A NOTE ON ANGULAR MOMENTUM COMMUTATORS IN LIGHT-CONE FORMULATION OF OPEN BOSONIC STRING THEORY

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We recalculate, in a systematic and pedagogical way, one of the most important results of Bosonic open string theory in the light-cone formulation, namely the $[J^{-i}, J^{-j}]$ commutators, which together with Lorentz covariance, famously yield the critical dimension $D = 26$ and the normal order constant $a = 1$. We use traditional transverse oscillator mode expansions (avoiding the elegant but more advanced language of operator product expansions). We streamline the proof by introducing a novel bookkeeping/regularization parameter κ to avoid splitting into creation and annihilation parts, and to avoid sandwiching between bras and kets.

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1. Introduction

One of the most fundamental facts of Bosonic string theory, that a student of string theory would want to rederive for himself, is the critical dimension $D = 26$. There are traditionally four[1] ways to detect the critical dimension $D = 26$ in the Bosonic string theory:

1. Preservation of Lorentz covariance in the light-cone formulation [3,4].
2. No negative norm states/ghosts in the covariant formulation [5].
3. The vanishing of the conformal/Weyl anomaly in Polyakov's path integral formulation [6].
4. Nilpotency of the BRST generator in the covariant formulation [7].

[1] Plus various heuristic arguments typically involving zeta function regularization [1,2].

Here we will only consider the first method in the open string case in a flat Minkowski target space. Students of Zwiebach's book [2], which uses the light-cone formulation, will notice that the book in Section 12.5 stops short of proving[2] the critical dimension $D = 26$. The goal of the current paper is to fill that gap in a pedagogical and efficient manner. The main calculation is an evaluation of a commutator $[E^i, E^j]$ between two expressions E^i and E^j, which are cubic in the transverse α oscillator modes, see Sec. 8. The original papers of Goddard, Goldstone, Rebbi and Thorn [3,4], where the creation and annihilation parts were singled out, are sparse on details, although recently an explicit calculation appeared in Ref. [8] for the closed string case. Here, we will use a more efficient method by introducing a κ regularization parameter [9], see Sec. 3, so that we can rigorously calculate with symmetrized expressions via Wick's Theorem [10], see Sec. 4. We believe the techniques displayed here are interesting in their own right, applicable far beyond the shown calculations.

2. Basic settings

To keep this paper short, it is necessary to assume that the reader is familiar with the light-cone formulation of the Bosonic string theory. Here we will only briefly repeat all relevant definitions and formulas to set notations and conventions. For explanations and justifications, we defer to, *e.g.* Zwiebach's book [2]. The light-cone metric in flat Minkowski target space is

$$\eta_{\mu\nu} \equiv \left[\begin{array}{cc|ccc} 0 & -1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ \hline 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & & \ddots \end{array}\right], \quad \mu,\nu \in \{+,-,i\}, \quad i \in \{1,\ldots,D-2\}. \tag{2.1}$$

Here Greek indices $\mu, \nu, \ldots,$ run over all target space dimensions, while Latin indices $i, j, \ldots,$ only run over transversal directions. We normalize the center-of-mass position x_0^μ and the total momentum p^μ of the open string as follows[3]

$$\begin{aligned} q_0^\mu &\equiv \frac{x_0^\mu}{c\sqrt{2\hbar\alpha'}} \equiv \sqrt{\frac{\hbar}{2}}\frac{x_0^\mu}{\ell_s}\,, \qquad & \alpha_0^\mu \equiv c\sqrt{2\hbar\alpha'}p^\mu \equiv \sqrt{\frac{2}{\hbar}}\ell_s p^\mu\,, \\ \alpha' &\equiv \frac{1}{2\pi\hbar c T_0}\,, & \ell_s \equiv \hbar c\sqrt{\alpha'}\,, \end{aligned} \tag{2.2}$$

[2] Ref. [1] gives a proof by sandwiching $[J^{-i}, J^{-j}]$ between a bra and a ket, which depend on the choice of vacuum.

[3] Note that Goddard, Goldstone, Rebbi and Thorn [4] call the center-of-mass position x_0^μ for q_0^μ.

where the string tension T_0 has dimension of force, and $1/\sqrt{\alpha'}$ has dimension of energy. The fundamental dynamical operators in the light-cone formalism (in light-cone gauge) are

$$q_0^- \,, \qquad \alpha_0^+ \,, \qquad q_0^i \,, \qquad \alpha_n^i \,, \tag{2.3}$$

where $n \in \mathbb{Z}$ and $i = 1, \ldots, D-2$. All the operators (2.3) have the same dimension as $\sqrt{\hbar}$. The notations for the commutator and the anti-commutator of two operators A and B are

$$[A, B] \equiv AB - BA \,, \qquad \{A, B\} \equiv AB + BA \,, \tag{2.4}$$

respectively. More generally, define the n-symmetrizer

$$\{A_1, \ldots, A_n\} \equiv \sum_{\pi \in S_n} A_{\pi(1)} \ldots A_{\pi(n)} \tag{2.5}$$

of n operators $A_1, \ldots, A_n$, as a sum of all possible permutations $\pi \in S_n$. Normal ordering (usually denoted with a double colon) moves all the annihilation operators $\alpha^i_{m>0}$ to the right of all the creation operators $\alpha^i_{m<0}$. Equivalently, in formula

$$: \alpha^i_m \alpha^j_n : \; = \; \theta(n-m)\alpha^i_m \alpha^j_n \; + \; \theta(m-n)\alpha^j_n \alpha^i_m \,, \tag{2.6}$$

where θ denotes the Heaviside step function with $\theta(0) = \frac{1}{2}$. Note that operators commute inside the normal order symbol, for instance $: AB : = : BA :$ for two operators A and B.

3. Fundamental commutator relations and the κ parameter

The non-zero commutator relations for the fundamental operators (2.3) are[4]

$$\begin{aligned} \left[q_0^-, \alpha_0^+\right] \; &= \; \mathrm{i}\hbar\eta^{-+} \overset{(2.1)}{=} -\mathrm{i}\hbar \,, \qquad \left[q_0^i, \alpha_0^j\right] = \mathrm{i}\hbar\eta^{ij} \,, \\ \left[\alpha^i_m, \alpha^j_n\right] \; &= \; \hbar m \kappa^{|m|} \delta^0_{m+n} \eta^{ij} \,. \end{aligned} \tag{3.1}$$

The parameter κ in the commutator relation (3.1) is a regularization parameter with $|\kappa| < 1$. In the end of the calculations, one should take the limit $\kappa \to 1$. The limit $\kappa \to 0$ corresponds to the classical limit $\hbar \to 0$. Now, why do we introduce the regularization parameter κ? To answer this question, imagine in the standard $\kappa = 1$ case, that we want to calculate the commutator $C = [A, B]$ of two normal-ordered operators A and B (which

[4] An "i" that is not an upper or lower index does always denote the imaginary unit.

are polynomials in the α oscillator modes) by carefully performing a minimal number of $\alpha\alpha$ commutations to bring $C = [A, B]$ on normal-ordered form. Imagine further that the result C happens to be finite. Then convergence can only improve if we repeat the calculation $C(\kappa) = [A, B]$ with $|\kappa| < 1$. Moreover, the result will depend continuously, $C(\kappa) \to C(1)$ as $\kappa \to 1$. This suggests a strategy. We first introduce the regularization parameter κ in the commutator relation (3.1) with $|\kappa| < 1$. As we shall soon see, the commutator $C(\kappa) = [A, B]$ will remain well-defined under a wider and more powerful class of mathematical manipulations as long as $|\kappa| < 1$. Thus, we can calculate the commutator $C(\kappa)$ more efficiently, and in the end, we take the limit $\kappa \to 1$.

4. Wick's Theorem

We list here some of the first few consequences of Wick's Theorem [10], which will be our main computational tool.

Theorem 4.1 (Wick's Theorem for symmetrization and normal order) *(i) The anti-commutator*

$$\frac{1}{2}\left\{\alpha^i_m, \alpha^j_n\right\} = \,:\alpha^i_m\alpha^j_n:\, + c^{ij}_{mn}\,, \qquad c^{ij}_{mn} \equiv \frac{\hbar}{2}|m|\kappa^{|m|}\delta^0_{m+n}\eta^{ij} \tag{4.1}$$

is a sum of a normal ordered term and a single contraction term.
(ii) The 4*-symmetrizer*

$$\begin{aligned}\tfrac{1}{24}\left\{\alpha^{i_1}_{n_1}, \alpha^{i_2}_{n_2}, \alpha^{i_3}_{n_3}, \alpha^{i_4}_{n_4}\right\} \;=\;& :\alpha^{i_1}_{n_1}\alpha^{i_2}_{n_2}\alpha^{i_3}_{n_3}\alpha^{i_4}_{n_4}: \\ &+\tfrac{6}{24}\sum_{\pi\in S_4} c^{i_{\pi(1)}i_{\pi(2)}}_{n_{\pi(1)}n_{\pi(2)}} :\alpha^{i_{\pi(3)}}_{n_{\pi(3)}}\alpha^{i_{\pi(4)}}_{n_{\pi(4)}}: \\ &+\tfrac{3}{24}\sum_{\pi\in S_4} c^{i_{\pi(1)}i_{\pi(2)}}_{n_{\pi(1)}n_{\pi(2)}} c^{i_{\pi(3)}i_{\pi(4)}}_{n_{\pi(3)}n_{\pi(4)}}\end{aligned} \tag{4.2}$$

is a sum of a normal ordered term, 6 *different single contraction terms and* 3 *different double contraction terms.*

Theorem 4.2 (Wick's Theorem for commutators and symmetrization) *(i) If $[A_a, B_b]$ are c-numbers, $a, b = 1, 2 \mod 2$, then the commutator of anti-commutators*

$$\tfrac{1}{2}\left[\tfrac{1}{2}\{A_1, A_2\}, \tfrac{1}{2}\{B_1, B_2\}\right] = \sum_{a,b=1}^{2} \tfrac{1}{2}[A_a, B_b]\tfrac{1}{2}\{A_{a+1}, B_{b+1}\} \tag{4.3}$$

is a sum of single commutator terms.
(ii) If $[A_a, B_b]$ are c-numbers, $a, b = 1, 2, 3 \mod 3$, then the commutator of 3*-symmetrizers*

$$
\begin{aligned}
&\tfrac{1}{2}\left[\tfrac{1}{6}\{A_1,A_2,A_3\},\tfrac{1}{6}\{B_1,B_2,B_3\}\right]\\
&=\sum_{a,b=1}^{3}\tfrac{1}{2}\left[A_a,B_b\right]\tfrac{1}{24}\{A_{a+1},A_{a+2},B_{b+1},B_{b+2}\}\\
&+\sum_{\pi\in S_3}\tfrac{1}{2}\left[A_1,B_{\pi(1)}\right]\tfrac{1}{2}\left[A_2,B_{\pi(2)}\right]\tfrac{1}{2}\left[A_3,B_{\pi(3)}\right]
\end{aligned}
\tag{4.4}
$$

is a sum of single and triple commutator terms.

Wick's Theorem 4.2 follows by expanding out to appropriate order the "little Baker–Campbell–Hausdorff formula" $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]}$ (which holds if $[A,B]$ is a c-number) with $A=\sum_a x^aA_a$ and $B=\sum_b y^bB_b$, where x^a and y^b are parameters, and then afterwards antisymmetrize with respect to $A\leftrightarrow B$ on both sides.

5. Transverse Virasoro generators $L_n^{\perp}$ and algebra

The α_n^- modes and the transverse Virasoro generators $L_n^{\perp}$ are defined as[5]

$$
\alpha_n^- \equiv \frac{1}{\alpha_n^+}\left(L_n^{\perp}-\hbar a\delta_n^0\right), \qquad n\in\mathbb{Z}\,, \tag{5.1}
$$

$$
L_n^{\perp} \equiv \tfrac{1}{2}\eta_{ij}\sum_{k\in\mathbb{Z}}:\alpha_{n-k}^i\alpha_k^j: \overset{k=\ell+\frac{n}{2}}{=} \tfrac{1}{2}\eta_{ij}\sum_{\ell\in\mathbb{Z}+\frac{n}{2}}:\alpha_{\frac{n}{2}-\ell}^i\alpha_{\frac{n}{2}+\ell}^j: \tag{5.2}
$$

$$
\overset{(4.1)}{=} \tfrac{1}{4}\eta_{ij}\sum_{k\in\mathbb{Z}}\left\{\alpha_{n-k}^i,\alpha_k^j\right\} - \hbar\frac{D-2}{4}\delta_n^0\sum_{k\in\mathbb{Z}}|k|\kappa^{|k|}\,. \tag{5.3}
$$

It may, at first, seem a bit cumbersome to sum over half-integers ℓ in Eq. (5.2), but it makes the symmetry $\ell\leftrightarrow-\ell$ manifest, which is sometimes convenient. Notice that the last c-number sum

$$
\sum_{k\in\mathbb{Z}}|k|\kappa^{|k|} = \kappa\frac{d}{d\kappa}\sum_{k\neq 0}\kappa^{|k|} \overset{|\kappa|<1}{=} \kappa\frac{d}{d\kappa}\frac{2\kappa}{1-\kappa} = \frac{2\kappa}{(1-\kappa)^2} \tag{5.4}
$$

in Eq. (5.3) is absolutely and unconditionally convergent for $|\kappa|<1$ but divergent for $|\kappa|>1$ and $\kappa=1$. (Zeta function regularization would suggest that one should assign the value $2\sum_{k>0}k\sim 2\zeta(-1)=-\frac{1}{6}$ to the sum (5.4) at $\kappa=1$.) We precisely introduced the regularization parameter κ to be able to rearrange expressions without encountering infinities. The non-zero

[5] Zwiebach [2] defines the normal ordering constant a with the opposite sign.

commutator relations between the transverse Virasoro generators $L^{\perp}_n$ and the fundamental variables read

$$\left[\alpha^i_m, L^{\perp}_n\right] = \hbar m\kappa^{|m|}\alpha^i_{m+n}\,, \qquad \left[q^i_0, L^{\perp}_n\right] = i\hbar\alpha^i_n\,. \tag{5.5}$$

As a warm-up exercise, let us derive the transverse Virasoro algebra with central charge $c = D-2$

$$\left[L^{\perp}_m, L^{\perp}_n\right] = \hbar(m-n)L^{\perp}_{m+n} + \hbar^2\frac{D-2}{12}m(m^2-1)\kappa^{|m|}\delta^0_{m+n} + \mathcal{O}(\kappa-1)\,. \tag{5.6}$$

Proof of Eq. (5.6). The commutator on the left-hand side of Eq. (5.6) is a sum of two terms

$$\begin{aligned} C_{mn} \;&\equiv\; \left[L^{\perp}_m, L^{\perp}_n\right] \overset{(5.3)}{=} \sum_{k,\ell\in\mathbb{Z}}\left[\tfrac{1}{4}\left\{\alpha^i_{m-k}, \alpha^i_k\right\},\; \tfrac{1}{4}\left\{\alpha^j_{n-\ell}, \alpha^j_\ell\right\}\right] \\ &\overset{(4.3)}{=} \sum_{k,\ell\in\mathbb{Z}}\left[\alpha^i_k, \alpha^j_\ell\right]\tfrac{1}{2}\left\{\alpha^i_{m-k}, \alpha^j_{n-\ell}\right\} \overset{(3.1)}{=} \hbar\sum_{k\in\mathbb{Z}} k\kappa^{|k|}\tfrac{1}{2}\left\{\alpha^i_{m-k}, \alpha^i_{n+k}\right\} \\ &\overset{(4.1)}{=} \hbar\sum_{k\in\mathbb{Z}} k\kappa^{|k|}\left(:\alpha^i_{m-k}\alpha^i_{n+k}: \;+\; c^{ii}_{m-k,n+k}\right) = C^{(2)}_{mn} \;+\; C^{(0)}_{mn}\,. \end{aligned} \tag{5.7}$$

The first term $C^{(2)}_{mn}$ is quadratic (hence the superscript "2") in the transverse α oscillator modes

$$\begin{aligned} C^{(2)}_{mn} \;&\equiv\; \hbar\sum_{k\in\mathbb{Z}} k\kappa^{|k|} :\alpha^i_{m-k}\alpha^i_{n+k}: \\ &\overset{k=\ell+\frac{m-n}{2}}{=} \hbar\sum_{\ell\in\mathbb{Z}+\frac{m+n}{2}}\left(\frac{m-n}{2}+\ell\right)\kappa^{\left|\frac{m-n}{2}+\ell\right|} :\alpha^i_{\frac{m+n}{2}-\ell}\alpha^i_{\frac{m+n}{2}+\ell}: \\ &\overset{\ell\leftrightarrow-\ell}{=} \frac{\hbar}{2}\sum_{\ell\in\mathbb{Z}+\frac{m+n}{2}}\left[\left(\frac{m-n}{2}+\ell\right)\kappa^{\left|\frac{m-n}{2}+\ell\right|}\right. \\ &\qquad \left. + \left(\frac{m-n}{2}-\ell\right)\kappa^{\left|\frac{m-n}{2}-\ell\right|}\right] :\alpha^i_{\frac{m+n}{2}-\ell}\alpha^i_{\frac{m+n}{2}+\ell}: \\ &\longrightarrow\; \hbar(m-n)L^{\perp}_{m+n} \quad \text{for} \quad \kappa\to 1\,. \end{aligned} \tag{5.8}$$

The second term $C^{(0)}_{mn}$ is the c-number anomaly term

$$\begin{aligned} C^{(0)}_{mn} \;&\equiv\; \hbar\sum_{k\in\mathbb{Z}} k\kappa^{|k|} c^{ii}_{m-k,n+k} \overset{(4.1)}{=} \frac{\hbar^2}{2}\sum_{k\in\mathbb{Z}} k\kappa^{|k|}|m-k|\kappa^{|m-k|}\delta^0_{m+n}\eta^{ii} \\ &\overset{(5.10)}{=} \hbar^2\frac{D-2}{2}A_m\delta^0_{m+n}\,, \end{aligned} \tag{5.9}$$

with anomaly

$$A_m \equiv \sum_{k\in\mathbb{Z}} k|m-k|\kappa^{|k|+|m-k|} \stackrel{\ell=m-k}{=} \sum_{\substack{k,\ell\in\mathbb{Z}\\ k+\ell=m}} k|\ell|\kappa^{|k|+|\ell|}$$
$$= \frac{m\left(m^2-1\right)}{6}\kappa^{|m|}\,. \tag{5.10}$$

Standard reasoning shows that the κ power series (5.10) is absolutely and unconditionally convergent for $|\kappa|<1$. However, one can say more. The following argument reveals that the κ power series (5.10) only has one non-zero coefficient, and therefore is just a monomial in κ, which makes sense for any $\kappa\in\mathbb{C}$. In the restricted double summation (5.10), note that the (k,ℓ)-th term is antisymmetric under a $(k\leftrightarrow\ell)$ exchange if the summation variables k and ℓ have opposite signs. Therefore, one only has to consider ks and ℓs with weakly the same sign. (The word *weakly* refers to that k or ℓ could be 0.) Since at the same time the sum $k+\ell=m$ of k and ℓ is held fixed, the restricted (k,ℓ) double sum contains only finitely many terms, all with the same power $|m|$ of κ, and which may be readily summed. Since $A_{-m}=-A_m$ is odd, it is enough to consider $m\geq 1$. Then

$$A_m = \kappa^m\sum_{k=1}^{m} k(m-k) = \frac{m(m^2-1)}{6}\kappa^m\,, \qquad m\geq 1\,, \tag{5.11}$$

which, *e.g.* follows from the fact that $\sum_{k=1}^m k=\frac{1}{2}m(m+1)$ and $\sum_{k=1}^m k^2=\frac{1}{3}m(m+\frac{1}{2})(m+1)$ for $m\geq 1$.

□

By similar arguments, one may derive that the following κ power series (5.12) is also just a monomial in κ,

$$B_m \equiv \sum_{k\in\mathbb{Z}} \mathrm{sgn}(k)\kappa^{|k|+|m-k|} \stackrel{\ell=m-k}{=} \sum_{\substack{k,\ell\in\mathbb{Z}\\ k+\ell=m}} \mathrm{sgn}(k)\kappa^{|k|+|\ell|} = m\kappa^{|m|}\,, \tag{5.12}$$

which we will need later in Eq. (8.4).

6. Angular momentum $J^{\mu\nu}$

The angular momentum $J^{\mu\nu}$ consists of a center-of-mass part $\ell^{\mu\nu}$ and an oscillator part $E^{\mu\nu}$

$$J^{\mu\nu} \equiv \ell^{\mu\nu} + E^{\mu\nu} = -(\mu \leftrightarrow \nu)\,, \tag{6.1}$$

$$\ell^{\mu\nu} \equiv \tfrac{1}{2}\{x_0^\mu, p_0^\nu\} - (\mu \leftrightarrow \nu) \stackrel{(2.2)}{=} \tfrac{1}{2}\{q_0^\mu, \alpha_0^\nu\} - (\mu \leftrightarrow \nu)\,, \tag{6.2}$$

$$\begin{aligned} E^{\mu\nu} &\equiv -\sum_{n\neq 0}\frac{\mathrm{i}}{n} :\alpha_{-n}^\mu \alpha_n^\nu: \stackrel{(2.6)}{=} -\sum_{n>0}\frac{\mathrm{i}}{n}\alpha_{-n}^\mu\alpha_n^\nu - (\mu \leftrightarrow \nu) \\ &\stackrel{(3.1)}{=} \sum_{n\neq 0}\frac{\mathrm{i}}{2n}\{\alpha_{-n}^\mu, \alpha_n^\nu\}\,, \end{aligned} \tag{6.3}$$

where $\mu, \nu \in \{-, i\}$. (Recall that x_0^+ and $J^{\mu+}$ are somewhat amputated in the light-cone formalism [2].) The angular momentum J^{-i} consists of three terms[6]

$$J^{-i} \equiv \ell_I^{-i} + \ell_{II}^{-i} + E^{-i}\,, \qquad \ell_I^{-i} \equiv q_0^- \alpha_0^i\,, \tag{6.4}$$

$$\ell_{II}^{-i} \equiv -\tfrac{1}{2}\left\{q_0^i, \alpha_0^-\right\} \stackrel{(5.1)}{=} -\frac{1}{2\alpha_0^+}\left\{q_0^i, L_0^\perp - a\hbar\right\}\,, \tag{6.5}$$

$$E^{-i} \equiv \sum_{n\neq 0}\frac{\mathrm{i}}{n} :\alpha_{-n}^i \alpha_n^-: \stackrel{(5.1)}{=} \frac{1}{\alpha_0^+}E^i\,, \tag{6.6}$$

$$\begin{aligned} E^i &\equiv \sum_{n\neq 0}\frac{\mathrm{i}}{n} :\alpha_{-n}^i L_n^\perp: \stackrel{(2.6)}{=} \sum_{n>0}\frac{\mathrm{i}}{n}\left(\alpha_{-n}^i L_n^\perp - L_{-n}^\perp \alpha_n^i\right) \\ &\stackrel{(5.5)}{=} \sum_{n\neq 0}\frac{\mathrm{i}}{2n}\left\{\alpha_{-n}^i, L_n^\perp\right\} \stackrel{(3.1)}{=} \sum_{n\neq 0}\frac{\mathrm{i}}{12n}\sum_{k\in\mathbb{Z}}\left\{\alpha_{-n}^i, \alpha_{n-k}^j, \alpha_k^{j'}\right\}\eta_{jj'} \\ &= \sum_{n\neq 0}\frac{\mathrm{i}}{12n}\sum_{\ell\in\mathbb{Z}+\frac{n}{2}}\left\{\alpha_{-n}^i, \alpha_{\frac{n}{2}-\ell}^j, \alpha_{\frac{n}{2}+\ell}^{j'}\right\}\eta_{jj'}\,. \end{aligned} \tag{6.7}$$

Note that we have two expressions for the E^i operator, either as an anti-commutator with $L_n^\perp$, or as a 3-symmetrizer, which follows from straightforward manipulations. Hermiticity is manifestly guaranteed by the anti-commutator (3-symmetrizer) form

$$q_0^{\mu\dagger} = q_0^\mu\,, \quad \alpha_n^{\mu\dagger} = \alpha_{-n}^\mu\,, \quad L_n^{\perp\dagger} = L_{-n}^\perp\,, \quad J^{\mu\nu\dagger} = J^{\mu\nu}\,, \quad E^{i\dagger} = E^i\,, \tag{6.8}$$

basically because the anti-commutator (3-symmetrizer) of two (three) Hermitian operators is again Hermitian, respectively.

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[6] Conventions differ slightly between various references, $\mathrm{i}E^i \equiv -\mathrm{i}E^i_{\mathrm{GSW}} \equiv E^i_{\mathrm{GRT}} \equiv E^i_{\mathrm{SCH}} \equiv -\sum_{n\neq 0}\frac{1}{n} :\alpha_{-n}^i L_n^\perp:$, and $\mathrm{i}E^{ij} \equiv \mathrm{i}E^{ij}_{\mathrm{GSW}} \equiv E^{ij}_{\mathrm{GRT}} \equiv E^{ij}_{\mathrm{SCH}} \equiv \sum_{n\neq 0}\frac{1}{n} :\alpha_{-n}^i\alpha_n^j:$, where GRT $\equiv$ Ref. [3], SCH $\equiv$ Ref. [11] and GSW $\equiv$ Ref. [1].

7. Commutator $\left[J^{-i}, J^{-j}\right]$

Let us now derive the sought-for commutator

$$\begin{aligned}\left[J^{-i}, J^{-j}\right] &= \frac{2\hbar^2}{(\alpha_0^+)^2}\sum_{n\neq 0} :\alpha_{-n}^i\alpha_n^j: \kappa^{|n|}\left[n\left(\frac{D-2}{24}-1\right)-\frac{1}{n}\left(\frac{D-2}{24}-a\right)\right] \\ &\quad +\mathcal{O}(\kappa-1)\,, \end{aligned} \tag{7.1}$$

which, in the limit $\kappa \to 1$, precisely vanishes for $D = 26$ and $a = 1$.
Proof of Eq. (7.1), part 1: We may assume that the external transverse indices $i \neq j$ are different (or else the commutator (7.1) vanishes trivially). Then the operator α_0^i in the first term of Eq. (6.4) commutes with everything in the commutator (7.1) (because it never meets q_0^i), so that one may treat that α_0^i as a c-number. In particular, the commutator between the two first terms in Eq. (6.4) vanishes

$$\left[\ell_I^{-i}, \ell_I^{-j}\right] \stackrel{(6.4)}{=} 0\,. \tag{7.2}$$

Also the operators q_0^- and α_0^+ commute with everything except each other. This produces the following commutator between the first term and the two other terms in Eq. (6.4)

$$\begin{aligned}\left[\ell_I^{-i}, \ell_{II}^{-j} + E^{-j}\right] &= \alpha_0^i\left[q_0^-, \frac{1}{\alpha_0^+}\right]\left(-\tfrac{1}{2}\left\{q_0^j, L_0^\perp - a\hbar\right\} + E^j\right) \\ &\stackrel{(3.1)}{=} \frac{i\hbar\alpha_0^i}{(\alpha_0^+)^2}\left(-\tfrac{1}{2}\left\{q_0^j, L_0^\perp - a\hbar\right\} + E^j\right). \end{aligned} \tag{7.3}$$

The commutator between the two second terms (6.5) becomes

$$\begin{aligned}\left[\ell_{II}^{-i}, \ell_{II}^{-j}\right] &\stackrel{(6.5)}{=} \frac{1}{\left(\alpha_0^+\right)^2}\left[\tfrac{1}{2}\left\{q_0^i, L_0^\perp - a\hbar\right\}, \tfrac{1}{2}\left\{q_0^j, L_0^\perp - a\hbar\right\}\right] \\ &\stackrel{(4.3)}{=} \frac{1}{\left(\alpha_0^+\right)^2}\left[L_0^\perp - a\hbar, q_0^j\right]\tfrac{1}{2}\left\{q_0^i, L_0^\perp - a\hbar\right\} - (i\leftrightarrow j) \\ &\stackrel{(5.5)}{=} -\frac{i\hbar\alpha_0^j}{2\left(\alpha_0^+\right)^2}\left\{q_0^i, L_0^\perp - a\hbar\right\} - (i\leftrightarrow j) \end{aligned} \tag{7.4}$$

which cancels against the $[\ell_I^{-i}, \ell_{II}^{-j}] - (i \leftrightarrow j)$ contribution in Eq. (7.3). In particular, the two center-of-mass parts commute

$$\left[\ell^{-i}, \ell^{-j}\right] \equiv \left[\ell_I^{-i} + \ell_{II}^{-i}, \ell_I^{-j} + \ell_{II}^{-j}\right] = 0\,. \tag{7.5}$$

Notice that the light-cone Hamiltonian $L_0^\perp - a\hbar$ commutes with the operators E^j and E^{ij},

$$\left[L_0^\perp - a\hbar, E^j\right] \stackrel{(5.5)}{=} 0\,, \qquad \left[L_0^\perp - a\hbar, E^{ij}\right] \stackrel{(5.5)}{=} 0\,. \tag{7.6}$$

Moreover,

$$\begin{aligned}
\left[q_0^i, E^j\right] &\stackrel{(6.7)}{=} \left[q_0^i, \sum_{n\neq 0} \frac{\mathrm{i}}{2n}\left\{\alpha_{-n}^j, L_n^\perp\right\}\right] \stackrel{(3.1)}{=} \sum_{n\neq 0} \frac{\mathrm{i}}{2n}\left\{\alpha_{-n}^j, \left[q_0^i, L_n^\perp\right]\right\} \\
&\stackrel{(5.5)}{=} -\sum_{n\neq 0} \frac{\hbar}{2n}\left\{\alpha_{-n}^j, \alpha_n^i\right\} \stackrel{(6.3)}{=} \mathrm{i}\hbar E^{ij}\,.
\end{aligned} \tag{7.7}$$

Therefore, the commutator between the ℓ_{II}^{-i} and E^{-j} becomes

$$\begin{aligned}
\left[\ell_{II}^{-i}, E^{-j}\right] &= -\frac{1}{2\left(\alpha_0^+\right)^2}\left[\left\{q_0^i, L_0^\perp - a\hbar\right\}, E^j\right] \\
&\stackrel{(7.6)}{=} -\frac{1}{2\left(\alpha_0^+\right)^2}\left\{\left[q_0^i, E^j\right], L_0^\perp - a\hbar\right\} \\
&\stackrel{(7.7)}{=} -\frac{\mathrm{i}\hbar}{2\left(\alpha_0^+\right)^2}\left\{E^{ij}, L_0^\perp - a\hbar\right\} \\
&\stackrel{(7.6)}{=} -\frac{\mathrm{i}\hbar}{\left(\alpha_0^+\right)^2}\left(L_0^\perp - a\hbar\right) E^{ij} = -(i \leftrightarrow j) \\
&\stackrel{(7.9)}{=} -\frac{\mathrm{i}\hbar}{\left(\alpha_0^+\right)^2} : \left(L_0^\perp - a\hbar\right) E^{ij} : \\
&\quad -\frac{\hbar^2}{\left(\alpha_0^+\right)^2}\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\kappa^{|n|} : \alpha_{-n}^i \alpha_n^j :\,,
\end{aligned} \tag{7.8}$$

where we in the last equality normal-ordered the expression by using

$$\begin{aligned}
\mathrm{i}L_0^\perp E^{ij} - \ :\mathrm{i}L_0^\perp E^{ij}: &\stackrel{(6.3)}{=} \sum_{n>0} \frac{1}{n}\left[L_0^\perp, \alpha_{-n}^i\right]\alpha_n^j \ - \ (i \leftrightarrow j) \\
&\stackrel{(5.5)}{=} \hbar\sum_{n>0} \kappa^n \alpha_{-n}^i \alpha_n^j \ - \ (i \leftrightarrow j) \\
&= \hbar\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\kappa^{|n|} : \alpha_{-n}^i \alpha_n^j : \,.
\end{aligned} \tag{7.9}$$

It remains to compute the commutator between two oscillator terms (6.6),

$$\left[E^{-i}, E^{-j}\right] \stackrel{(6.6)}{=} \frac{1}{\left(\alpha_0^+\right)^2}\left[E^i, E^j\right]\,, \tag{7.10}$$

which we will do in the last section, *cf.* Eq. (8.1).

8. Commutator $[E^i, E^j]$ via 3-symmetrizer

Finally, let us derive, with the help of Wick's Theorem, that

$$\begin{aligned}\left[E^i, E^j\right] &= \mathrm{i}\hbar\left(2 : L_0^{\perp} E^{ij} : \; - \; \alpha_0^i E^j \; + \; \alpha_0^j E^i\right) \\ &\quad + 2\hbar^2 \sum_{n\neq 0} : \alpha_{-n}^i \alpha_n^j : \kappa^{|n|}\left[n\left(\frac{D-2}{24} - 1\right) + \operatorname{sgn}(n) - \frac{D-2}{24n}\right] \\ &\quad + \mathcal{O}(\kappa - 1)\,. \end{aligned} \tag{8.1}$$

Proof of Eq. (7.1), part 2: If one adds up contributions from Eq. (8.1), Eq. (7.8), and the last term in Eq. (7.3), one derives precisely the $[J^{-i}, J^{-j}]$ commutator (7.1).

□

Proof of Eq. (8.1). Recall that the operator E^i is cubic in the transverse α oscillator modes, *cf.* Eq. (6.7). The fact that the external transverse indices $i \neq j$ are different implies that there cannot be a triple commutator term in Eq. (4.4), nor double contraction terms in Eq. (4.2). Thus the commutator

$$\begin{aligned} C^{ij} &\equiv -\left[E^i, E^j\right] \\ &\overset{(6.7)}{=} \sum_{m\neq 0\neq n} \sum_{k,\ell\in\mathbb{Z}} \left[\frac{1}{12n}\left\{\alpha_{-n}^i, \alpha_{n-k}^{j'}, \alpha_k^{j'}\right\}, \frac{1}{12m}\left\{\alpha_{-m}^j, \alpha_{m-\ell}^{i'}, \alpha_\ell^{i'}\right\}\right] \\ &\overset{(4.4)}{=} \sum_{m\neq 0\neq n} \frac{1}{4mn} \sum_{k,\ell\in\mathbb{Z}} \left(2\left[\alpha_k^{j'}, \alpha_{-m}^j\right] \frac{1}{24}\left\{\alpha_{-n}^i, \alpha_{n-k}^{j'}, \alpha_{m-\ell}^{i'}, \alpha_\ell^{i'}\right\}\right. \\ &\qquad \left. + 2\left[\alpha_k^{j'}, \alpha_\ell^{i'}\right] \frac{1}{24}\left\{\alpha_{-n}^i, \alpha_{-m}^j, \alpha_{n-k}^{j'}, \alpha_{m-\ell}^{i'}\right\}\right) \; - \; (i \leftrightarrow j) \\ &\overset{(3.1)}{=} \sum_{m\neq 0\neq n} \frac{\hbar}{2n} \sum_{\ell\in\mathbb{Z}} \kappa^{|m|} \frac{1}{24}\left\{\alpha_{-n}^i, \alpha_{n-m}^j, \alpha_{m-\ell}^{i'}, \alpha_\ell^{i'}\right\} \\ &\qquad + \sum_{m\neq 0\neq n} \frac{\hbar}{2mn} \sum_{k\in\mathbb{Z}} k\kappa^{|k|} \frac{1}{24}\left\{\alpha_{-n}^i, \alpha_{-m}^j, \alpha_{n-k}^{i'}, \alpha_{m+k}^{i'}\right\} \; - \; (i \leftrightarrow j) \\ &\overset{(4.2)}{=} C_{(4)}^{ij} \; + \; C_{(2)}^{ij} \end{aligned} \tag{8.2}$$

is a sum of normal ordered terms $C_{(4)}^{ij}$, quartic in the transverse α oscillator modes; and single contraction terms $C_{(2)}^{ij}$, quadratic in the transverse α oscillator modes. The single contraction terms $C_{(2)}^{ij} = C_{(2')}^{ij} + C_{(2'')}^{ij}$ come in two types. One type $C_{(2')}^{ij}$ has a trace over transverse directions

$$
\begin{aligned}
C^{ij}_{(2')} \quad &\equiv \quad \sum_{m\neq 0\neq n} \frac{\hbar}{2n} \sum_{\ell\in\mathbb{Z}} \kappa^{|m|} c^{i'i'}_{m-\ell,\ell} : \alpha^i_{-n}\alpha^j_{n-m} : \\
&\quad + \sum_{m\neq 0\neq n} \frac{\hbar}{2mn} \sum_{k\in\mathbb{Z}} k\kappa^{|k|} c^{i'i'}_{n-k,m+k} : \alpha^i_{-n}\alpha^j_{-m} : \; - (i\leftrightarrow j) \\
&\overset{(4.1)}{=} \; 0 \; - \; \frac{D-2}{4} \sum_{n\neq 0} \frac{\hbar^2}{n^2} \sum_{k\in\mathbb{Z}} k|n-k|\kappa^{|k|+|n-k|} : \alpha^i_{-n}\alpha^j_n : \; - (i\leftrightarrow j) \\
&\overset{(5.10)}{=} \; -\frac{D-2}{4} \sum_{n\neq 0} \frac{\hbar^2}{n^2} A_n : \alpha^i_{-n}\alpha^j_n : \; - (i\leftrightarrow j) \\
&\overset{(5.10)}{=} \; \hbar^2 \frac{D-2}{12} \sum_{n\neq 0} \left(\frac{1}{n} - n\right) \kappa^{|n|} : \alpha^i_{-n}\alpha^j_n :
\end{aligned}
\tag{8.3}
$$

which becomes proportional to the number $D-2$ of transverse directions. The other type $C^{ij}_{(2'')}$ does not carry a trace over transverse directions

$$
\begin{aligned}
C^{ij}_{(2'')} \quad &\equiv \quad \sum_{m\neq 0\neq n} \frac{\hbar}{2n} \sum_{\ell\in\mathbb{Z}} \kappa^{|m|} \left(2c^{ji'}_{n-m,\ell} : \alpha^i_{-n}\alpha^{i'}_{m-\ell} : + 2c^{ii'}_{-n,\ell} : \alpha^{i'}_{m-\ell}\alpha^j_{n-m} :\right) \\
&\quad + \sum_{m\neq 0\neq n} \frac{\hbar}{2mn} \sum_{k\in\mathbb{Z}} k\kappa^{|k|} \Big(c^{i'j}_{n-k,-m} : \alpha^i_{-n}\alpha^{i'}_{m+k} : + c^{i'j}_{m+k,-m} : \alpha^i_{-n}\alpha^{i'}_{n-k} : \\
&\quad + c^{ii'}_{-n,m+k} : \alpha^{i'}_{n-k}\alpha^j_{-m} + c^{ii'}_{-n,n-k} : \alpha^{i'}_{m+k}\alpha^j_{-m} :\Big) - (i\leftrightarrow j) \\
&\overset{(4.1)}{=} \quad \sum_{m\neq 0\neq n} \frac{\hbar^2}{2n} \kappa^{|m|} \Big(|n-m|\kappa^{|n-m|} : \alpha^i_{-n}\alpha^j_n : \\
&\quad + |n|\kappa^{|n|} : \alpha^i_{m-n}\alpha^j_{n-m} : \Big) + 0 + 0 \\
&\quad + \sum_{m\neq 0\neq n} \frac{\hbar^2}{4mn} (n-m) \Big(|m|\kappa^{|m|+|n-m|} : \alpha^i_{-n}\alpha^j_n : \\
&\quad + |n|\kappa^{|n|+|m-n|} : \alpha^i_m\alpha^j_{-m} : \Big) - (i\leftrightarrow j) \\
&\overset{k=n-m}{=} \; \frac{\hbar^2}{2} \sum_{k\neq n\neq 0} \frac{|k|}{n} \kappa^{|k|+|n-k|} : \alpha^i_{-n}\alpha^j_n : \\
&\quad + \frac{\hbar^2}{2} \sum_{k\neq n\neq 0} \mathrm{sgn}(n) \kappa^{|n|+|k-n|} : \alpha^i_{-k}\alpha^j_k : \\
&\quad + \frac{\hbar^2}{2} \sum_{m\neq 0\neq n} \left(\mathrm{sgn}(m) - \frac{|m|}{n}\right) \kappa^{|m|+|n-m|} : \alpha^i_{-n}\alpha^j_n : - (i\leftrightarrow j)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\hbar^2}{2}\sum_{0\neq k\neq n\neq 0}\frac{|k|}{n}\kappa^{|k|+|n-k|} :\alpha^i_{-n}\alpha^j_n: \\
&\quad +\frac{\hbar^2}{2}\sum_{0\neq k\neq n\neq 0}\mathrm{sgn}(n)\kappa^{|n|+|k-n|} :\alpha^i_{-k}\alpha^j_k: \\
&\quad +\frac{\hbar^2}{2}\sum_{0\neq m\neq n\neq 0}\left(\mathrm{sgn}(m)-\frac{|m|}{n}\right)\kappa^{|m|+|n-m|} :\alpha^i_{-n}\alpha^j_n: -(i\leftrightarrow j) \\
&\overset{n\leftrightarrow k\leftrightarrow m}{=} \hbar^2\sum_{0\neq k\neq n\neq 0}\mathrm{sgn}(k)\kappa^{|k|+|n-k|} :\alpha^i_{-n}\alpha^j_n: \;-\;(i\leftrightarrow j) \\
&\overset{(5.12)}{=} \hbar^2\sum_{n\neq 0}\left(B_n-\mathrm{sgn}(n)\kappa^{|n|}\right) :\alpha^i_{-n}\alpha^j_n: \;-\;(i\leftrightarrow j) \\
&\overset{(5.12)}{=} 2\hbar^2\sum_{n\neq 0}(n-\mathrm{sgn}(n))\kappa^{|n|} :\alpha^i_{-n}\alpha^j_n: .
\end{aligned}
\tag{8.4}
$$

The normal-ordered terms $C^{ij}_{(4)}$ in Eq. (8.2) read

$$
\begin{aligned}
C^{ij}_{(4)} &\equiv \sum_{m\neq 0\neq n}\frac{\hbar}{2n}\sum_{\ell\in\mathbb{Z}}\kappa^{|m|} :\alpha^i_{-n}\alpha^j_{n-m}\alpha^{i'}_{m-\ell}\alpha^{i'}_{\ell}: \\
&\quad +\sum_{m\neq 0\neq n}\frac{\hbar}{2mn}\sum_{k\in\mathbb{Z}}k\kappa^{|k|} :\alpha^i_{-n}\alpha^j_{-m}\alpha^{i'}_{n-k}\alpha^{i'}_{m+k}: -(i\leftrightarrow j) \\
&= \sum_{-k\neq n\neq 0}\frac{\hbar}{2n}\sum_{\ell\in\mathbb{Z}+\frac{n+k}{2}}\kappa^{|n+k|} :\alpha^i_{-n}\alpha^j_{-k}\alpha^{i'}_{\frac{n+k}{2}-\ell}\alpha^{i'}_{\frac{n+k}{2}+\ell}: \\
&\quad +\sum_{m\neq 0\neq n}\frac{\hbar}{2mn}\sum_{\ell\in\mathbb{Z}+\frac{m+n}{2}}\left(\ell+\frac{n-m}{2}\right)\kappa^{|\ell+\frac{n-m}{2}|} \\
&\quad \times :\alpha^i_{-n}\alpha^j_{-m}\alpha^{i'}_{\frac{m+n}{2}-\ell}\alpha^{i'}_{\frac{m+n}{2}+\ell}: -(i\leftrightarrow j)\,,
\end{aligned}
\tag{8.5}
$$

where we in the first term replaced $k=m-n$ and shifted $\ell\to\ell+\frac{n+k}{2}$, while we replaced $k=\ell+\frac{n-m}{2}$ in the second term. The term in Eq. (8.5) with ℓ downstairs is odd under $\ell\leftrightarrow-\ell$ in the limit $\kappa\to 1$, so we can ignore them from now on. The terms in Eq. (8.5) corresponding to $k=0$ yield

$$
\begin{aligned}
&\sum_{n\neq 0}\frac{\hbar}{2n}\sum_{\ell\in\mathbb{Z}+\frac{n}{2}}\kappa^{|n|} :\alpha^i_{-n}\alpha^j_0\alpha^{i'}_{\frac{n}{2}-\ell}\alpha^{i'}_{\frac{n}{2}+\ell}: \;-\;(i\leftrightarrow j) \\
&\overset{(5.2)}{=} \sum_{n\neq 0}\frac{\hbar\kappa^{|n|}\alpha^j_0}{n} :\alpha^i_{-n}L^{\perp}_n: \;-\;(i\leftrightarrow j) \\
&\overset{(6.7)}{\longrightarrow} -\mathrm{i}\hbar\alpha^j_0E^i \;-\;(i\leftrightarrow j)\;\text{ for }\;\kappa\;\to\;1\,.
\end{aligned}
\tag{8.6}
$$

The terms in Eq. (8.5) corresponding to $m+n=0$ yield

$$
\begin{aligned}
& -\sum_{n\neq 0}\frac{\hbar}{2n}\sum_{\ell\in\mathbb{Z}}\kappa^{|\ell+n|} :\alpha^i_{-n}\alpha^j_n\alpha^{i'}_{-\ell}\alpha^{i'}_{\ell}: \;-\;(i\leftrightarrow j) \\
& \longrightarrow -\mathrm{i}\hbar :E^{ij}L^{\perp}_0: \;-\;(i\leftrightarrow j) \quad \text{for} \quad \kappa \;\to\; 1\,.
\end{aligned}
\tag{8.7}
$$

The remaining terms in Eq. (8.5) vanish

$$
\begin{aligned}
& \sum_{0\neq -k\neq n\neq 0}\frac{\hbar}{2n}\sum_{\ell\in\mathbb{Z}+\frac{n+k}{2}}\kappa^{|n+k|} :\alpha^i_{-n}\alpha^j_{-k}\alpha^{i'}_{\frac{n+k}{2}-\ell}\alpha^{i'}_{\frac{n+k}{2}+\ell}: \\
& + \sum_{0\neq -m\neq n\neq 0}\left(\frac{\hbar}{4m}-\frac{\hbar}{4n}\right)\sum_{\ell\in\mathbb{Z}+\frac{m+n}{2}}\kappa^{|\ell+\frac{n-m}{2}|} :\alpha^i_{-n}\alpha^j_{-m}\alpha^{i'}_{\frac{m+n}{2}-\ell}\alpha^{i'}_{\frac{m+n}{2}+\ell}: \\
& -\;(i\leftrightarrow j) \longrightarrow 0 \quad \text{for} \quad \kappa \;\to\; 1
\end{aligned}
\tag{8.8}
$$

which can be seen by renaming $n \leftrightarrow m$ in the term containing $\frac{\hbar}{4m}$ in Eq. (8.8). Finally, if one adds up contributions from Eqs. (8.3), (8.4), (8.6) and (8.7), one derives precisely the commutator (8.1).

□

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