A COMMENT ON THE GENERALIZATION OF THE MARINATTO–WEBER QUANTUM GAME SCHEME

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Iqbal and Toor in *Phys. Rev.* **A65**, 022306 (2002) and *Commun. Theor. Phys.* **42**, 335 (2004) generalized the Marinatto–Weber quantum scheme for 2×2 games in order to study bimatrix games of 3×3 dimension, in particular the Rock–Paper–Scissors game. In our paper, we show that Iqbal and Toor's generalization exhibits certain undesirable property that can considerably influence the game result. To support our argumentation, in the further part of the paper we construct the protocol corresponding to the MW concept for any finite bimatrix game that is free from the fault.

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1. Introduction

The Marinatto–Weber (MW) scheme [1] has become one of the most frequently used quantum game schemes. Though it was created for research on Nash equilibria in quantum 2×2 games, it has also found application in studying some of the refinements of a Nash equilibrium like evolutionarily stable strategies [2]. Moreover, it has been proved that the MW scheme is applicable to finite extensive games [3] and even various problems of duopoly [4–6]. These recent papers show uninterrupted interest in research on quantum games played according to the MW idea and they provide sufficient motivation to improve the existing results.

Our comment mainly concerns frequently cited paper [7], where the concept of quantum 3×3 game was introduced. However, it also relates to the later paper [8], where the same concept was used.

2. Comment on the Iqbal and Toor's quantum 3×3 game

The possibility of recovering a classical game from its quantum counterpart is a necessary condition for any quantum game scheme to be treated as a correct one. In the case of the MW scheme, the classical game is obtained by putting the initial state $|\psi_{\rm in}\rangle = |00\rangle$. In fact, the Marinatto–Weber construction allows us to recover the classical game by putting any basis state $|ij\rangle$, $i, j = 0, 1$ as the initial state. If, for example, the initial state equals $|01\rangle$ we obtain the classical bimatrix with the only difference that the columns of the bimatrix are permuted. Another case, in which the initial state $(|00\rangle + |11\rangle)/\sqrt{2}$ is considered, could be interpreted that players play with equal probability the classical game and the game, where both rows and columns were permuted. Of course, such interpretation corresponds to outcomes given by the MW protocol, where the payoff pair is the same irrespective of choosing $I \otimes I$ or $C \otimes C$ (see, for instance, the Battle of the Sexes game studied in [1]).

It turns out that this natural feature of the MW scheme is not transferred to Iqbal and Toor's generalization. For technical convenience, let us number the computational states from 0 to 2 instead of Iqbal and Toor's numbering from 1 to 3. Definitions of unitary strategies I, D and C introduced in [7], but with respect to numbering from 0 to 2 are as follows

$$\begin{array}{lll} I|0\rangle = |0\rangle \qquad & C|0\rangle = |2\rangle \qquad & D|0\rangle = |1\rangle \\ I|1\rangle = |1\rangle & C|1\rangle = |1\rangle & D|1\rangle = |0\rangle \\ I|2\rangle = |2\rangle & C|2\rangle = |0\rangle & D|2\rangle = |2\rangle \end{array} \tag{1}$$

and they mean, respectively, the first, the second and the third strategy for each player. Although their protocol yields the classical 3×3 game if $|\psi_{\rm in}\rangle = |00\rangle$, it gives nonequivalent games for other basis states. Putting, for example, $|\psi_{\rm in}\rangle = |01\rangle$, the bimatrix of a 3×3 game changes as follows:

$$\begin{pmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{pmatrix} \xrightarrow{|\psi_{\rm in}\rangle = |01\rangle} \begin{pmatrix} P_{01} & P_{00} & P_{01} \\ P_{11} & P_{10} & P_{11} \\ P_{21} & P_{20} & P_{21} \end{pmatrix}, \tag{2}$$

where $P_{ij} := (a_{ij}, b_{ij})$. The output game is not equivalent (up to the order of the columns) to the input one. Instead, we obtain the game in which the outcomes P_{02}, P_{12} and P_{22} are no longer available. Obviously, such a change affects a game significantly. However, in our opinion, transformation (2) should not be caused by another basis state. According to theory of quantum correlations, superior results can be created only by entangled states [9]. Therefore, non-classical transformation (2) obtained simply by a separable basis state suggests some irregularity in the Iqbal and Toor's approach. Obviously, the undesirable property (2) translates into any superpositions of states.

It is not difficult to note that the problem lies in the definition of operators D and C. Although the players are provided with the identity operator I, the operators C and D act like the identity operator for states $|1\rangle$ and $|2\rangle$, respectively. In consequence, player 2 is not able to change his/her state from $|1\rangle$ to $|2\rangle$ which makes some of the outcomes unavailable.

3. The extension of the Marinatto–Weber scheme to any finite bimatrix game

Consider an $n \times m$ bimatrix game

$$\begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0,m-1} \\ P_{10} & P_{11} & \cdots & P_{1,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n-1,0} & P_{n-1,1} & \cdots & P_{n-1,m-1} \end{pmatrix}, \tag{3}$$

where $P_{ij} \in \mathbb{R} \times \mathbb{R}$ and define l unitary operators V_k for $k = 0, 1, \ldots, l-1$, that act on states of the computational basis $\{|0\rangle, |1\rangle, \ldots |l-1\rangle\}$ as follows:

$$\begin{aligned} V_0|i\rangle &= |i\rangle\,, \\ V_1|i\rangle &= |i+1 \bmod l\rangle\,, \\ &\vdots \\ V_{l-1}|i\rangle &= |i+(l-1) \bmod l\rangle\,. \end{aligned} \tag{4}$$

According to this extension of the MW scheme, for the game (3) one obtains the final state

$$\rho_{\text{fin}} = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} p_i q_j V_i \otimes V_j \rho_{\text{in}} V_i^{\dagger} \otimes V_j^{\dagger}\,, \tag{5}$$

where $\{p_i\}$ and $\{q_j\}$ are probability distributions over $\{V_i\}$ and $\{V_j\}$ and ρ_{in} is the density operator for a state $|\psi_{\text{in}}\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$, and the payoff operator

$$X = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} P_{ij}|ij\rangle\langle ij|\,. \tag{6}$$

Then the average payoff pair is given by the formula

$$E = \operatorname{tr}(X\rho_{\text{fin}})\,. \tag{7}$$

Let us determine the output bimatrix corresponding to the initial state $|ij\rangle$ in order to prove the correctness of the scheme (3)–(7). Equations (5)–(7) imply the following bimatrix:

$$\begin{pmatrix} P_{ij} & \cdots & P_{i,j+m-1 \bmod m} \\ P_{i+1 \bmod n,j} & \cdots & P_{i+1 \bmod n,j+m-1 \bmod m} \\ \vdots & \ddots & \vdots \\ P_{i+n-1 \bmod n,j} & \cdots & P_{i+1 \bmod n,j+m-1 \bmod m} \end{pmatrix}. \tag{8}$$

As a result, we obtain the bimatrix which is equivalent to (3) from a game-theoretic point of view. The only difference between them is that now rows and columns are numbered from i and j instead of zero. In particular, in the problem (2) the bimatrix on the right hand side takes now the following form:

$$\begin{pmatrix} P_{01} & P_{02} & P_{00} \\ P_{11} & P_{12} & P_{10} \\ P_{21} & P_{22} & P_{20} \end{pmatrix} \tag{9}$$

and it differs from the original one in the numbering order.

4. Conclusion

The idea of playing 3×3 games and the examination of evolutionarily stable strategies in the language of quantum games propounded by Iqbal and Toor in [7] and [8] show a great area of research into quantum games. It suggests that the number of open problems in theory of quantum games may be equal to the number of classical game theory concepts that have not been studied yet in the quantum mechanics environment.

We have refined Iqbal and Toor's generalization to inherit the MW-scheme features. Surprisingly, the output game (8) coincides with Iqbal and Toor's one [7] if the input game is their modified Rock–Scissors–Paper game and the final state is $1/2(|01\rangle + |10\rangle + |02\rangle + |20\rangle)$. It follows from the fact that the Rock–Scissors–Paper game is defined only by three different numbers. In general, bimatrix (8) provides us with a quite different output game than the one defined by Iqbal and Toor. Therefore, we claim that our scheme ought to be used in the case of studying more complex games.

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