}^2), \\ \frac{E}{p} &= 1 + \mathcal{O}(\tilde{M}^2), \\ E_G \frac{\partial}{\partial E_G} \mathfrak{M}_0(E_G) &= \frac{\partial}{\partial E_G} E_G \mathfrak{M}_0(E_G) - \mathfrak{M}_0(E_G), \end{aligned} \quad (39)$$

and we get

$$\begin{aligned} \epsilon_L^\mu \mathfrak{M}_\mu(E) - \frac{1}{\tilde{M}} p^\mu \mathfrak{M}_\mu(E_G) &= \frac{1}{\tilde{M}} \left([p - E] \mathfrak{M}_0(E_G) - p_i \left(\frac{E}{p} - 1 \right) \mathfrak{M}_i(E_G) \right. \\ &\left. + (E - E_G) \left\{ \frac{\partial}{\partial E_G} E_G \mathfrak{M}_0(E_G) - \frac{\partial}{\partial E_G} p_i \mathfrak{M}_i(E_G) \right\} \right) + \mathcal{O}(\tilde{M}^3) \\ &= -\frac{\tilde{M}}{2p} (p \mathfrak{M}_0(E_G) + p_i \mathfrak{M}_i(E_G)) + \frac{(E - E_G)}{\tilde{M}} \frac{\partial}{\partial E_G} \\ &\times [E_G \mathfrak{M}_0(E_G) - p_i \mathfrak{M}_i(E_G)] + \mathcal{O}(\tilde{M}^3). \end{aligned} \quad (40)$$

Thus, if Eq. (36) is valid, the derivative term can be neglected.

However, the expression in Eq. (40) may enter into some product with terms of order $\tilde{M}^{-1}$ (as for instance ϵ_L^μ) thus producing a nonvanishing result. Typically, this happens by evaluating the sum over final states as for instance

$$-\frac{d^3 p}{2E} \mathfrak{M}_\mu^* \epsilon_L^{*\mu} \epsilon_L^\nu \mathfrak{M}_\nu, \quad (41)$$

where the $\mathcal{O}(\tilde{M})$ term in Eq. (37) gives a finite contribution when multiplied by $\tilde{M}^{-1}$ in $\epsilon_L^{*\mu}$. Finally, the problem consists in evaluating

$$\frac{1}{\tilde{M}} (E_G \mathfrak{M}_0(E_G, \vec{p}) - p_i \mathfrak{M}_i(E_G, \vec{p})) \quad (42)$$

for $\tilde{M} = 0$. Then, the terms in Eq. (40) are expected to contribute, if the behavior of the expression in Eq. (42) is like $\tilde{M}^{-1}$ (as in matrix elements with unphysical modes). Otherwise, they yield vanishing products (as for matrix elements with only physical modes).

One can approach the problem of the decoupling of the longitudinal polarization in a different but, nevertheless, interesting way: by considering the sum of the contribution of Eq. (41) and that of the spin zero part

$$\frac{d^3p}{2E_G} \frac{1}{\tilde{M}^2} \mathfrak{M}_\mu^* p^\mu p^\nu \mathfrak{M}_\nu. \quad (43)$$

With an algebra similar to the one in Eq. (41), one concludes that in the limit of $\tilde{M} = 0$ the contributions of the longitudinal polarization and of the spin zero cancel if only physical states are present (*i.e.* Eq. (36) is valid). A scholarly example in Appendix A, based on the free fields, illustrates some of the features of the limit $\tilde{M} = 0$, discussed in the present section.

Now, we compare the two quite different situations present in the Abelian and the non-Abelian gauge theories.

The Lagrange multiplier b , used for the gauge fixing, decouples from the physical modes, as can be seen by using the STI. In the Abelian case, this yields Eq. (36). Consequently, the contribution of the longitudinal polarization can be replaced according to Eq. (35) in the zero mass limit, since the expression in the second term of Eq. (37) will never multiply a $\tilde{M}^{-1}$ factor.

In non-Abelian gauge theories, the decoupling does not happen. In fact, the decoupling of the Lagrange multipliers does not bring to the Eq. (36), but instead to a relation involving the Goldstone bosons, as discussed later on. This is the source of many problems. For a single external gauge particle with longitudinal polarization, the replacement (35) does yield the correct result. However, already for two gauge particles with longitudinal polarization, the replacement (35) might give results that depend on the gauge or on the order of the replacements. The first replacement gives no problems because all other particles are physical. After the first replacement, the $\mathcal{O}(\frac{\tilde{M}}{E})$ term in Eq. (35) might yield nonzero contributions involving the Faddeev–Popov ghosts, since the spin zero part of the gauge boson ($\epsilon_\mu \simeq p_\mu, p^2 = \tilde{M}_G^2$) is an unphysical mode.

This fact has further unpleasant consequences. For instance, in the proof of physical unitarity, the sum over final states is only on physical modes. After the replacement (35) (a physical mode by an unphysical one), this necessary property is lost, if no further condition is introduced to cancel out the spurious terms.

6. Metamorphosis in the Higgs mechanism scenario

For $\tilde{M} = 0$, only the transverse polarizations are physical, thus there is a problem in the limit. In the massless case, the two unphysical modes of the vector fields conspire with the Faddeev–Popov ghosts in order to cancel out in the cutting rule, *i.e.* in the equation of perturbative unitarity. Instead, for every finite value of $\tilde{M}$ the net balance to zero involves the spin zero part of the vector mesons, the Goldstone bosons and the Faddeev–Popov ghosts. Therefore, if the longitudinal polarization does not decouple, unitarity is violated in the limit. This has been noticed a long time ago [8, 9].

The conceptual difficulty of the limit disappears if we accept a scenario, where the longitudinal mode transforms into the former Goldstone boson for zero vector meson mass. In fact, the Goldstone boson field describes a physical mode at $\tilde{M} = 0$. This scenario is in agreement with the discussion of Sec. 5 about the impossibility to distinguish the modes at very high energy.

The metamorphosis scenario has a further advantage for practical calculations: one can use the limit theory in order to evaluate the amplitudes involving the longitudinal modes. This statement is very close to the CLTCG theorem which relates the S-matrix elements of the longitudinal modes to those of the Goldstone boson. This advantage, however, is limited by the infrared divergences, that eventually will emerge (for instance in self-energies). Our scenario provides a more flexible setup, where the objections on the zero mass limit are removed (metamorphosis *versus* decoupling) and a proper use of the theory is established (only at $M \neq 0$ we have a *bona fide* theory and the S-matrix elements at $M = 0$ can be evaluated by using M as an infrared regulator). In the next section, we give some comments about the CLTCG theorem.

7. Comments on the CLTCG theorem

In this section, we provide the general formulation of the CLTCG theorem in any covariant 't Hooft gauge.

Let $|\vec{p}L\rangle$ denotes an asymptotic state longitudinally polarized with momentum $\vec{p}$. Since it is a physical state, then, it must be annihilated by the operator F that generates the BRST transformations on the fields of the non-Abelian gauge theory (internal index is not displayed) [35]

$$F|\vec{p}L\rangle = 0. \quad (44)$$

However, the state is also represented by any element of the equivalent class made of vectors like

$$|\vec{p}L\rangle + F|X\rangle, \quad (45)$$

where X is an arbitrary state. Due to the nilpotency of F , Eq. (44) is still valid

$$F\left(|\vec{p}L\rangle + F|X\rangle\right) = 0. \quad (46)$$

By the standard proof of physical unitarity, the states $|\vec{p}L\rangle$ and $|\vec{p}L\rangle + F|X\rangle$ have the same S-matrix elements.

We shall use this freedom in describing the physical modes of the gauge fields in order to evaluate the behavior for $\tilde{M} \rightarrow 0$, without using the replacement (35) and encountering some pitfalls. The recipe is the following: any longitudinal mode is replaced by

$$|\vec{p}L\rangle + \frac{1}{\tilde{M}} F|\vec{p}\bar{c}\rangle, \quad (47)$$

where $|\vec{p}\bar{c}\rangle$ is a single-mode anti-ghost state, and the relative weight is chosen in order to reproduce Eq. (34), when the wave functions are exhibited by the reduction formulas. We construct the in- and out-states in the Fock space by using the recipe in Eq. (47).

The S-matrix element for the longitudinal mode is constructed by using the amputated connected Green function defined by⁷

$$W_{A_\mu} = \sum_{\psi} W_{A_\mu\psi} W_{\psi}^{\widehat{}}. \quad (51)$$

The asymptotic states are described by the eigenvectors $\epsilon^{(r)}$ and eigenvalues $\lambda^{(r)}$ of the residuum matrix of the two-point connected function $-W(p)$ at the physical pole $p^2 = m_r^2$. The construction of the S-matrix element, where the state $|\vec{p}L\rangle$ appears as a factor in the final state proceed via the usual procedure (wave function renormalization factor and internal indexes are omitted)

⁷ ψ is an irreducible set of fields. Throughout the paper, we use the notation

$$W_{A_{a\mu}^* \dots} \equiv \frac{\delta^n W}{\delta A_{a\mu}^* \dots} = i^{n-1} \langle 0 | T((D^\mu[A]c)_a \dots) | 0 \rangle_C \quad (48)$$

for composite fields, while for elementary fields

$$W_{\underbrace{b_a \dots}_n} \equiv i^{n-1} \langle 0 | T(b_a \dots) | 0 \rangle_C. \quad (49)$$

For the effective action, we use a similar short notation

$$\Gamma_X \equiv \frac{\delta \Gamma}{\delta X}. \quad (50)$$

$$S_{\vec{p}L\dots} \simeq \epsilon_{L\mu}(\vec{p}) \, i \, \widehat{W_{A_\mu(p)***}} \Big|_{p^2=\tilde{M}^2}. \quad (52)$$

With these notations, the residuum of the b -field (the Lagrange multiplier of the 't Hooft gauge) yields (see Eq. (108))

$$\lim_{p^2=\tilde{M}_G^2} \left(p^2 - \tilde{M}_G^2 \right) W_{b(p)***} = \left(i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \widehat{W_{\phi(p)***}} + i p^\mu \widehat{W_{A^\mu(p)***}} \right) \Big|_{p^2=\tilde{M}_G^2} = 0, \quad (53)$$

where *** denotes more b and physical mode insertions. It is worth noticing that from Eq. (21)

$$i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \Big|_{p^2=\tilde{M}_G^2} = \frac{\xi}{M} \tilde{M}_G^2 \quad (54)$$

and at the tree level

$$i \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \Big|_{p^2=M_G^2} = M. \quad (55)$$

In order to reproduce the pattern of Eq. (33), the Feynman amplitude is replaced, according to (47)

$$\begin{aligned} \epsilon_{L\mu}(p) \widehat{W_{A_\mu(p)***}} \Big|_{p^2=\tilde{M}^2} &= \lim_{p^2=\tilde{M}^2} \epsilon_{L\mu} \widehat{W_{A_\mu(p)***}} \\ &+ i \lim_{p^2=\tilde{M}_G^2} \left(p^2 - \tilde{M}_G^2 \right) \frac{1}{\tilde{M}} W_{b(p)***} \end{aligned} \quad (56)$$

and by using Eq. (53)

$$\begin{aligned} \epsilon_{L\mu}(p) \widehat{W_{A_\mu(p)***}} \Big|_{p^2=\tilde{M}^2} &= \lim_{p^2=\tilde{M}^2} \epsilon_{L\mu} \widehat{W_{A_\mu(p)***}} \\ &- \lim_{p^2=\tilde{M}_G^2} \frac{1}{\tilde{M}} \left(p^\mu \widehat{W_{A^\mu(p)***}} + \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \widehat{W_{\phi(p)***}} \right). \end{aligned} \quad (57)$$

The above replacement (47) may be repeated for every external gauge line with longitudinal polarization, since on both terms in Eq. (56) this replacement is allowed. In fact, the external legs are either physical states or b -lines and, therefore, the STI (97) guarantees the validity of Eq. (53).

In the limit $M = 0$, the first two terms in the r.h.s. of Eq. (57) cancel out

$$\lim_{p^2=\tilde{M}^2} \epsilon_L^\mu \widehat{W_{A_\mu(p)***}} - \lim_{p^2=\tilde{M}_G^2} \frac{1}{\tilde{M}} p^\mu \widehat{W_{A^\mu(p)***}} = 0 + \mathcal{O} \left(\frac{\tilde{M}}{E} \right). \quad (58)$$

Thus, finally, we get

$$\epsilon_L^\mu(p) \widehat{W_{A_\mu(p)^{***}}} \Big|_{p^2=\tilde{M}^2} = i \frac{p^\nu \Gamma_{\phi A^\nu}}{\tilde{M} \Gamma_{\phi\phi}} \widehat{W_{\phi(p)^{***}}} \Big|_{p^2=\tilde{M}_G^2} + \mathcal{O}(\tilde{M}). \quad (59)$$

The procedure can be repeated for every vector boson in the longitudinal mode. The feared occurrence of cross terms $\mathcal{O}(M) \times \frac{1}{M}$ vanishes since all external modes are either physical or b -insertions.

Few comments are in order on our proof of the CLTCG theorem in Eq. (59).

1. The exact knowledge of the two-point functions in the unphysical sector, as displayed in Sec. 3, allows the correct formulation of the CLTCG theorem at any number of loops. The quantities needed are $\Gamma_{\phi\phi}$ and $p^\mu \Gamma_{\phi A^\mu}$.
2. The CLTCG theorem, as in Eq. (59), concerns the amputated connected amplitudes. In order to formulate the theorem for the S-matrix elements, one has to introduce the wave-function normalization of the asymptotic states. In the limit $M = 0$, the normalization of the longitudinal modes equals that of the Goldstone boson, since W_L approaches the free-field value, as displayed in Eq. (31). Here, we are not going into further details on this problem.
3. After we introduce the necessary wave function renormalization factor in the l.h.s. of Eq. (59), the S-matrix elements are gauge invariant. This is valid for any finite value of $\tilde{M}$. Thus also the limit, when it exists, is gauge invariant. The property has been verified in explicit calculations [6].
4. For generic S-matrix elements, the limit of zero mass is expected to be infrared divergent. Then, Eq. (59) is of no use. However, one might consider quantities that are infrared finite as, for instance, some transition probabilities. On those quantities, the theorem in Eq. (59) might apply. A further possibility is to work in generic dimension D , whenever it is possible.
5. Physical unitarity is valid at every value of $\tilde{M}$. In the case of the Higgs mechanism scenario, we get the hint for considering a metamorphosis of the longitudinally polarized vector meson into a massless scalar at $\tilde{M} = 0$.
6. Within the Higgs mechanism scenario, the limit $\tilde{M} = 0$ can be performed on the perturbative expansion by providing a consistent picture of the metamorphosis of the longitudinal mode. Thus Eq. (59) can be read in the other way around: the $\tilde{M} \neq 0$ theory provides a doable infrared regulator for the $v = 0$ theory (symmetric phase), when on-shell amplitudes are needed.

The CLTCG theorem, as in Eq. (59), provides a tool for solving the problem associated to the longitudinal mode that does not decouple from physical states in the zero mass limit. The scenario of a metamorphosis of this mode into the Goldstone boson field, which can be consistently taken as a physical mode at zero mass, looks very promising for satisfying perturbative unitarity and the set of relations derived from the STI, gauge fixing equation and anti-ghost equation. This setting is compatible with the picture of a symmetry restoration through the limit $v = 0$ on the perturbative series of the effective action. It is tempting to argue that this setting allows the limit $v = 0$ in a non-Abelian gauge theory coupled with scalars (the former, *i.e.* $v \neq 0$, Higgs field), *i.e.* the generating functionals are continuous in $v = 0$. Such dynamical theory remains nonasymptotically free, according to the classification of Refs. [18, 19, 20].

8. Zero mass limit with a Stückelberg gauge invariant term

The equations used to support the metamorphosis scenario are still valid in the case of a Yang–Mills theory with mass *à la* Stückelberg. In particular, one has the same STI, gauge-fixing equation and anti-ghost equation. However, the use of Eq. (59) is now in question: although the formal derivation is the same, the non-existence of a zero mass limit in the loop expansion makes the CLTCG theorem inapplicable.

For small M , many terms of perturbative expansion have singular M^{-1} behavior. In fact, in the nonlinear theories the perturbative expansion in the loop number works for momenta small with respect to M . It is not known how this region is connected to the one around $M = 0$. Consequently, the extension of the CLTCG theorem to the massless limit becomes questionable.

The study of the zero-mass region implies a typical strong-coupling limit: the M^{-2} factor in the ϕ propagator is responsible for the presence of many divergent terms in the perturbative expansion. This means that one cannot explore the $M = 0$ region by starting from the series expansion in the number of loops.

One can make an educated guess on the small mass behavior of the theory on the basis of some naïve considerations.

(i) M controls in some way the spontaneous breakdown of the global $SU(2)_L \otimes SU(2)_R$ transformations. In fact, the Stückelberg mass term is the source of the interaction of ϕ with the rest of the fields. The order parameter $\langle \phi_0 \rangle$ for the SBS has no effect at $M = 0$. Thus, a limit theory is expected to be symmetric.

(ii) BRST properties of the asymptotic fields (if they can be defined) indicate that the fields $\vec{\phi}$ remain unphysical for any value of M , since the vacuum expectation value of ϕ_0 is expected to remain nonzero.

(iii) One might consider a resummation of the series, by performing first the integration over the $SU(2)$ group, thus taking A_μ as an external source. One gets an expansion in powers of M , where the coefficients are invariant under local gauge transformations. In this setting, the path integral on the fields $\vec{\phi}$ is performed on a lattice, with spacing a ,

$$\int \mathcal{D}[\phi] \exp \left(\int_E d^4x \frac{M^2}{2g^2} (A_{a\mu} - F_{a\mu})^2 \right) \\ \simeq \int \mathcal{D}\Omega \exp \sum_{x\mu} \left[\frac{2M^2}{g^2} a^2 \Re \sum_{x\mu} \text{Tr} \left\{ \Omega(x)^\dagger U(x, \mu) \Omega(x + \mu) - 1 \right\} \right], \quad (60)$$

where link variable is function solely of the classical field A_μ , the remaining integration variable in the final expression of the generating functional. The path integral integration is over a compact set for each site, therefore, we can expand in powers of M

$$\frac{1}{\int \prod_x \mathcal{D}[\Omega(x)]} \int \prod_x \mathcal{D}[\Omega(x)] \left[1 + \frac{1}{2} \left(\frac{\beta M^2 a^2}{2} \right)^2 \right. \\ \times \left(\sum_{x\mu} \text{Tr} \left\{ \Omega^\dagger(x) U(x, \mu) \Omega(x + \mu) \right\} \right)^2 + \frac{1}{4!} \left(\frac{2M^2 a^2}{g^2} \right)^4 \\ \times \prod_{j=1}^4 \left(\sum_{x_j \mu_j} \text{Tr} \left\{ \Omega(x_j)^\dagger U(x_j, \mu_j) \Omega(x_j + \mu_j) \right\} \right) \Big] \\ = 1 + \frac{1}{2} \left(\frac{\beta M^2 a^2}{2} \right)^2 \text{DN} + \frac{1}{8} \left(\frac{\beta M^2 a^2}{2} \right)^4 (\text{DN})^2 + \frac{1}{4} \left(\frac{M^2 a^2}{g^2} \right)^4 \sum_{\square} \text{Tr} \{ U_{\square} \}, \quad (61)$$

where $U_{\square}$ is the $SU(2)$ matrix associated to the plaquette $\square$ and DN is the number of degrees of freedom of the gauge bosons times the number of sites. Subsequently, we take into account the integration over the link variables. Finally, we take the logarithm of the partition function

$$\ln Z = \ln Z_0 + \frac{1}{2} \left(\frac{\beta M^2 a^2}{2} \right)^2 \text{DN} + \frac{1}{4} \left(\frac{\beta M^2 a^2}{2} \right)^4 \left\langle \sum_{\square} \text{Tr} \{ U_{\square} \} \right\rangle \Big|_{M^2 a^2 = 0}, \quad (62)$$

where Z_0 is the partition function of the massless theory. Thus, the final result is a Yang-Mills theory with local insertions.

There is another point in favor of this guess and it comes from Eq. (59). From completely general consideration (*i.e.* no approximations are needed) we have argued that the limit of zero mass can be performed by keeping BRST invariance, gauge invariance of the S-matrix and perturbative unitarity. If the limit implied by the CLTCG theorem exists in the form of a well-defined local theory represented on a Fock space of asymptotic fields, then both the longitudinal polarization mode and the Goldstone boson must decouple in the limiting region. If not, perturbative unitarity is violated in default of cancellation of the unphysical modes. BRST transformations on the asymptotic fields show that $\vec{\phi}$ remains an unphysical mode, if not decoupled. Therefore, the metamorphosis of a physical mode (longitudinal polarization) into an unphysical mode (the Goldstone boson) cannot occur. This is an educated guess saying that the longitudinal polarization mode decouples in the zero-mass limit.

The resulting massless Yang–Mills theory is supposed to describe events where the mass is negligible (with respect to energies, momentum transfers and any other dimensionful quantity). This point should be made clear: no confinement is implied of any sort.

These arguments indicate that Yang–Mills with mass generated by the Higgs mechanism or introduced by the the Stückelberg term might be compared on phenomenological ground since at very high energy the number of degrees of freedom are different. Thus, the two theories can be tested not only by the direct detection of the Higgs boson but also by this new very important difference in processes at high energy.

9. An example

The difference between non-Abelian *massive* gauge theories with Higgs mechanism and with nonlinear realization can be shown in many realms. The one-loop corrections in the two theories have been discussed and analytically evaluated in Ref. [28]. Moreover, the two models show marked differences in the celebrated processes WW , WZ and ZZ elastic scattering. In fact, according to the previously presented arguments, the limit $M = 0$ of the nonlinear theory is a pure non-Abelian gauge model without Higgs scalars and vector meson longitudinal polarizations.

In the present section, we consider a process involving quarks or leptons in order to illustrate the metamorphosis and its consequences. In particular, we focus on a process, where no Higgs boson is mediating, in order to have a signature which is not directly connected to its existence [36].

The present example is not intended as a quantitative argument for future measurements. For the last scope, one needs more involved calculations including, for instance, the loop contributions of the top and the corrections due to the running of the constants. This part of the research is not considered in the present work.

We consider the following process [37]

$$d + \bar{u} \rightarrow b + \bar{t}, \quad (63)$$

where the intervening quarks might be easily replaced by other constituents (*e.g.* $l, \bar{\nu}$). We follow the conventions of Ref. [38] for the fermion sector. In the case of Higgs mechanism, we have a Drell–Yan process mediated by W^- and ϕ^- . In the Landau, gauge we have (we consider only the s -channel graph)

$$\begin{aligned} \mathfrak{M}(M) = & \frac{g^2 V_{ud} V_{tb}^*}{2} \bar{v}_u \gamma_\mu \frac{1 - \gamma_5}{2} u_d \frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}}{q^2 - M^2} \bar{u}_b \gamma_\nu \frac{1 - \gamma_5}{2} v_t \\ & + V_{ud} V_{tb}^* \bar{v}_u \left[f_u \frac{1 - \gamma_5}{2} - f_d \frac{1 + \gamma_5}{2} \right] u_d \frac{1}{q^2} \bar{u}_b \left[f_t \frac{1 - \gamma_5}{2} - f_b \frac{1 + \gamma_5}{2} \right] v_t \end{aligned} \quad (64)$$

with $M = gv$ and $\sqrt{2}f_x v = m_x$. Unitarity is preserved on-shell; in fact, at $q^2 = M^2$ the only pole is in the propagator of the W^- with a residuum that projects on the physical polarizations. Moreover, there is no pole at $q^2 = 0$, since the Goldstone boson cancels the spin zero part of the vector meson.

Eq. (64) shows in detail what happens in the limit $v = 0$: we perform the limit in two different ways. First, we add the contributions of the Goldstone part to the vector meson propagator to obtain the unitary gauge amplitude. On that amplitude the limit is performed to discover that the longitudinal polarizations yield a finite result. Second, the limit is taken separately on the two terms of Eq. (64). The gauge term $q^\mu q^\nu$ of the W -propagator vanishes in the limit, while the Goldstone contribution survives to match the longitudinal polarization's of the previous limit procedure, as in the mechanism described in Eq. (40). The $q^\mu q^\nu$ term in the W -propagator vanishes via Dirac equation, while the “Goldstone” field contribution survives, as in the mechanism described in Eq. (40). This exemplifies the metamorphosis of the longitudinal polarization into the physical massless scalar mode, originally associated to the unphysical Goldstone boson for $v \neq 0$.

By taking the limit in the first fashion for every value of M , the two terms add to

$$\mathfrak{M}(M) = \frac{g^2 V_{ud} V_{tb}^*}{2} \bar{v}_u \gamma_\mu \frac{1 - \gamma_5}{2} u_d \frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{M^2}}{q^2 - M^2} \bar{u}_b \gamma_\nu \frac{1 - \gamma_5}{2} v_t. \quad (65)$$

The M^{-2} term does survive in the limit of zero mass, since the $q^\mu q^\nu$ produces a quadratic term in the quark or lepton masses and, therefore, the v^2 dependence disappears in the ratio. While in the second way, we take the $v = 0$ limit on the longitudinal part of the propagator in the Landau gauge ($q^\mu q^\nu / q^2$): the result is zero. But the Goldstone contribution is finite and adds to the total amplitude in Eq. (65) taken at $v^2 = 0$.

In the nonlinear theory, according to our guess, such terms are not present and the W -propagator in the Landau gauge is as usual

$$-i \frac{g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}}{q^2}, \quad (66)$$

where the $\frac{q^\mu q^\nu}{q^2}$ vanishes on the quark and lepton chiral currents, since the $M = 0$ limit has been already taken. Finally, the difference between the two scenarios (linear *versus* nonlinear) are traced simply by the presence of a $\frac{1}{M^2}$ factor.

Now, we look whether the difference is of some phenomenological relevance in order to show the origin of the difficulty to find a measurable signature. The square modulus of the amplitude (65) summed over the polarizations of the incoming and outgoing particles yields

$$\begin{aligned} \sum_{\text{POL}} |\mathfrak{M}(M)|^2 &= \frac{1}{(q^2 - M^2)^2} \left\{ 16(p_b p_u)(p_t p_d) \right. \\ &- \frac{8}{M^2} \left[m_t^2 m_u^2 (p_b p_d) + m_t^2 m_d^2 (p_b p_u) + m_b^2 m_u^2 (p_t p_d) + m_b^2 m_d^2 (p_t p_u) \right] \\ &\left. + \frac{4}{M^4} \left[2m_t^2 m_b^2 + (m_b^2 + m_t^2)(p_b p_t) \right] \left[2m_u^2 m_d^2 + (m_u^2 + m_d^2)(p_u p_d) \right] \right\}. \quad (67) \end{aligned}$$

It is clear that the M^{-2} and M^{-4} terms are negligible and, therefore, one cannot discriminate the linear model (with Higgs boson) from the nonlinear one (without Higgs boson) in this process.

10. Conclusions

We consider the massive Yang–Mills theory at very high momenta both in the case of a Higgs mechanism generated mass and of a Stückelberg mass term. The kinematical set up is reproduced by the $M = 0$ limit, by assuming that only one energy scale is present in the physical process. In this limit, the number of degrees of freedom of vector mesons changes. This fact poses a problem for unitarity, since the longitudinal modes do not decouple in non-Abelian gauge theories for $M = 0$.

In the first case, we suggest the metamorphosis of the longitudinal modes into the Goldstone scalar bosons when the limit $M = 0$ is taken. The scenario is supported by the CLTCG theorem. *In passing* we present some improvements on the CLTCG theorem. According to this proposal, the unitarity equation is consistently satisfied both for $M \neq 0$ and $M = 0$: no mismatch of degrees of freedom shows up and, moreover, the symmetric limit $v = 0$ looks smooth. The limit theory consists of a massless gauge

Yang–Mills in interaction with a doublet of physical complex scalar fields (the Higgs and the Goldstone bosons). The theory is expected to be non-asymptotically free.

In the Stückelberg mass case, the limit $M = 0$ on the perturbative series is not possible due to very singular terms. We suggest that the perturbative region is separated from the $M \sim 0$ region by some singularity line between different phases. We envisage the scenario, where at $M \sim 0$ both the Goldstone bosons and the longitudinal modes decouple and the theory is realized in a confined phase, typical of a massless gauge theory. However, the properties of the confinement phase are relevant only for extremely high momenta ($M \sim 0$) processes (*e.g.* asymptotic freedom) and not for low energy states.

The conjecture establishes a ground for developing experimental tests capable of distinguishing a linearly (Higgs formalism) from a nonlinearly (Stückelberg mass) realized non-Abelian massive gauge theory. However, this aspect of the work is beyond the scope of the present paper.

In the present paper, we use the covariant 't Hooft gauge. Several exact results are derived for the two-point functions in the unphysical sector. The limit at $M = 0$ of these two-point functions is evaluated in the Higgs mechanism. This limit tells that the equivalence theorem (CLTCG) in its tree-level formulation is not modified by loop corrections.

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Appendix A

Free field example

In this appendix, we evaluate the contributions of the longitudinal mode and of the spin zero part of the vector meson to the unitarity sum in the free case. Consider the *free* vector meson propagator in the generic ξ gauge

$$-\frac{g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}}{p^2 - M^2} - \frac{1}{\xi} \frac{1}{p^2 - \frac{M^2}{\xi}} \frac{p_\mu p_\nu}{p^2}. \quad (68)$$

We use the residua of the poles both for $p^2 = M^2$ and $p^2 = \frac{M^2}{\xi}$ and at the end, we take the limit $M^2 = 0$. The contribution of the longitudinal polarization on the pole $p^2 = M^2$ as in Eq. (32) is

$$\frac{1}{M^2} \begin{pmatrix} p^2 & E p_j \\ E p_i & \frac{E^2}{p^2} p_i p_j \end{pmatrix}, \quad (69)$$

while the spin zero at $p^2 = \frac{M^2}{\xi}$

$$- \frac{1}{M^2} \begin{pmatrix} E_G^2 & E_G p_j \\ E_G p_i & p_i p_j \end{pmatrix}. \quad (70)$$

The contribution of the two transverse polarizations is

$$\begin{pmatrix} 0 & 0 \\ 0 & \delta_{ij} - \frac{1}{p^2} p_i p_j \end{pmatrix}. \quad (71)$$

The expression in Eq. (69) is multiplied by some quantity $M^{\mu\nu}(E, \vec{p})$, while that in Eq. (70) by $M^{\mu\nu}(E_G, \vec{p})$.

Now, we add the contributions with the front factors $1/E$ and $1/E_G$ originating from the Lorentz invariant measure. Thus, we have to add the terms

$$\frac{1}{2EM^2} \left(p^2 M^{00}(E, \vec{p}) + E p_j M^{0j}(E, \vec{p}) + E p_i M^{i0}(E, \vec{p}) + \frac{E^2}{p^2} p_i p_j M^{ij}(E, \vec{p}) \right) \quad (72)$$

and

$$\begin{aligned} & - \frac{1}{2E_G M^2} (E_G^2 M^{00}(E_G, \vec{p}) + E_G p_j M^{0j}(E_G, \vec{p}) + E_G p_i M^{i0}(E_G, \vec{p}) + p_i p_j M^{ij}(E_G, \vec{p})) \\ & = - \frac{1}{2EM^2} \left(1 - \frac{E_G - E}{E} \right) \left[\left(p^2 + \frac{M^2}{\xi} \right) \left(M^{00}(E) + (E_G - E) \frac{\partial}{\partial E} M^{00}(E) \right) \right. \\ & + p_j \left(E M^{0j}(E) + (E_G - E) \left(M^{0j}(E) + E \frac{\partial}{\partial E} M^{0j}(E) \right) \right) \\ & + p_i \left(E M^{i0}(E) + (E_G - E) \left(M^{i0}(E) + E \frac{\partial}{\partial E} M^{i0}(E) \right) \right) \\ & \left. + p_i p_j \left(M^{ij}(E) + (E_G - E) \frac{\partial}{\partial E} M^{ij}(E) \right) \right]. \quad (73) \end{aligned}$$

Now we have

$$E_G - E \simeq -M^2 \frac{1 - \frac{1}{\xi}}{2p} \quad (74)$$

and, therefore, we get

$$\begin{aligned} & \frac{1}{2p} \left[\frac{p}{2} \left(1 - \frac{1}{\xi} \right) \frac{\partial}{\partial E} M^{00}(E) - \frac{1}{\xi} M^{00}(E) + \frac{p_j}{2p} \left(1 - \frac{1}{\xi} \right) \left(M^{0j}(E) + E \frac{\partial}{\partial E} M^{0j}(E) \right) \right. \\ & + \frac{p_i}{2p} \left(1 - \frac{1}{\xi} \right) \left(M^{i0}(E) + E \frac{\partial}{\partial E} M^{i0}(E) \right) + \frac{p_i p_j}{p^2} M^{ij}(E) + \frac{p_i p_j}{2p} \left(1 - \frac{1}{\xi} \right) \frac{\partial}{\partial E} M^{ij}(E) \left. \right] \\ & - \frac{1}{4p^3} \left(1 - \frac{1}{\xi} \right) \{ p^2 M^{00} + p p_j M^{0j} + p p_i M^{i0} + p_i p_j M^{ij} \}. \quad (75) \end{aligned}$$

If we add the transverse part (71), we get

$$\begin{aligned} & \frac{1}{2p} \left[\frac{p}{2} \left(1 - \frac{1}{\xi} \right) \frac{\partial}{\partial E} M^{00}(E) - \frac{1}{\xi} M^{00}(E) + \frac{p_j}{2p} \left(1 - \frac{1}{\xi} \right) \left(M^{0j}(E) + E \frac{\partial}{\partial E} M^{0j}(E) \right) \right. \\ & + \frac{p_i}{2p} \left(1 - \frac{1}{\xi} \right) \left(M^{i0}(E) + E \frac{\partial}{\partial E} M^{i0}(E) \right) + \delta_{ij} M^{ij}(E) + \frac{p_i p_j}{2p} \left(1 - \frac{1}{\xi} \right) \frac{\partial}{\partial E} M^{ij}(E) \\ & \left. - \frac{1}{2p^2} \left(1 - \frac{1}{\xi} \right) \{ p^2 M^{00} + p p_j M^{0j} + p p_i M^{i0} + p_i p_j M^{ij} \} \right] \end{aligned} \quad (76)$$

that can be written

$$\begin{aligned} & \frac{1}{2p} \left(-g_{\mu\mu} M^{\mu\nu}(E) + \left(1 - \frac{1}{\xi} \right) \left[M^{00}(E) + \frac{p}{2} \frac{\partial}{\partial E} M^{00}(E) \right. \right. \\ & + \frac{p_j}{2p} \left(M^{0j}(E) + E \frac{\partial}{\partial E} M^{0j}(E) \right) \\ & + \frac{p_i}{2p} \left(M^{i0}(E) + E \frac{\partial}{\partial E} M^{i0}(E) \right) + \frac{p_i p_j}{2p} \frac{\partial}{\partial E} M^{ij}(E) \\ & \left. \left. - \frac{1}{2p^2} \{ p^2 M^{00} + p p_j M^{0j} + p p_i M^{i0} + p_i p_j M^{ij} \} \right] \right) \Big|_{E=|\vec{p}|} \\ & = \frac{1}{2p} \left(-g_{\mu\mu} M^{\mu\nu}(E) + \left(1 - \frac{1}{\xi} \right) \left[\frac{1}{2} M^{00}(E) - \frac{p_i p_j}{2p^2} M^{ij} + \frac{p}{2} \frac{\partial}{\partial E} M^{00}(E) \right. \right. \\ & \left. \left. + \frac{p_j}{2} \frac{\partial}{\partial E} M^{0j}(E) + \frac{p_i}{2} \frac{\partial}{\partial E} M^{i0}(E) + \frac{p_i p_j}{2p} \frac{\partial}{\partial E} M^{ij}(E) \right] \right) \Big|_{E=|\vec{p}|}. \end{aligned} \quad (77)$$

This agrees with the $M = 0$ limit propagator

$$- \frac{g_{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \frac{p_\mu p_\nu}{p^2}}{p^2}. \quad (78)$$

In fact, the positive frequency pole gives the first term in Eq. (77), while the double pole gives

$$\begin{aligned} & \left(1 - \frac{1}{\xi} \right) \frac{\partial}{\partial p_0} \frac{p_\mu p_\nu}{(p_0 + p)^2} M^{\mu\nu}(p_0) \Big|_{p_0=|\vec{p}|} \\ & = \left(1 - \frac{1}{\xi} \right) \left[\frac{1}{2p} M^{00}(p) - \frac{1}{4p} M^{00}(p) + \frac{1}{4} \frac{\partial}{\partial p_0} M^{00}(p) \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{p_j}{4p^2} M^{0j} - \frac{p_j}{4p^2} M^{0j} + \frac{p_j}{4p} \frac{\partial}{\partial p_0} M^{0j}(p) \\
 & + \frac{p_i}{4p} \frac{\partial}{\partial p_0} M^{i0}(p) - \frac{p_i p_j}{4p^3} M^{ij}(p) + \frac{p_i p_j}{4p^2} \frac{\partial}{\partial p_0} M^{ij}(p) \Bigg] \\
 & = \frac{1}{2p} \left(1 - \frac{1}{\xi} \right) \left[\frac{1}{2} M^{00}(p) + \frac{p}{2} \frac{\partial}{\partial p_0} M^{00}(p) + \frac{p_j}{2} \frac{\partial}{\partial p_0} M^{0j}(p) \right. \\
 & \left. + \frac{p_i}{2} \frac{\partial}{\partial p_0} M^{i0}(p) - \frac{p_i p_j}{2p^2} M^{ij}(p) + \frac{p_i p_j}{2p} \frac{\partial}{\partial p_0} M^{ij}(p) \right]. \quad (79)
 \end{aligned}$$

The r.h.s. of Eq. (79) is zero in presence of the conservation laws $p^\mu M_{\mu\nu} = p^\nu M_{\mu\nu} = 0$.

This appendix shows that terms in Eq. (37), otherwise neglected on-shell, give essential contributions for the limit $M = 0$.

Appendix B

BRST transformations and STI (Higgs)

We discuss the BRST transformations for both models. We derive the STI and the relations that are needed for the CLTCG theorem. First, we consider the case where the mass is generated by the Higgs mechanism.

The necessity to maintain physical unitarity after the introduction of a gauge fixing term requires the use of the Faddeev–Popov ghosts. This is easily done by imposing BRST invariance of the action under the transformations

$$\begin{aligned}
 \mathfrak{s} A_{a\mu} &= (D_\mu[A]c)_a, & \mathfrak{s} \phi_0 &= -\frac{1}{2} \phi_a c_a, \\
 \mathfrak{s} \phi_a &= \frac{1}{2} \phi_0 c_a + \frac{1}{2} \epsilon_{abc} \phi_b c_c, \\
 \mathfrak{s} \bar{c}_a &= b_a, & \mathfrak{s} b_a &= 0. \quad (80)
 \end{aligned}$$

In the above equation, $D_\mu[A]$ denotes the covariant derivative w.r.t. $A_{a\mu}$

$$(D_\mu[A])_{ac} = \delta_{ac} \partial_\mu + \epsilon_{abc} A_{b\mu}. \quad (81)$$

The BRST transformation of c_a then follows by nilpotency

$$\mathfrak{s} c_a = -\frac{1}{2} \epsilon_{abc} c_b c_c. \quad (82)$$

Now, we can easily obtain a BRST invariant action by making invariant the gauge fixing term. This is achieved by using the nilpotency of $\mathfrak{s}$. For the generic covariant 't Hooft gauge, we have

$$S_{\text{H}_{\text{gf}}} \rightarrow S_{\text{H}_{\text{GF}}} = \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \mathfrak{s} \left[\bar{c}_a \left(\frac{b_a}{2\xi} + \frac{M}{\xi} \phi_a + \partial_\mu A_a^\mu \right) \right]. \quad (83)$$

The STI associated to the above BRST transformations, in the notations of Batalin and Vilkovisky [39, 40] can be easily derived. By introducing the external sources

$$\int d^D x \left(A_{a\mu}^* \mathfrak{s} A_a^\mu + \phi_a^* \mathfrak{s} \phi_a + \phi_0^* \mathfrak{s} \phi_0 + c_a^* \mathfrak{s} c_a \right). \quad (84)$$

For the 1-PI functional one gets

$$\int d^D x \left(\Gamma_{A_{a\mu}^*} \Gamma_{A_a^\mu} + \Gamma_{\phi_a^*} \Gamma_{\phi_a} + \Gamma_{\phi_0^*} \Gamma_{\phi_0} + \Gamma_{c_a^*} \Gamma_{c_a} + b_a \Gamma_{\bar{c}_a} \right) = 0. \quad (85)$$

While for the generating functional of the connected amplitudes one has

$$\int d^D x \left(-W_{A_{a\mu}^*} J_{a\mu} - W_{\phi_a^*} K_a - W_{\phi_0^*} K_0 + W_{c_a^*} \bar{\eta}_a - W_{b_a} \eta_a \right) = 0. \quad (86)$$

The equation associated to the gauge fixing gives

$$\Gamma_{b_a} = \Lambda_g \left(\frac{b_a}{\xi} + \frac{M}{\xi} \phi_a + \partial_\mu A_a^\mu \right), \quad (87)$$

$$-J_{b_a} = \Lambda_g \left(\frac{1}{\xi} W_{b_a} + \frac{M}{\xi} W_{\phi_a} + \partial_\mu W_{A_a^\mu} \right). \quad (88)$$

The anti-ghost equation

$$\begin{aligned} \Gamma_{\bar{c}_a} &= \Lambda_g \left[-\frac{M}{2\xi} (c_a \phi_0 + \epsilon_{abc} c_c \phi_b) - \partial_\mu (D^\mu [A] c)_a \right], \\ &= \Lambda_g \left[-\frac{M}{\xi} \Gamma_{\phi_a^*} - \partial_\mu \Gamma_{A_{a\mu}^*} \right] \end{aligned} \quad (89)$$

$$\eta_a = \Lambda_g \left[\frac{M}{\xi} W_{\phi_a^*} + \partial_\mu W_{A_{a\mu}^*} \right]. \quad (90)$$

Appendix C

BRST transformations and STI (Stückelberg)

In the case, where the mass of the Yang–Mills fields comes from a Stückelberg term, the same BRST apply as in Eqs. (80) and (82). Moreover, the STI and the gauge fixing equation are akin to those of Yang–Mills with Higgs mechanism. Here, we illustrate this property, which is not trivial due to the fact that Stückelberg’s mass term makes the theory nonrenormalizable.

Eq. (16) becomes

$$S_{S_{\text{gf}}} \rightarrow S_{S_{\text{GF}}} = \frac{\Lambda^{(D-4)}}{g^2} \int d^D x \mathfrak{s} \left[\bar{c}_a \left(\frac{b_a}{2\xi} + 2\frac{M^2}{\xi} \phi_a + \partial_\mu A_a^\mu \right) \right]. \quad (91)$$

The process of removal of the divergences of this nonrenormalizable field theory [27], requires a much wider set of external sources ($K_0, V_{a\mu}, z_0, z_a$) than in Eq. (84)

$$\int d^D x \left(A_{a\mu}^* \mathfrak{s} A_a^\mu + \phi_a^* \mathfrak{s} \phi_a + \phi_0^* \mathfrak{s} \phi_0 + c_a^* \mathfrak{s} c_a + V_{a\mu} \mathfrak{s} D^\mu [A]_{ab} \bar{c}_b + z_0 \chi_0 + z_a \chi_a + K_0 \phi_0 \right), \quad (92)$$

where

$$\begin{aligned} \bar{c} &\equiv \bar{c}_a \frac{\tau_a}{2}, \\ \chi_0 + i\tau_a \chi_a &\equiv 2i \mathfrak{s} \bar{c} \Omega = 2i (b \Omega - \bar{c} \mathfrak{s} \Omega). \end{aligned} \quad (93)$$

By construction, the fields

$$\begin{aligned} \chi_0 &= -b_a \phi_a + \frac{1}{2} \bar{c}_a c_a \phi_0 + \frac{1}{2} \epsilon_{abc} \bar{c}_b c_a \phi_c, \\ \chi_c &= \phi_0 b_c - \epsilon_{abc} b_a \phi_b + \frac{1}{2} \epsilon_{abc} \bar{c}_b c_a \phi_0 + \frac{1}{2} (\bar{c}_c c_a \phi_a + \bar{c}_a c_a \phi_c - \bar{c}_a \phi_a c_c) \end{aligned} \quad (94)$$

transform like ϕ_0, ϕ_a . The source K_0 is needed in order to perform the insertion of the composite operator ϕ_0 [21], ϕ_0^* and $V_{a\mu}$ are the external sources for the BRST transform of ϕ_0 and for some operator entering in the gauge fixing [27], and finally, z_0, z_a are necessary in order to deal with the 't Hooft gauge [41].

In a standard way, one gets the STI

$$\int d^D x \left(\Gamma_{A_{a\mu}^*} \Gamma_{A_a^\mu} + \Gamma_{\phi_a^*} \Gamma_{\phi_a} - K_0 \Gamma_{\phi_0^*} + \Gamma_{c_a^*} \Gamma_{c_a} + b_a \Gamma_{\bar{c}_a} \right) = 0, \quad (95)$$

and the gauge fixing equation

$$\Gamma_{b_a} = \Lambda_g \left[\frac{b_a}{\xi} + 2 \frac{M^2}{\xi} \phi_a + \partial^\nu A_a^\mu \right] - z_0 \phi_a + z_c (\Gamma_{K_0} - \epsilon_{abc} \phi_b). \quad (96)$$

For the connected amplitudes, we have

$$\int d^D x \left(-J_{a\mu} W_{A_{a\mu}^*} - K_a W_{\phi_a^*} - K_0 W_{\phi_0^*} + \bar{\eta}_a W_{c_a^*} - \eta_a W_{b_a} \right) = 0 \quad (97)$$

and the gauge-fixing equation

$$-J_{b_a} = \Lambda_g \left[\frac{1}{\xi} W_{b_a} + 2 \frac{M^2}{\xi} W_{\phi_a} + \partial^\nu W_{A_a^\mu} \right] - z_0 W_{\phi_a} + z_c (W_{K_0} - \epsilon_{abc} W_{\phi_b}). \quad (98)$$

The presence of the sources z_a upsets the anti-ghost equation, in fact, $\Gamma_{\bar{c}_a}$ contains the insertion of composite operators that are not associated to the listed external sources. The use of the sources $K_0, V_{a\mu}, z_0, z_a$ is necessary for the subtraction of the infinities [41]. In this work, we evaluate only the two-point function of b, A_μ, ϕ , then, we can put to zero all the external source $K_0, V_{a\mu}, z_0, z_a$ and, consequently, the usual anti-ghost equation (see Eqs. (89) and (90)) is at our disposal.

In this subsection, we have shown that in the massive Yang–Mills theory the two-point function W and Γ in the unphysical sector obey the same equations with the Higgs mechanism as well as with the Stückelberg term, apart from an unessential rescaling of the field $\phi \rightarrow 2M\phi$. Also the results of Appendix E apply, with the same rescaling.

Appendix D

Properties of the two-point functions (Higgs)

The results of this section apply to both Higgs and Stückelberg scenario, since the STI (Eqs. (85), (86), (95) and (97)) and the gauge fixing equations (Eqs. (87), (88), (96) and (98)) are the same after rescaling the ϕ field.

Now we derive some consequences of the above equations.

From Eq. (87), we get

$$\Gamma_{bb} = \Lambda_g \frac{1}{\xi}, \quad \Gamma_{b\phi} = \Lambda_g \frac{M}{\xi}, \quad \Gamma_{A^\mu b} = i\Lambda_g p_\mu. \quad (99)$$

From the STI in Eq. (86)

$$\begin{aligned} W_{b_a b_b} &= 0, \\ W_{A_{a\mu}^* \bar{c}_b} &= W_{A_{a\mu} b_b}, \\ W_{\phi_a^* \bar{c}_b} &= W_{\phi_a b_b}, \end{aligned} \quad (100)$$

and from Eq. (88)

$$\Lambda_g \left[\frac{M}{\xi} W_{\phi_a^* \bar{c}_b} - ip^\mu W_{A_{a\mu}^* \bar{c}_b} \right] = -\delta_{ab}. \quad (101)$$

From the STI in Eq. (89)

$$\Gamma_{c_a A_{a'\mu}^*} \Gamma_{b_b A_{a'\mu}} + \Gamma_{c_a \phi_{a'}^*} \Gamma_{b_b \phi_{a'}} + \Gamma_{c_a \bar{c}_b} = 0 \quad (102)$$

i.e.

$$\Lambda_g \left[-ip_\mu \Gamma_{c_a A_{b\mu}^*} + \frac{M}{\xi} \Gamma_{c_a \phi_b^*} \right] = -\Gamma_{c_a \bar{c}_b}. \quad (103)$$

Moreover, from the STI in Eq. (85), we get (we drop unnecessary indexes)

$$\Gamma_{cA_\mu^*} \Gamma_{A^\mu \phi} + \Gamma_{c\phi^*} \Gamma_{\phi\phi} = 0, \quad (104)$$

$$\Gamma_{cA_\mu^*} \Gamma_{A^\mu A^\nu} + \Gamma_{c\phi^*} \Gamma_{\phi A^\nu} = 0. \quad (105)$$

Eqs. (104) and (105) are compatible if the Jacobian is zero

$$(p^\nu \Gamma_{A^\nu \phi})^2 + p^2 \Gamma_L \Gamma_{\phi\phi} = 0. \quad (106)$$

Now, we solve the linear system given by Eqs. (103) and (104)

$$\begin{aligned} \Lambda_g p_\mu \Gamma_{cA_\mu^*} &= -\frac{\Gamma_{c\bar{c}} \Gamma_{\phi\phi}}{i\Gamma_{\phi\phi} - \frac{M}{\xi p^2} p^\nu \Gamma_{A^\nu \phi}} = ip^2 \frac{\Gamma_{c\bar{c}}}{p^2 + i\frac{M}{\xi \Gamma_{\phi\phi}} p^\nu \Gamma_{A^\nu \phi}}, \\ \Lambda_g \Gamma_{c\phi^*} &= -\frac{p^\nu \Gamma_{A^\nu \phi}}{p^2 \Gamma_{\phi\phi}} p_\mu \Gamma_{cA_\mu^*} = -i \frac{p^\nu \Gamma_{A^\nu \phi}}{\Gamma_{\phi\phi}} \frac{\Gamma_{c\bar{c}}}{p^2 + i\frac{M}{\xi \Gamma_{\phi\phi}} p^\nu \Gamma_{A^\nu \phi}}. \end{aligned} \quad (107)$$

Thus Eq. (100) gives

$$\begin{aligned} W_{A^\mu b} &= -\frac{i}{\Lambda_g} \frac{p^\mu}{p^2 - i\frac{M}{\xi} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}, \\ W_{\phi b} &= \frac{i}{\Lambda_g} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \frac{1}{p^2 - \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}. \end{aligned} \quad (108)$$

The two-point functions have the pole in the same position, *i.e.* the solution of

$$p^2 = \frac{M}{\xi} \frac{ip^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}. \quad (109)$$

Two-point functions

From the STI Eq. (86), one gets ($k > 1$)

$$W_{b_1 \dots b_k ***} = 0, \quad (110)$$

where *** indicates any reduction formula operator (on-shell) for *physical* modes.

From Eqs. (88) and (110) we get

$$\frac{M}{\xi} W_{\phi b} - ip_\mu W_{A^\mu b} = -\frac{1}{\Lambda_g}, \quad (111)$$

$$\frac{1}{\xi} W_{b\phi} + \frac{M}{\xi} W_{\phi\phi} - ip_\mu W_{A^\mu \phi} = 0, \quad (112)$$

$$\frac{1}{\xi} W_{bA^\nu} + \frac{M}{\xi} W_{\phi A^\nu} - ip_\mu W_{A^\mu A^\nu} = 0. \quad (113)$$

Now, we use

$$\Gamma W = -\mathbb{I} \quad (114)$$

and Eqs. (110)–(113) in order to derive the two-point function W in terms of the two-point function Γ . We explicit write some elements of the matrix in Eq. (114)

$$(\Gamma W)_{\phi b} = 0 \implies \Gamma_{\phi\phi} W_{\phi b} + \Gamma_{\phi A^\mu} W_{A^\mu b} = 0, \quad (115)$$

$$(\Gamma W)_{A^\nu b} = 0 \implies \Gamma_{A^\nu\phi} W_{\phi b} + \Gamma_{A^\nu A^\mu} W_{A^\mu b} = 0, \quad (116)$$

$$(\Gamma W)_{\phi\phi} = -1 \implies \Gamma_{\phi b} W_{b\phi} + \Gamma_{\phi\phi} W_{\phi\phi} + \Gamma_{\phi A^\mu} W_{A^\mu\phi} = -1, \quad (117)$$

$$(\Gamma W)_{\phi A^\nu} = 0 \implies \Gamma_{\phi b} W_{bA^\nu} + \Gamma_{\phi\phi} W_{\phi A^\nu} + \Gamma_{\phi A^\mu} W_{A^\mu A^\nu} = 0, \quad (118)$$

$$(\Gamma W)_{A^\nu\phi} = 0 \implies \Gamma_{A^\nu b} W_{b\phi} + \Gamma_{A^\nu\phi} W_{\phi\phi} + \Gamma_{A^\nu A^\mu} W_{A^\mu\phi} = 0, \quad (119)$$

$$(\Gamma W)_{A^\nu A^\mu} = -g_{\mu\nu} \implies \Gamma_{A^\nu b} W_{bA^\mu} + \Gamma_{A^\nu\phi} W_{\phi A^\mu} + \Gamma_{A^\nu A^\sigma} W_{A^\sigma A^\mu} = -g_{\mu\nu}. \quad (120)$$

Now, we solve the linear system given by the Eqs. (112) and (117). The Jacobian of the homogeneous system is

$$J_\Delta = p^\nu \Gamma_{\phi A^\nu} \frac{M}{\xi p^2} + i \Gamma_{\phi\phi} = i \frac{\Gamma_{\phi\phi}}{p^2} \left(p^2 - i p^\nu \Gamma_{\phi A^\nu} \frac{M}{\xi \Gamma_{\phi\phi}} \right). \quad (121)$$

The straightforward solutions are

$$W_{A^\mu\phi} = i \frac{1}{\xi \Gamma_{\phi\phi}} \left(\frac{i}{\Lambda_g} p^\nu \Gamma_{A^\nu\phi} + M p^2 \right) \frac{p^\mu}{\left(p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2} \quad (122)$$

and

$$W_{\phi\phi} = -\frac{p^2}{\Gamma_{\phi\phi}} \left(p^2 - \frac{1}{\Lambda_g \xi} \Gamma_L \right) \frac{1}{\left(p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2}. \quad (123)$$

A similar calculation for the linear system given by the Eqs. (108), (113) and (122) yields

$$W_L = \frac{p^2}{\xi \Gamma_{\phi\phi}} \frac{\frac{1}{\Lambda_g} \Gamma_{\phi\phi} - \frac{M^2}{\xi}}{\left(p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} \right)^2}. \quad (124)$$

The presence of double poles is the source of some technical problems in dealing with the proof of Physical Unitarity.

For comparison, we evaluate the relevant quantities for the free fields

$$\begin{aligned} \Gamma_{bb} &= \frac{\Lambda_g}{\xi}, & \Gamma_{\phi b} &= \Lambda_g \frac{M}{\xi}, & \Gamma_{A^\nu b} &= i\Lambda_g p_\nu, \\ \Gamma_{A^\nu \phi} &= i\Lambda_g M p_\nu, & \Gamma_{\phi\phi} &= \Lambda_g p^2, & \Gamma_L &= \Lambda_g M^2. \end{aligned} \quad (125)$$

Then

$$W_{A^\mu \phi} = 0 \quad (126)$$

and

$$\begin{aligned} W_{A^\mu b} &= -i \frac{p^\mu}{\Lambda_g(p^2 - \frac{M^2}{\xi})}, & W_{\phi b} &= \frac{M}{\Lambda_g(p^2 - \frac{M^2}{\xi})}, \\ W_L &= \frac{1}{\Lambda_g \xi} \frac{1}{p^2 - \frac{M^2}{\xi}}, & W_{\phi\phi} &= -\frac{1}{\Lambda_g(p^2 - \frac{M^2}{\xi})}. \end{aligned} \quad (127)$$

Only simple poles appear in the free field approximation.

Appendix E

Zero eigenvalue mode

We construct the mode corresponding to the zero eigenvalue of the matrix $\mathcal{G}$ in (26) when on-shell.

First, consider

$$\mathcal{r}_a \equiv \frac{1}{\xi} b_a + \frac{M}{\xi} \phi_a + \partial_\mu A_a^\mu. \quad (128)$$

From Eq. (88), we see that the field decouples from every other

$$W_{\mathcal{r}b} = -1, \quad W_{\mathcal{r}\phi} = 0, \quad W_{\mathcal{r}A_\mu} = 0. \quad (129)$$

A next interesting local field is (used in Ref. [34])

$$X_1 \equiv \frac{1}{\xi} \phi + \frac{1}{M} \partial_\mu A^\mu. \quad (130)$$

From Eqs. (112), (113) and (108), we have

$$\begin{aligned} W_{X_1 \phi} &= -\frac{1}{\xi M} W_{b\phi} = -\frac{i}{\Lambda_g} \frac{p^\nu \Gamma_{\phi A^\nu}}{\xi M \Gamma_{\phi\phi}} \frac{1}{p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}, \\ W_{X_1 A^\nu} &= -\frac{1}{\xi M} W_{b A^\nu} = -\frac{i}{\Lambda_g \xi M} \frac{p^\nu}{p^2 - i \frac{M}{\xi} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}}. \end{aligned} \quad (131)$$

From Eq. (131), we see that X_1 is the mode corresponding to the zero eigenvalue of $\mathcal{G}$ when taken on-shell. Finally, one has

$$\begin{aligned} W_{X_1 X_1} &= \frac{1}{\xi} W_{X_1 \phi} + i \frac{1}{M} p^\nu W_{X_1 A^\nu} \\ &= -\frac{1}{\Lambda_g \xi M^2} \left(i \frac{M}{\xi} \frac{p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}} - p^2 \right) \frac{1}{p^2 - \frac{M}{\xi} \frac{i p^\nu \Gamma_{\phi A^\nu}}{\Gamma_{\phi\phi}}} = \frac{1}{\Lambda_g \xi M^2}. \end{aligned} \quad (132)$$

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