BOSE–EINSTEIN CONDENSATION IN ARBITRARY DIMENSIONS

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The density of bosonic states is calculated for spinless free massive bosons in generalised d dimensions. The number of bosons is calculated in the lowest energy state. The Bose–Einstein condensation was found in generalised d dimensions (at and above $d = 3$) and the condensation temperature is calculated. It is observed to drop abruptly above three dimensions and decreases monotonically as the dimensionalities of the system increase. The rate of fall of the condensation temperature decreases as the dimensionality increases. Interestingly, in the limit $d \to \infty$, the condensation temperature is observed to approach a nonzero finite value.

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Bose–Einstein condensation (BEC) [1,2] is a very interesting and important subject of modern research. The slowing of atoms by the use of cooling apparatus produces a singular quantum state known as a Bose condensate or Bose–Einstein condensate. Einstein demonstrated that cooling of bosonic atoms to a very low temperature would cause them to fall (or "condense") into the lowest accessible quantum state, resulting in a new form of matter. This transition occurs below a critical temperature, which for a uniform three-dimensional gas consists of non-interacting particles with no apparent internal degrees of freedom. After its experimental discovery [3] the BEC becomes much more challenging in modern research. Very recently, the condensation of photons (massless bosons) was observed experimentally [4] in an optical microcavity.

Recently, the static properties of positive ions in atomic BEC [5], the band structures, elementary excitations, stability of BEC in periodic potentials [6] and the collisional relaxation in diffuse clouds of trapped bosons [7] are studied in physical systems.

However, all these studies are done in three dimensions. Although, it is a pedagogical matter, one may try to see the condensation in higher dimensions or, more generally, in d dimensions.

In this paper, we study the Bose–Einstein condensation in generalised d dimensions. We hope, that the present study will motivate the researchers to do such type of generalisation of some other physical phenomena in higher dimensions.

In d dimensions, the density of states for non-interacting spinless bosons is the same as that for free fermions [8]. The energy of a free boson can be written as [2] (Appendix C)

$$E = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2 + \ldots + p_d^2\right) . \tag{1}$$

Here, the above relation represents an equation of d-dimensional hypersphere (in momentum space) having radius $R = \sqrt{2mE}$. The density of single bosonic states will be proportional to the volume of the spherical shell bounded between energy E and $E + dE$. This can be calculated easily [8] and gives

$$g_d(E)dE = C(m,V)E^{\frac{d-2}{2}}dE\,, \tag{2}$$

where the constant $C(m,V)$ is given by

$$C(m,V) = \frac{V}{\Gamma(d/2)}\left(\frac{2m\pi}{h^2}\right)^{d/2} . \tag{3}$$

It may be readily checked that in three dimensions ($d = 3$), one gets the well known and widely used result of density of states, *i.e.*, $g(E) \sim E^{1/2}$.

The thermodynamics of the system can be easily obtained from the grand canonical partition function, $Z_{\mathrm{g}}(z,V,T)$, which is given by

$$Z_{\mathrm{g}}(z,V,T) = \prod\nolimits_i \left(1 - ze^{-\beta E_i}\right)^{-1} , \tag{4}$$

where $\beta = 1/kT$ and z is the fugacity of the gas and related to the chemical potential μ by the relation, $z = e^{-\beta\mu}$. Here the suffix i refers to the single particle state, having energy E_i and the product is over all the single particle states.

The total number of particles of the system are given by the following formula

$$N = \Sigma_i \frac{1}{z^{-1} e^{\beta E_i} - 1} . \tag{5}$$

It is to be noted that $ze^{-\beta E} < 1$ for all E, and since for free particle minimum value of E is zero, the maximum value of z is 1. The number of particle N_0 at ground state $E = 0$ is

$$N_0 = \frac{1}{z^{-1} - 1} . \tag{6}$$

For large volume, the spectrum of free particle becomes almost continuous and hence the summation in the above equation could be represented by $\int g_d(E) dE$. However, at $E = 0$, $g_d(E)$ becomes zero (Eq. (2)), hence the contribution of the number of particles (N_0) in the ground state ($E = 0$) to the total number of particles (N) could not be obtained in this way. So, one has to separate the number N_0 from the total number N. This trick is applied here since one finds that N_0 becomes considerably large as $z \to 1$. Thus, the total number N is written as

$$N = N_0 + \int_{0^+}^{\infty} \frac{g_d(E) dE}{z^{-1} e^{\beta E} - 1} . \tag{7}$$

From the expression of N_0, it is quite clear that as z approaches to one, the $E = 0$ state starts to populate. This phenomenon of accumulation of particles in the ground state (even at $T \neq 0$) is known as Bose condensation

$$N - N_0 = C(m, V) \int_0^{\infty} \frac{E^{d/2-1} dE}{z^{-1} e^{\beta E} - 1} , \tag{8}$$

$$N - N_0 = \frac{C(m, V)}{\beta^{d/2}} \int_0^{\infty} \frac{X^{d/2-1} dX}{z^{-1} e^{X} - 1} , \tag{9}$$

where $X = \beta E$. Substituting the value of $C(m, V)$ from Eq. (3), one obtains

$$\frac{N - N_0}{V} = \frac{1}{\lambda^d} g_{\frac{d}{2}}(z) \tag{10}$$

where λ is given by

$$\lambda^{-1} = \sqrt{\frac{2m\pi kT}{h^2}} , \tag{11}$$

and $g_\nu(z)$ is the Bose–Einstein function, given by

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int\limits_0^\infty \frac{x^{\nu-1}dx}{z^{-1}e^x - 1}\,. \tag{12}$$

Since, for bosons, the value of z is always ≤ 1, $g_\nu(z)$ can be expressed as a power series

$$g_\nu(z) = \Sigma_{l=1} \frac{z^l}{l^\nu}\,. \tag{13}$$

The Bose condensation temperature ($T_{\rm B}$) is obtained here from the equation for $z = 1$ (since $\mu = 0$) and $N_0 = 0$. Hence $T_{\rm B}$ is given by [2]

$$T_{\rm B}{}^{\frac{d}{2}} = \frac{N/V}{\zeta(d/2)} \left(\frac{h^2}{2m\pi k}\right)^{d/2}, \tag{14}$$

where $\zeta(\nu) = g_\nu(z = 1)$ and is given by

$$\zeta(\nu) = \Sigma_{l=1}^\infty \frac{1}{l^\nu}\,. \tag{15}$$

For $\nu \leq 1$, $g_\nu(z)$ diverges [2] (Appendix D) as $z \rightarrow 1$, thus for one and two dimensions Bose condensation temperature($T_{\rm B}$) (see Eq. (14)) is zero. And for $d = 3$ one obtains the well known result

$$T_{\rm B} = \left(\frac{h^2}{2m\pi k}\right) \left(\frac{N/V}{2.612}\right)^{2/3}\,. \tag{16}$$

The condensation temperature $T_{\rm B}$ can be calculated in any dimension (d), from equation (14). We have estimated, the condensation temperature $T_{\rm B}$, for helium in various dimensions and shown in Table I.

TABLE I

Dimensionality (d)	Condensation temperature ($T_{\rm B}$) (in Kelvin)
$d = 3$	3.13269 (see Ref. [2])
$d = 4$	8.76365×10^{-4}
$d = 5$	5.83222×10^{-6}
$d = 6$	1.99971×10^{-7}
$d = 7$	1.77437×10^{-8}
$d = 8$	2.86711×10^{-9}
$d = 9$	6.92417×10^{-10}
$d = 10$	2.21788×10^{-10}

Here, we have used the value of $V = 27.6$ c.c./mole (see Ref. [2]) and $m = 6.65 \times 10^{-24}$. The results given in Table I show that the condensation temperature falls abruptly in $d = 4$. As the dimensionalities of the system increase, the condensation starts at lower temperature. It is also evident from the data that the rate of fall of the condensation temperature with respect to the dimensionalities decreases as the dimensionality increases. Now the question is, should we find the BEC at zero temperature in infinite dimensions? Interestingly, the answer is NO. Taking the limit $d \to \infty$ in the expression (14), we get $T_{\mathrm{B}}(d \to \infty)$ is equal to $h^2/(2\pi mk)$ [9]. Here, it may be noted that $(N/(V\zeta(d/2)))^{\frac{2}{d}}$ approaches unity as $d \to \infty$.

In conclusion, we can say that the present study of BEC in generalised d dimensions is a new one. The condensation temperature was found to decrease abruptly (see Table I) in $d = 4$ and decrease monotonically (and slowly) as the dimensionality of the system increases. Additionally, it may also be noted (from Table I) that the rate of fall of condensation temperature, decreases as the dimensionality increases. *In addition, it is quite interesting that Bose–Einstein condensation occurs at any nonzero finite temperature [9], in any dimensions, even at* $d \to \infty$*, except* $d = 1$ *and* $d = 2$*.*

It may be mentioned here that non-interacting electrons show some interesting unusual behaviours in higher dimensions [8, 10, 11]. In infinite dimensions, all electron posses Fermi momentum [8]. Pauli spin susceptibility becomes temperature independent [10] *only* in two dimensions, the form of electrical conductivity remains invariant [11] in generalised d dimensions. All these interesting results could not be obtained unless studied in generalised d dimensions.

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