NONLINEAR SPREAD OF RUMOR AND INOCULATION STRATEGIES IN THE NODES WITH DEGREE DEPENDENT TIE STRENGTH IN COMPLEX NETWORKS

Anurag Singh, Yatindra Nath Singh

Department of Electrical Engineering, IIT Kanpur, 208016, India

(Received September 6, 2012; revised version received December 27, 2012)

In earlier rumor spreading models of the real world complex networks, nodes contact all of their neighbors at each time step. In more realistic scenario, a node may contact only some of its neighbors to spread the rumor. The rumor spreading rate may also depend on the degree of the spreader and ignorant nodes. We have given a new modified rumor spreading model to accommodate these facts. This new model has been studied for rumor spreading in scale free networks model of real world complex networks. Nonlinear rumor spread exponent α and degree dependent tie strength exponent β of nodes affect the rumor threshold. By using the given two exponents, rumor threshold has some finite value. This was not observed in the earlier models for scale free networks. The rumor threshold becomes independent of network size when α and β parameters are tuned to appropriate value. In any social network, rumors can spread and may have undesirable effect. One of the possible solutions to control rumor spread is to inoculate a certain fraction of nodes against rumors. We have used modified rumor spreading model over scale free networks to investigate the efficacy of random and targeted inoculation schemes. It has been observed that rumor threshold in targeted inoculation scheme is higher than in the random inoculation. Therefore, it is hard to spread rumors using modified rumor spreading model in scale free networks using targeted inoculation scheme. The proposed hypothesis is also verified by the simulation results.

DOI:10.5506/APhysPolB.44.5

PACS numbers: 05.45.–a, 89.75.–k, 05.10.–a, 64.60.aq

1. Introduction

Many researchers have discussed how the properties of networks affect the dynamical processes taking place in the networks. In the recent years, complex network structures and their dynamics have been studied extensively [1–8]. By analyzing different real world networks, *e.g.* Internet, the `www`, social network and so on, researchers have identified different topological characteristics of complex networks such as the small world phenomenon and scale free property. An interesting dynamical process in complex networks is the epidemic spreading. Work on epidemic spreading has been done by many researchers [5, 9, 10]. There are two models, susceptible-infected-susceptible (SIS) [11, 12] and susceptible-infected-recovered [5, 9] for the epidemic spreading.

In order to improve the resistance of the community against undesirable rumors, it is essential to develop deep understanding of the mechanism and underlying laws involved in rumor spreading and establish an appropriate prevention and control to generate social stability. First time Sudbury studied the spread of rumors based on SIR model [13]. Another standard model of rumor spreading was introduced many years ago by Daley and Kendal [14]. Its variant was introduced by Maki–Thomsan [15]. In Daley–Kendal (DK) model, homogeneous population is subdivided into three groups: ignorant (who do not know about the rumor), spreaders (who know about the rumor) and stifler (who know rumor but do not want to spread it). The rumor is propagated throughout the population by pairwise contacts between spreaders and other individuals in the population. Any spreader involved in a pairwise meeting attempts to infect other individual with the rumor. In the case this other individual is an ignorant, it becomes a spreader. If other individual is a spreader or stifler, it finds that rumor is known and decides not to spread rumor anymore, thereby turning into stifler. In Maki–Thomsan (MK) model, when spreader contacts another spreader, only the initiating spreader becomes a stifler. DK and MK models have an important shortcoming that they do not take into account the topology of the underlying social interconnection networks along which rumors spread. These models are restricted in explaining real world scenario for rumor spreading. By considering the topology of network, rumor model on small world network [4, 16, 17] and scale free networks [18] has been defined. Therefore, as long as one knows the structure of spreading networks, he can figure out variables and observable to conduct quantitative analysis, forecast and control the rumor spreading. The most important conclusion of classical propagation theory is the existence of critical point of rumor transmission intensity. When an actual intensity is greater than critical value, the rumors can spread in networks and persistently exist. When the actual intensity is less than the critical value, rumors decay at an exponential rate and this critical value is called rumor threshold. In Internet, weight implies the knowledge of its traffic flow or the bandwidth of routers [19], in the world wide airport networks it can define the importance of an airport [20] and so on. In the case of rumor spreading, the strength can indicate the frequency of the contact between two nodes in the scale free networks. The greater the strength, the more intensely the two nodes are communicating. Chances of spreading rumors tend to differ among individuals and laws of spreading in social networks with different topologies are also different. Each informed node can be assumed to make contacts with all of its neighbors in a single time step. In other words, we can say that each informed node can spread information to nodes equal to its degree. In real case, an informed node cannot make contact to all of its neighbors in single time step. Studies on small world networks found that compared to regular network, small world network has smaller transmission threshold and faster dissemination. Even at small spreading rates, rumors can exist for long. Studies on infinite-size scale free networks have also revealed that no matter how small transmission intensity be, the rumors can be persistent as positive critical threshold does not exist [7, 21].

In previous studies on rumor spreading in scale free networks, it has been assumed that the larger the nodal degree, the greater the rumor spreading from the informed node, *i.e.* the rumor spread is proportional to the nodal degree. With these assumptions for SIR model, in scale free networks with sufficiently large size, the rumor threshold λ_{c} can be zero. Yan *et al.* [22] have demonstrated that the asymmetry of infection plays an important role. They redistributed the asymmetry to balance the degree heterogeneity of the network and found the finite value of epidemic threshold. Zhou *et al.* [23, 24] concluded that this hypothesis is not always correct. In rumor spreading, the hub nodes have many acquaintances; however, they cannot contact all their acquaintances in single time step. They assumed that the rumor spreadness is not equal to the degree but identical for all nodes of the scale free networks and obtained the threshold $\lambda_{\mathrm{c}} = \frac{1}{A}$, where A is the constant infectivity of each node and is not equal to the degree of node. Recently, Fu *et al.* [25] have defined piecewise linear infectivity. They suggested if the degree k, of a node is small, its infectivity is $\alpha' k$, otherwise its infectivity is a saturated value A when k is beyond a constant A/α'. In both constant and piecewise linear infectivity, the heterogeneous infectivity of the nodes due to different degrees has not been considered. While in scale free networks heterogeneity in nodal degree is very common. There may be nodes with different degrees which have the same infectivity, and there will be a large number of such nodes if infectivity does not saturate or the size of network is infinite.

In order to control the spread of rumors, inoculating the nodes is an option. Although random immunization strategy works very well in homogeneous random networks, this strategy is not effective in preventing a rumor in scale free networks [31]. Hence, the new immunization strategies need to be developed which are able to recover from the rumor spreading. One of the efficient approach is to immune the high degrees nodes, or, more specifically, to immune those nodes (hereafter termed as hubs or hub nodes) which have degrees higher than a preset cut-off value k_{c}. Such a strategy is known as targeted immunization [12, 21, 26–30]. Targeted inoculation is successful in arresting the rumor spread in scale free networks [7, 31].

Random inoculation usually requires inoculation of large number of nodes for being effective. If nodes with higher connectivity are targeted for inoculation, the same effectiveness can be achieved with smaller number of inoculated nodes. But it needs the knowledge of nodes which have higher connectivity [32]. Chen *et al.* [33] suggested that identifying influential nodes using betweenness centrality and closeness centrality can lead to faster and wider spreading in complex networks. This approach cannot be, however, applied in large-scale networks due to the computational complexity involved in identifying such nodes. While higher degree nodes can be identified with much less efforts. The best spreaders using various measures of centrality can be identified in a network to ensure the more efficient spread of information. The inoculation of these efficient spreaders can also stop the rumor spreading efficiently [34, 35].

In this work, we have investigated rumor spread in the scale free network considering the varying tie strengths between nodes. Further, we have assumed that a non-linearly varying number of neighbors are infected with the rumor in each time step. While in the earlier models [4, 9] tie strength has been considered to be uniform and a constant number of neighbors have been assumed to be infected in each time step by each node. In the earlier models, if a node has K neighbors in each time step, all the K neighbors will be infected. We have modified the earlier SIR model given by Nevokee [4] and included a rumor spreading exponent α. In this work, K^{α} neighboring nodes will be infected in each time step. Here, α is the spreading exponent, where $0 < \alpha \leq 1$. The tie strength between two nodes is $(k_i k_j)^{\beta}$, where k_i and k_j are degrees of node i and j, and β is tie strength exponent. We have used Barabasi–Albert (BA) model [2] to create scale free networks with power law distribution of nodal degree, and then used the proposed strategy in them to study the rumor spread. Scale free networks have been specifically chosen as they are much more heterogeneous than the small world or the random network models, and thus a good candidate for testing our proposition.

The dynamical differential equations have been used to represent the modified model for information spread. The equations have been used to find the threshold and study the rumor propagation behavior. The results have also been verified by the simulations. By choosing the appropriate values of α and β, finite non-zero threshold value can be found for the scale free networks. It is found that the rumor spreading threshold is more sensitive to α than β in a large scale free network. Finally, the rumor threshold has been calculated after applying the random and targeted inoculation of the nodes to suppress the rumor in the scale free networks. In the targeted inoculation scheme, the rumor threshold has been found to be larger. The targeted inoculation has also been effective in the scale free networks to suppress the rumor spread.

2. Classical SIR model

Classical SIR model is one of the most investigated rumor spreading models for complex networks. In this model, nodes are in one of the three categories: ignorants (the nodes who are ignorant of the rumor), spreaders (those who hear the rumor and also actively spread it) and stifler (the nodes who hear the rumor but do not spread it further). The rumor is propagated through the nodes by pairwise contacts between the spreaders and other nodes in the network. Following the law of mass action, the spreading process evolves with direct contact of the spreaders with others in the population. These contacts can only take place along the edges of undirected graph of complex network. If the other node is the spreader or stifler, then the initiating spreader becomes the stifler. The classical SIR model has been studied by Nekovee *et al.* [3, 4] for heterogeneous population (nodes having different degrees). In this paper, $I(k,t)$, $S(k,t)$, $R(k,t)$ are expected values of ignorants, spreaders and stifler nodes in network with degree k at time t. Above rumor spreading process can be summarized by following set of pairwise interactions:

$S_1 + I_2 \xrightarrow{\lambda} S_1 + S_2$
(when spreader meets with the ignorant, it makes them spreader at rate λ),

$S_1 + R_2 \xrightarrow{\sigma} R_1 + R_2$
(when a spreader contacts a stifler, the spreader becomes a stifler at the rate σ),

$S_1 + S_2 \xrightarrow{\sigma} R_1 + S_2$
when a spreader contacts with another spreader, initiating spreader becomes a stifler at the rate σ),

$S \xrightarrow{\delta} R$
(δ is the rate at which spreaders change state to stifler spontaneously and stop spreading of a rumor).

Let $\rho^{\mathrm{i}}(k,t) = I(k,t)/N(k)$, $\rho^{\mathrm{s}}(k,t) = S(k,t)/N(k)$, $\rho^{\mathrm{r}}(k,t) = R(k,t)/N(k)$ are the fraction, of ignorant, spreaders and stifler nodes, respectively, with degree k at time t. These fractions of the nodes satisfy the normalization condition, $\rho^{\mathrm{i}}(k,t) + \rho^{\mathrm{s}}(k,t) + \rho^{\mathrm{r}}(k,t) = 1$, where $N(k)$ represents the total number of nodes with degree k in the network. Nekovee *et al.* [4] proposed the formulation of this model for analyzing complex networks as interacting Markov chains. They used the framework to derive from the first-principles, the mean-field equations for the dynamics of rumor spreading in the complex networks with arbitrary correlations. These are given below:

$$\frac{d\rho^{\mathrm{i}}(k,t)}{dt} = -k\lambda\rho^{\mathrm{i}}(k,t)\sum_{l} P(l|k)\rho^{\mathrm{s}}(l,t)\,, \tag{1}$$

$$\begin{aligned}\frac{d\rho^{\mathrm{s}}(k,t)}{dt} &= k\lambda\rho^{\mathrm{i}}(k,t)\sum_{l} P(l|k)\rho^{\mathrm{s}}(l,t) - k\sigma\rho^{\mathrm{s}}(k,t)\sum_{l}\left(\rho^{\mathrm{s}}(l,t)\right.\\ &\quad \left.+\rho^{\mathrm{r}}(l,t)\right)P(l|k) - \delta\rho^{\mathrm{s}}(k,t)\,,\end{aligned} \tag{2}$$

$$\frac{d\rho^{\mathrm{r}}(k,t)}{dt} = k\sigma\rho^{\mathrm{s}}(k,t)\sum_{l}(\rho^{\mathrm{s}}(l,t)+\rho^{\mathrm{r}}(l,t))P(l|k) + \delta\rho^{\mathrm{s}}(k,t)\,, \tag{3}$$

where conditional probability $P(l|k)$ is the degree–degree correlation function that a randomly chosen edge emanating from a node of degree k leads to a node of degree l. Here, it has been assumed that the degree of nodes in the whole network are uncorrelated. Therefore, degree–degree correlation is $P(l|k) = \frac{lP(l)}{\langle k\rangle}$, where $P(l)$ is the degree distribution and $\langle k\rangle$ is the average degree of the network. Nekovee *et al.* [4] have shown that the critical threshold for rumor spreading is independent of the stifling mechanism. The critical threshold found by him was $\lambda_{\mathrm{c}} = \frac{\langle k\rangle}{\langle k^2\rangle}$. It is the same as found for SIR model [9, 36]. Hence, it implies that epidemic threshold is absent in large size scale free networks ($\langle k^2\rangle \to \infty, \lambda_{\mathrm{c}} \to 0$). This result is not good for epidemic control, since the epidemics will exist in the real networks for any non-zero value of spreading rate λ.

3. Modified rumor spreading model

The real world networks can have the intimacy, confidence, *etc.* between the nodes. Unlike previous studies, where each node can spread the rumor with constant transmission rate λ, in this study, we have considered a rumor spreading model with nonlinear rumor spread. The transmission rate between two connected nodes has been considered a function of their degrees. Based on this assumption, we can write the rate equations as follows:

$$\frac{d\rho^{\mathrm{i}}(k,t)}{dt} = -k\rho^{\mathrm{i}}(k,t)\sum_{l} P(l|k)\rho^{\mathrm{s}}(l,t)\frac{\Phi(l)}{l}\lambda_{lk}\,, \tag{4}$$

$$\begin{aligned}\frac{d\rho^{\mathrm{s}}(k,t)}{dt} &= k\rho^{\mathrm{i}}(k,t)\sum_{l} P(l|k)\rho^{\mathrm{s}}(l,t)\frac{\Phi(l)}{l}\lambda_{lk} - k\rho^{\mathrm{s}}(k,t)\sum_{l}\left(\rho^{\mathrm{s}}(l,t)\right.\\ &\quad \left.+\rho^{\mathrm{r}}(l,t)\right)P(l|k)\frac{\Phi(l)}{l}\sigma_{lk} - \delta\rho^{\mathrm{s}}(k,t)\,,\end{aligned} \tag{5}$$

$$\frac{d\rho^{\mathrm{r}}(k,t)}{dt} = k\rho^{\mathrm{s}}(k,t)\sum_{l}\left(\rho^{\mathrm{s}}(l,t)+\rho^{\mathrm{r}}(l,t)\right)P(l|k)\frac{\Phi(l)}{l}\sigma_{lk} + \delta\rho^{\mathrm{s}}(k,t)\,, \tag{6}$$

where $\Phi(l)$ represents the rumor spreadness of a node with degree l, λ_{lk} and σ_{lk} represents the rumor spreading rate and stifling rate from nodes of degree l to nodes with degree k, respectively, and $P(l|k)$ replaced by $\frac{\Phi(l)P(l|k)}{l}$.

3.1. Tie strength in complex networks

The topological properties of a graph are fully encoded in its adjacency matrix $\boldsymbol{A}$ whose elements a_{ij} $(i \neq j)$ are 1 if a link connects node i to node j, and 0 otherwise. The indices i, j run from 1 to N, where N is the size of the network. Similarly, a weighted network is entirely described by a matrix $\boldsymbol{W}$ whose entry w_{ij} gives the weight on the edge connecting the vertices i and j ($w_{ij} = 0$, if the nodes i and j are not connected). In this study, we will consider only the case of symmetric weights ($w_{ij} = w_{ji}$) while the undirected case of the network is considered [20]. In a call network, if two nodes call each other for a long duration, then weight of the connecting edge will be high and it shows high tie strength between them [37]. Here, weight of the edge in terms of total call duration defines the tie strength between the nodes. It has also been observed in the dependence of the edge weight w_{ij} to define strength between nodes with end point degrees k_i and k_j. Weight as a function of the end-point degrees can be well approximated by a power-law dependence

$$w_{ij} = b(k_i k_j)^{\beta}\,,$$

where β is the degree influenced real exponent which depends on the type of complex networks and b is a positive quantity. When $\beta > 0$, then rumor transmit to high degree nodes and when $\beta < 0$, then rumor will transmit to low degree nodes. Further, if $\beta = 0$ there will be the degree independent transmission.

It has been observed that the individual edge weight does not provide clear view of network's complexity. A detailed measurement of tie strength using the actual weights is obtained by enhancing the property of a vertex degree $k_i = \sum_j a_{ij}$ in terms of the vertex strength $S_i = \sum_{j=1}^{N} a_{ij} w_{ij}$ (total weights of their neighbors). Therefore, there is a coupling between interaction strengths of the nodes with the counterintuitive consequence that social networks are robust enough to the removal of the strong ties but fall apart after a phase transition if the weak ties are removed [20]. Therefore, we can measure the strength of a node of degree k for scale free network

$$S_k \;=\; k\sum_l P(l|k)w_{kl} = k\sum_l \frac{lP(l)}{\langle k\rangle} w_{kl} = b\frac{k^{1+\beta}}{\langle k\rangle}\left\langle k^{1+\beta}\right\rangle\,. \qquad (7)$$

Here, it has been considered that the rumor spreading model, where rumor transmission rate in contact process between a spreader node and an ignorant node is influenced by their degrees. If w_{kl} is the tie strength between k-degree node and l-degree node for $(k,\ l)$ edge, S_k is the node strength with degree k. In scale free network, each node of degree k is a constant rumor transmission rate λk. Therefore, rumor transmission rate from k-degree node to l-degree node is given by the proportion of w_{kl} to S_k. Hence, λ_{kl} can be defined as

$$\lambda_{kl} = \lambda k \frac{w_{kl}}{S_k} \,. \tag{8}$$

We can see in Eq. (8) that by increasing the proportion of w_{kl}/S_k, the more possibility of rumor transmission rate can be increased through the edge. In the present work, uncorrelated networks have been considered, hence $\lambda_{kl} = \lambda l^{\beta}\langle k\rangle/\langle k^{1+\beta}\rangle$. In this model, rumor spreadness $\Phi(k) = k^{\alpha}$, where $0 < \alpha \leq 1$, it defines that each spreader node may contact with k^{α} neighbors within one time step. Therefore, spreadness of a rumor will vary nonlinearly with the growing degree k. In Eqs. (4)–(6), we can write rumor equation for $\Phi(k)$ and λ_{lk} can be written as

$$\frac{d\rho^{\mathrm{i}}(k,t)}{dt} = \frac{\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_{l} l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t)\,, \tag{9}$$

$$\begin{aligned}\frac{d\rho^{\mathrm{s}}(k,t)}{dt} &= \frac{\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_{l} l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_{l}(\rho^{\mathrm{s}}(k,t) \\ &\quad +\rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) - \delta\rho^{\mathrm{s}}(k,t)\,,\end{aligned} \tag{10}$$

$$\frac{d\rho^{\mathrm{r}}(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_{l}(\rho^{\mathrm{s}}(l,t) + \rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) + \delta\rho^{\mathrm{s}}(k,t)\,. \tag{11}$$

After solving rumor Eqs. (9)–(11) with initial conditions $\rho^{\mathrm{i}}(k,0) \simeq 1$, $\rho^{\mathrm{s}}(k,0) \simeq 0$, $\rho^{\mathrm{r}}(k,0) \simeq 0$, we get

$$\rho^{\mathrm{i}}(k,t) = e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta(t)}\,, \tag{12}$$

where an auxiliary function is

$$\Theta(t) = \sum_{k} \frac{kP(k)}{\langle k\rangle} \int_{0}^{t} \rho^{\mathrm{s}}(k,t')dt'. \tag{13}$$

4. Rumor threshold of the modified model

In the infinite time limit, *i.e.*, at the end of rumor spreading, we will have $\rho^{\mathrm{s}}(k,\infty) = 0$, $\lim_{t\to\infty}\Theta(t) \to \Theta$ and $\lim_{t\to\infty} d\Theta/dt = 0$. Near the critical threshold, Θ will be small as $\rho^{\mathrm{s}}(k,\infty) = 0$. After solving Eqs. (9)–(11) and Eq. (13)

$$\Theta = \frac{\left(\lambda\frac{\langle k^{\alpha+\beta+1}\rangle}{\langle k^{1+\beta}\rangle} - \delta\right)}{\lambda^2\frac{\langle k^{\alpha+2\beta+2}\rangle}{\langle k^{1+\beta}\rangle^2}\left(1/2 + \sigma\delta\frac{\langle k^{\alpha+\beta+1}\rangle}{\langle k^{1+\beta}\rangle}I\right)}\,, \tag{14}$$

where $I = \int_0^t e^{\lambda}(t - t\prime)f(t')dt\prime$ is the finite and positive integral and $\Theta(t) = \Theta f(t)$, where $f(t)$ is a finite function. Eq. (14) will give positive value for Θ, when

$$\lambda\frac{\left\langle k^{\alpha+\beta+1}\right\rangle}{\left\langle k^{1+\beta}\right\rangle} - \delta \geq 0\,, \qquad \frac{\lambda}{\delta} \geq \frac{\left\langle k^{1+\beta}\right\rangle}{\left\langle k^{\alpha+\beta+1}\right\rangle}\,. \tag{15}$$

Therefore, to leading order in σ, the critical threshold is independent of the stifling mechanism, for $\delta = 1$ the critical rumor spreading threshold is given by

$$\lambda_{\mathrm{c}} = \frac{\left\langle k^{1+\beta}\right\rangle}{\left\langle k^{\alpha+\beta+1}\right\rangle}\,. \tag{16}$$

It is interesting to note that, by putting $\alpha = 1$ and $\beta = 0$ in Eq. (16), the threshold for this model reduces to $\langle k\rangle/\langle k^2\rangle$ for classical rumor spreading model [7].

When $t \to \infty$ spreader nodes will be 0, ($\rho^{\mathrm{s}}(k,\infty) = 0$) and from Eq. (12), $\rho^{\mathrm{i}}(k,\infty) = e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta}$. Therefore, the final size of rumor R at $t \to \infty$ ($\lim_{t\to\infty} \rho^{\mathrm{r}}(k,t) = R$)

$$\begin{aligned} R &= \sum_k P(k)\rho^{\mathrm{r}}(k,\infty) = \sum_k P(k)(1 - \rho^{\mathrm{s}}(k,\infty)) \\ &= \sum_k P(k)\left(1 - e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta}\right) = 1 - \sum_k P(k)e^{\frac{-\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta}\,. \end{aligned} \tag{17}$$

In the real world, complex networks rumor spreads on a finite size complex networks. It may be possible that size of scale free network is very large. The maximum or minimum degree of scale free network is mentioned by $k_{\max}$ or $k_{\min}$. Pastor *et al.* [21] found that the epidemic threshold λ_{c} for $k_{\max}$ for SIS model on bounded SF networks with $P(k) \sim k^{-2-\gamma'}$, $0 < \gamma' \leq 1$. They assumed that with the soft and hard cut-off $k_{\min}$ and $k_{\max}$, when $\alpha = 1$. The hard cut-off denotes that a network does not possess any node with degree $k > k_{\max}$. As $k_{\max}$ of a node is network age, defined in the terms of number of nodes N

$$k_{\max} = k_{\min}N^{\frac{1}{\gamma'+1}}\,. \tag{18}$$

The normalized degree distribution is defined by

$$P(k) = \frac{(1+\gamma')k_{\mathrm{min}}^{1+\gamma'}}{1-(k_{\mathrm{max}}/k_{\mathrm{min}})^{-1-\gamma'}} k^{-2-\gamma'} \theta(k_{\mathrm{max}}-k)\,. \tag{19}$$

Here $\theta(x)$ is a Heaviside step function [21].

In modified rumor spreading model, if $\alpha = 1$ and $\beta = 0$, then it converges to classic rumor spreading model. As the degree distribution in scale free networks $P(k) = k^{-\gamma}$, where $2 \leq \gamma \leq 3$, therefore,

$$\begin{aligned} \lambda_{\mathrm{c}}{}'(k_{\mathrm{max}}) &= \frac{\langle k \rangle}{\langle k^2 \rangle} && (20)\\ &= \frac{\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} k^{1-\gamma} dk}{\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} k^{2-\gamma} dk} && (21)\\ &\simeq \frac{3-\gamma}{(\gamma-2)k_{\mathrm{min}}} (k_{\mathrm{max}}/k_{\mathrm{min}})^{\gamma-3}\,. && (22) \end{aligned}$$

Eq. (18) gets modified for the given scale free network as

$$\begin{aligned} k_{\mathrm{max}} &= k_{\mathrm{min}} N^{\frac{1}{\gamma-1}}\,, && (23)\\ \lambda_{\mathrm{c}}{}'(N) &\simeq \frac{3-\gamma}{(\gamma-2)k_{\mathrm{min}}} (N)^{(\gamma-3)/(\gamma-1)}\,, && (24) \end{aligned}$$

for $\gamma = 3$

$$\gamma_{\mathrm{c}}{}'(N) \simeq 2[k_{\mathrm{min}} \ln(N)]^{-1}\,. \tag{25}$$

Eqs. (24)–(25) show that $\lambda_{\mathrm{c}}' \to 0$ if $N \to \infty$.

In modified rumor spreading model, nonlinear rumor spread is considered using $\Theta(k) = k^{\alpha}$. The rumor threshold is given by

$$\begin{aligned} \lambda_{\mathrm{c}}{}^{\#}(k_{\mathrm{max}}) &= \frac{\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} k^{\beta+1-\gamma} dk}{\int_{k_{\mathrm{min}}}^{k_{\mathrm{max}}} k^{\alpha+\beta+1-\gamma} dk} \\ &= k_{\mathrm{min}}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\beta-\gamma+2} \frac{\left[(k_{\mathrm{max}}/k_{\mathrm{min}})^{\beta-\gamma+2}-1\right]}{\left[(k_{\mathrm{max}}/k_{\mathrm{min}})^{\alpha+\beta-\gamma+2}-1\right]}\,. && (26) \end{aligned}$$

Theorem 4.1 *In classic rumor spread model ($\alpha = 1, \beta = 0$) threshold is smaller than the modified rumor spread model ($0 < \alpha < 1$ and $\beta \neq 0$).*

The proof is given in the next section after lemmas.

Lemma 4.2 *When the size of network (N) increases, the value of critical threshold $\lambda_c^\# > 0$ for $\alpha + \beta + 2 < \gamma$, otherwise it will approach to 0.*

Proof Since $k_{\max}/k_{\min} = N^{\frac{1}{\gamma-1}}$, therefore, $k_{\max}/k_{\min}$ increases when N increases, it becomes infinity when $N \to \infty$. When, $\alpha + \beta + 2 < \gamma$, $(\frac{k_{\max}}{k_{\min}})^{\beta-\gamma+2} = (\frac{k_{\max}}{k_{\min}})^{\alpha+\beta-\gamma+2} = 0$. The value of $\lambda_c^\#$ will be positive. Here, $\beta < 0$ (rumor transmission influenced to low degree nodes) is considered. Now from Eq. (26), we can conclude that $\lambda_c^\#$ will be positive. For $\alpha + \beta + 2 \geq \gamma$, $\lambda_c^\# \to 0$ when N increases. It can be summarized as

$$\lambda_c{}^\#(k_{\max}) = \begin{cases} k_{\min}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\gamma-\beta-2} (k_{\max}/k_{\min})^{\gamma-\alpha-\beta-2}, & \alpha+\beta+2 > \gamma\,, \\ k_{\min}^{(-\alpha)} \frac{\gamma-\alpha-\beta-2}{\gamma-\beta-2}\,, & \alpha+\beta+2 < \gamma\,, \\ k_{\min}^{(-\alpha)} \frac{1}{\alpha ln(k_{\max}/k_{\min})}\,, & \alpha+\beta+2 = \gamma\,. \end{cases} \tag{27}$$

Lemma 4.3 *In given rumor spreading model when $\alpha+\beta+2 < \gamma$, then rumor spreading threshold $\lambda^\#$ is independent from the size of scale free network (N).*

Proof It may also be defined using Eqs. (23)–(27) in the term of the number of nodes N

$$\lambda_c{}^\#(N) = \begin{cases} k_{\min}^{(-\alpha)} \frac{\alpha+\beta-\gamma+2}{\gamma-\beta-2} (N)^{(\gamma-\alpha-\beta-2)/(\gamma-1)}, & \alpha+\beta+2 > \gamma\,, \\ k_{\min}^{(-\alpha)} \frac{\gamma-\alpha-\beta-2}{\gamma-\beta-2}\,, & \alpha+\beta+2 < \gamma\,, \\ k_{\min}^{(-\alpha)} \frac{\gamma-1}{\alpha ln(N)}\,, & \alpha+\beta = \gamma\,. \end{cases} \tag{28}$$

Here, it is found that for $\alpha + \beta + 2 < \gamma$, $\lambda_c^\#$ is independent of N.

Proof Now using lemmas (4.2) and (4.3) the theorem can be proved for $\alpha + \beta + 2 > \gamma$. The ratio of rumor threshold in classic model and given model is given as

$$\frac{\lambda_c'(N)}{\lambda_c{}^\#(N)} = \frac{(2-\gamma)(\gamma-\beta-2)}{(\gamma-2)k_{\max}^{(1-\alpha)}(\alpha+\beta-\gamma+2)N^(1-\gamma-\beta+2)/(\gamma-3)}\,. \tag{29}$$

It has been found from Eq. (29) that $\frac{\lambda_c'(N)}{\lambda_c{}^\#(N)} < 1$ for finite scale free networks. Therefore, it has been justified that rumor threshold $\lambda_c^\#(N)$ is greater than the $\lambda_c'(N)$ in finite size scale free networks. In finite size scale free networks for which $0 < \alpha < 1$, $\beta \neq 0$ and $\alpha + \beta + 2 > \gamma$, it is harder to spread rumor in comparison to networks which have $\alpha = 1$ and $\beta = 0$. Finite rumor threshold is possible for any size of networks as seen in Eq. (28). However, it will be 0 when N approaches infinity.

5. Random inoculation

In random inoculation strategy, randomly selected node will be inoculated Fig. 1. This approach inoculates a fraction of the nodes randomly, without any information of the network. Here, variable g ($0 \leq g \leq 1$) defines the fraction of inoculative nodes. In the presence of random inoculation, the rumor spreading rate λ is reduced by a factor $(1-g)$. In mean field level, for the scale free networks in the case of random inoculation, the rumor equations are modified using initial conditions as

$$\frac{d\rho^{\mathrm{i}}(k,t)}{dt} = \frac{(1-g)\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_{l} l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t)\,, \tag{30}$$

$$\frac{d\rho^{\mathrm{s}}(k,t)}{dt} = \frac{(1-g)\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_{l} l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_{l}(\rho^{\mathrm{s}}(l,t) +\rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) - \delta\rho^{\mathrm{s}}(k,t)\,, \tag{31}$$

$$\frac{d\rho^{\mathrm{r}}(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_{l}(\rho^{\mathrm{s}}(l,t)+\rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) + \delta\rho^{\mathrm{s}}(k,t)\,. \tag{32}$$

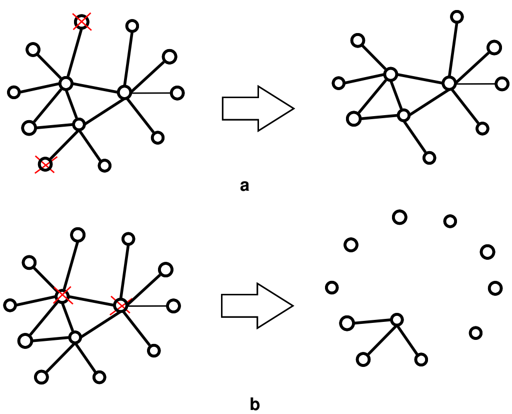


Fig. 1. Modified network after inoculation: (a) random inoculation (red crossed nodes inoculated), (b) targeted inoculation (red crossed nodes inoculated).

Therefore, final size of informed nodes (R) is

$$R = 1 - \sum_{k} P(k)(1-g)e^{\frac{-\lambda(1-g)k^{1+\beta}}{\langle k^{1+\beta}\rangle}\Theta} - g\,. \tag{33}$$

The rumor spreading threshold in the case of random inoculation is obtained from Eq. (14) as

$$\hat{\lambda_{\mathrm{c}}} = \frac{\langle k^{\beta+1}\rangle}{(\langle k^{\alpha+\beta+1}\rangle)\,(1-g)}\,. \tag{34}$$

The relation between rumor spreading threshold with inoculation ($\hat{\lambda_{\mathrm{c}}}$) and without inoculation (λ_{c}) can be defined as

$$\hat{\lambda_{\mathrm{c}}} = \frac{\lambda_{\mathrm{c}}}{1-g} \,. \tag{35}$$

It is to note that by applying random inoculation, the rumor spreading threshold ($\hat{\lambda_{\mathrm{c}}}$) can be increased as seen in Eq. (35) (*i.e.*, $\hat{\lambda_{\mathrm{c}}} > \lambda_{\mathrm{c}}$).

6. Targeted inoculation

Scale free networks permit efficient strategies and depend upon the hierarchy of nodes. It has been shown that SF networks show robustness against random inoculation. It shows that the high fraction of inoculation of nodes can be resisted without loosing its global connectivity, see Fig. 1. But on the other hand, SF networks are strongly affected by targeted inoculation of nodes. The SF network suffers an interesting reduction of its robustness to carry information. In targeted inoculation, the high degree nodes have been inoculated progressively, *i.e.* more likely to spread the information. In SF networks, the robustness of the network decreases at the effect of a tiny fraction of inoculated individuals.

Let us assume that fraction g_k of nodes with degree k are successfully inoculated. An upper threshold of degree k_{t}, such that all nodes with degree $k > k_{\mathrm{t}}$ get inoculated. Fraction g_k of nodes with the degree k are successfully inoculated. The fraction of inoculated nodes is given by

$$g_k = \begin{cases} 1\,, & k > k_{\mathrm{t}}\,, \\ f\,, & k = k_{\mathrm{t}}\,, \\ 0\,, & k < k_{\mathrm{t}}\,, \end{cases} \tag{36}$$

where $0 < f \leq 1$, and $\sum_k g_k P(k) = \bar{g}$, where $\bar{g}$ is the average inoculation fraction. Therefore, now rumor spreading equation is defined for targeted inoculation as

$$\frac{d\rho^{\mathrm{i}}(k,t)}{dt} = \frac{(1-g_k)\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_l l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t)\,, \tag{37}$$

$$\begin{aligned} \frac{d\rho^{\mathrm{s}}(k,t)}{dt} &= \frac{(1-g_k)\lambda k^{1+\beta}}{\langle k^{1+\beta}\rangle}\rho^{\mathrm{i}}(k,t)\sum_l l^{\alpha}P(l)\rho^{\mathrm{s}}(l,t) - \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_l(\rho^{\mathrm{s}}(l,t) \\ &\quad +\rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) - \delta\rho^{\mathrm{s}}(k,t)\,, \end{aligned} \tag{38}$$

$$\frac{d\rho^{\mathrm{r}}(k,t)}{dt} = \frac{\sigma k^{1+\beta}}{\langle k^{1+\beta}\rangle}\sum_l(\rho^{\mathrm{s}}(l,t)+\rho^{\mathrm{r}}(l,t))l^{\alpha}P(l) + \delta\rho^{\mathrm{s}}(k,t)\,. \tag{39}$$

Next, rumor spreading threshold in the case of targeted inoculation is obtained from Eq. (14) as

$$\tilde{\lambda_c} = \frac{\langle k^{\beta+1}\rangle}{\langle k^{\alpha+\beta+1}\rangle - \langle g_k k^{\alpha+\beta+1}\rangle}\,. \tag{40}$$

Here $\langle g_k k^{\alpha+\beta+1}\rangle = \bar{g}\langle k^{\alpha+\beta+1}\rangle + \eta\prime$, where $\eta\prime = \langle (g_k - \bar{g})[\langle k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1}\rangle]\rangle$ is the covariance of g_k and $k^{\alpha+\beta+1}$. The cut-off degree k_t is large enough, where $\eta\prime < 0$, but for small k_t, $g_k - \bar{g}$ and $k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1}\rangle$ have the same signs except for ks, where $g_k - \bar{g}$ and/or $k^{\alpha+\beta+1} - \langle k^{\alpha+\beta+1}\rangle$ is 0.

Hence, $\eta\prime > 0$ for appropriate k_t

$$\tilde{\lambda_c} > \frac{1-g}{1-\bar{g}}\hat{\lambda_c}\,. \tag{41}$$

If average inoculation fraction of nodes in targeted inoculations is the same as fraction of nodes in the random inoculations, then $g = \bar{g}$

$$\tilde{\lambda_c} > \hat{\lambda_c}\,. \tag{42}$$

The above relation shows that in scale free networks targeted inoculation is more effective than the random inoculation.

7. Numerical simulations: results and discussion

The studies of uncorrelated networks have been performed using the degree distribution of scale free network. The size of the network is considered to be $N = 10^5$, the degree exponent $(\gamma) = 2.4$, $\delta = 1$ and $\sigma = 0.2$. At the starting of rumor spreading, the spreaders are randomly chosen. In Fig. 2, the final size of rumor R is plotted against rumor transmission rate for $N = 100000$, 1000 and 100 by tuning α and β as:

- $\alpha+\beta = 0$: Finite rumor threshold has been found. It has been observed that for different size of networks, constant threshold is there (after fixing the value of α and β). For the case, $\alpha+\beta+2 < \gamma$, since $\gamma = 2.4$. Therefore, here it is interesting to see that a finite threshold has been found which is independent from the size of network, the same as obtained from Eqs. (27) and (28).

- $\alpha+\beta = -1$: The simulation results are found the same as above since $\alpha+\beta+2 < \gamma$ with finite threshold and constant for any network size (for fixed values of α and β).

- $\alpha+\beta = 1$: In this case, the rumor threshold has non-zero value but it tends to 0 when the network size increases. For this case, $\alpha+\beta+2 > \gamma$ since $\gamma = 2.4$. Therefore, the threshold approaches to 0 as network size increases. Similar results have been obtained from Eqs. (27)–(28).

- $\alpha + \beta = 2$: The simulation results are found the same as above since $\alpha + \beta + 2 > \gamma$ with threshold approaches to 0 as size of network increases.

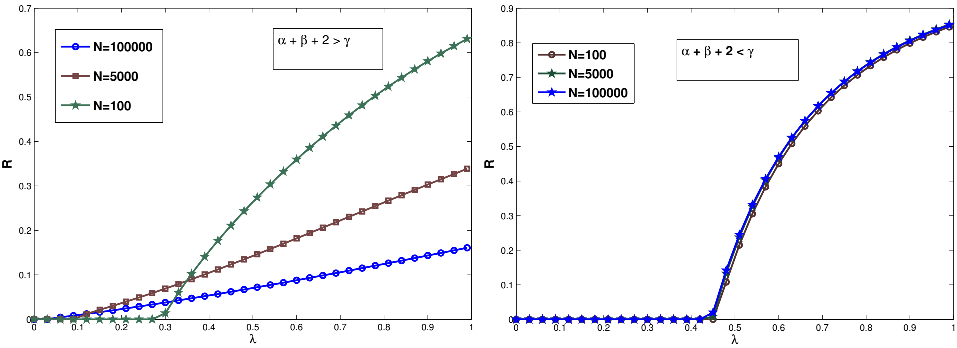


Fig. 2. *R versus* λ with $\alpha + \beta + 2 > \gamma$ (left) and $\alpha + \beta + 2 < \gamma$ (right) for different size of scale free networks.

Final size of the rumor (R) obtained in numerical simulation is plotted against time (t) in Fig. 3. It has been observed that rumor size increases exponentially as time increases and after some time it approaches a steady state, that will remain constant, since spreader density is 0 at that time. It has also been observed that when $\alpha + \beta$ is low, then rumor size initially increases slowly but when $\alpha + \beta$ increases, rumor size increases rapidly against time. While tuning the parameter α and β from $\alpha + \beta = -1$ to $\alpha + \beta = 2$ rumor increments faster initially (Fig. 3). When ratio of α and β is high, then the rumor size is also high. This result justifies that the α affects more final rumor size R than β. It is seen from Eq. (16) that when α is very small (0.1–0.3), then the rumor threshold will be high. Then, it is seen that the final size of rumor will be too small when rumor transmission rate (λ) is less than the rumor threshold (λ_c).

Critical rumor threshold is plotted against α in Fig. 4 while considering $\beta = 0$. Here, λ_c decreased exponentially with the increase of α. It is maximum for $\alpha = 0.1$ and almost 0 at $\alpha = 1$ for $N = 100000$. Interestingly, this also happens in real life situation, when an informed node pass information to its maximum number of neighbors then rumor spreading will get outbreak in the network. However, the outbreak is hard to achieve when it passes rumor to less number of neighbors ($\simeq$ 10–30%). Similarly, λ_c has been studied against β at $\alpha = 1$ in Fig. 4. It is found that β affects less rumor threshold for entire range except when $\beta > 0$. Further, it approaches to 0. When size of the network (N) increases, then rumor threshold is decreased as shown in Fig. 4 (left) and (right).

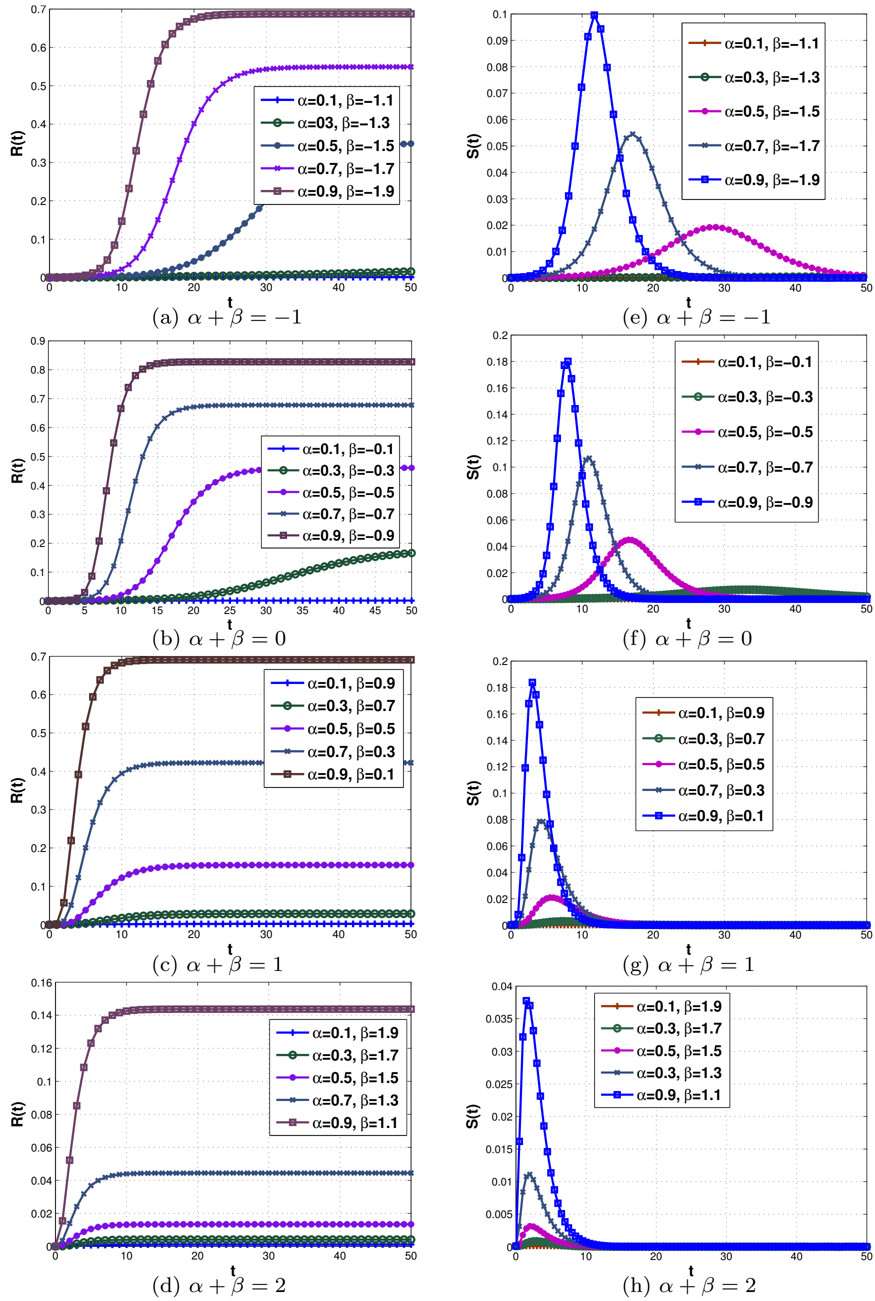


Fig. 3. $R(t)$ *versus* t (a)–(d) and $S(t)$ *versus* t (e)–(h) for $\lambda = 0.8$ and different combinations of α and β parameters.

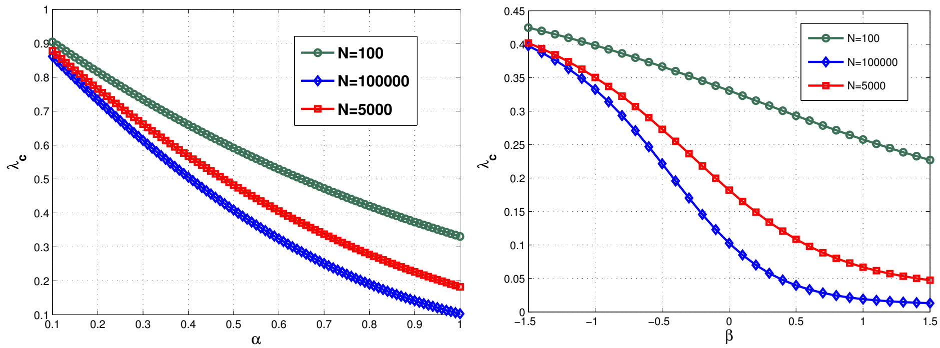
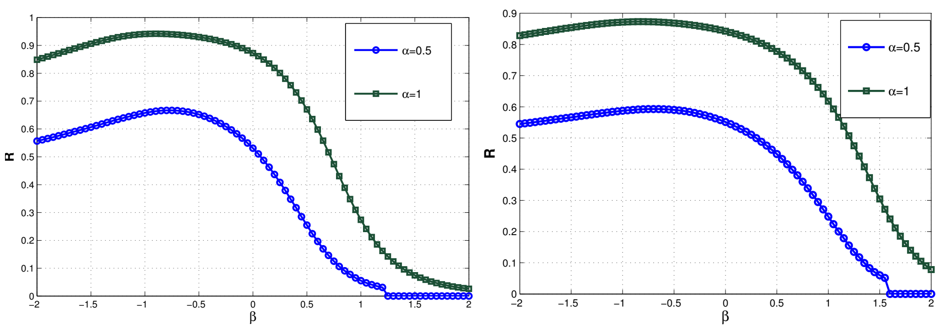


Fig. 4. Threshold (λ_c) *versus* α (left) and β (right) for $\gamma = 2.4$ for different size of scale free networks.

In Fig. 5, final size of rumor has been plotted against β for $\alpha = 1$, $\gamma = 2.4$, 3, and $\lambda = 1$. It is seen that R is maximum when $\beta = -1$. Initially rumor size R increases with β but after achieving a maximum value for $\beta = -1$ it decays exponentially. Further, for $\alpha = 1$, $\gamma = 2.4$ final rumor size R approaches to 0 (beyond $\beta = 1.5$). Furthermore, it is interesting to



$\gamma = 2.4$ $\gamma = 3$

Fig. 5. Final size of rumor R vs β ($N = 10^5$ nodes, $\alpha = 0.5$ and $\alpha = 1$) for different values of γ.

note that final rumor size R increased with increase of α, see Fig. 6. For random inoculation $g = 0.1$, 0.3, 0.5, 0.7, 0.9, the final rumor size R has been plotted against β (Fig. 7). It is observed that to get maximum value of R, β increases when g increases. Also, maximum size of rumor decreases with the increase of g. It is because in random inoculation rumor threshold value is larger than the threshold in model without inoculation as inferred from Eq. (35) ($\hat{\lambda}_c > \lambda_c$). The sharp decrease in the value of R is seen when rumor transmission rate (λ) is decrease by 0.5 in comparison to the case, where decrease of R is shallow when α is decreased by 0.5. Since for $\lambda \simeq 1$ there may be a chance that rumor will spread to some extent at any value of $g < 1$, for very large N. Similar results have been observed in

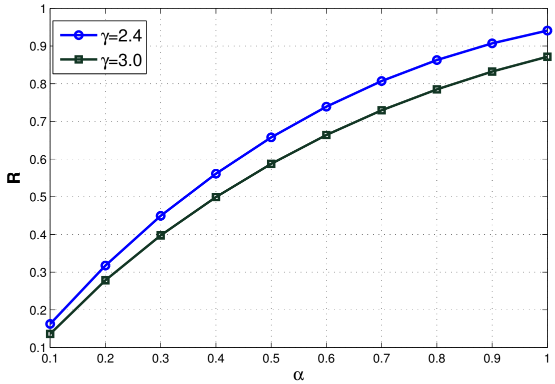
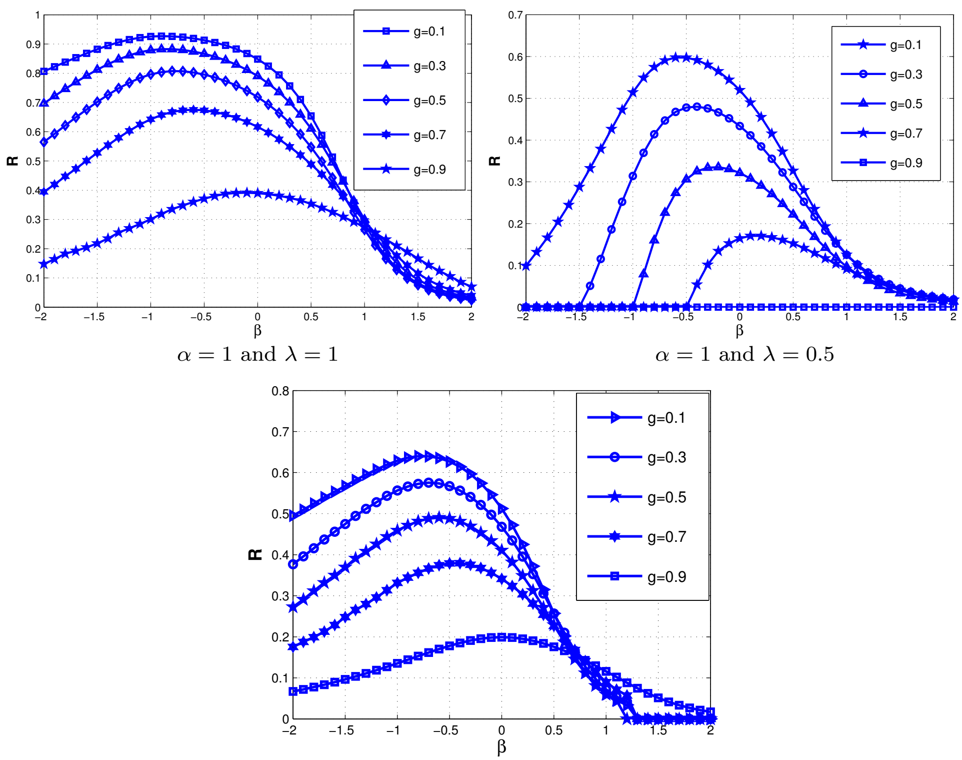


Fig. 6. Final size of rumor R *versus* α ($N = 10^5$ nodes, $\beta = -1$ and $\lambda = 1$) for $\gamma = 2.4$, 3.



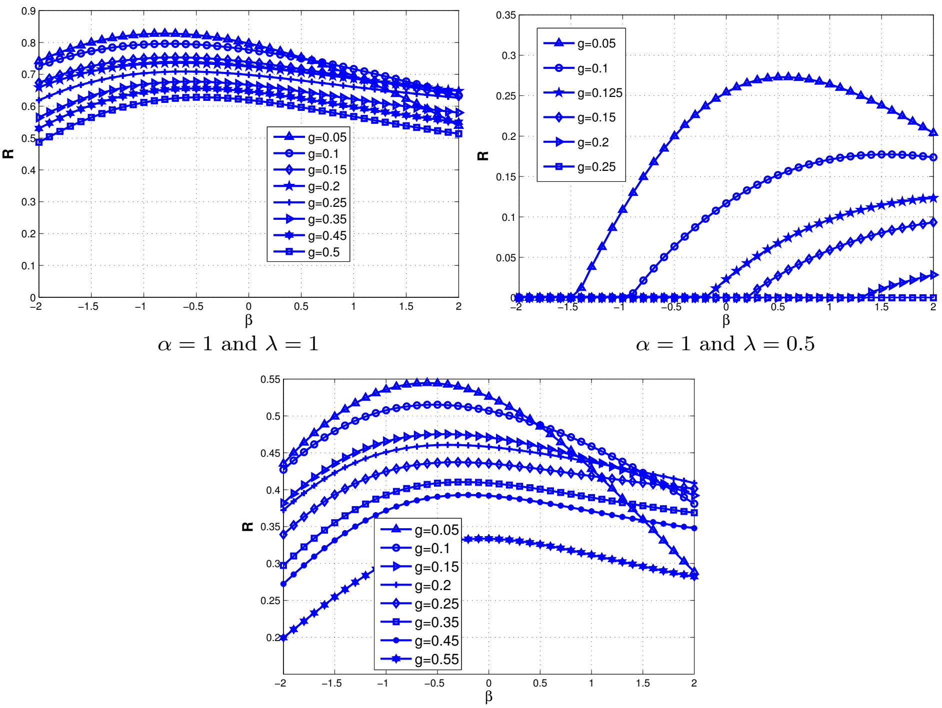
$\alpha = 1$ and $\lambda = 1$

$\alpha = 1$ and $\lambda = 0.5$

$\alpha = 0.5$ and $\lambda = 1$

Fig. 7. Final size of rumor R *versus* β in random inoculation scheme for different fraction of inoculation (g).

the case of targeted inoculation in Fig. 8. But here maximum rumor size is much smaller with the inoculation of very less fraction of nodes (*e.g.* for $g = 0.25$ the final rumor size R is almost suppressed), since rumor threshold in targeted inoculation is larger than the random inoculation.



$\alpha = 1$ and $\lambda = 1$ $\alpha = 1$ and $\lambda = 0.5$

$\alpha = 0.5$ and $\lambda = 1$

Fig. 8. Final size of rumor R *versus* β in targeted inoculation scheme for different fraction of inoculation (g).

For random inoculation strategy, the rumor spreading is plotted against time evolution using modified model through simulation results. It is found in Figs. 9–12, for $g = 0.1$, 0.3, 0.5, 0.7 that, if $\alpha + \beta$ increases from -1 to 2, R will increase since rumor threshold decreases. Further, it can be observed from Eq. (28) that, for $\alpha + \beta = -1$ and 0, λ_c is finite and higher than the case, where $\alpha + \beta = 1$ and 2. Therefore, $R(t)$ is almost 0 and grows slowly with time when $\alpha + \beta = -1$. The growth in $R(t)$ is higher for lower values of g, but the case is reversed for the higher values of g. Similarly, in the case of targeted inoculation scheme using lower values of $g = 0.05$, 0.1, 0.15, 0.2 for $\alpha + \beta = -1$ to 2, rumor threshold found more than the random inoculation scheme (Eq. (42)) and rumor spreading is suppressed for inoculation less number of nodes than the random inoculation scheme (Figs. 13–16).

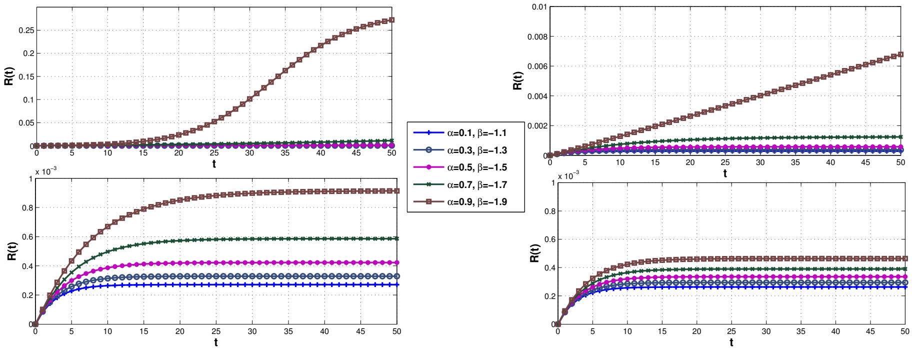


Fig. 9. $R(t)$ *versus* t with $\alpha + \beta = -1$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).

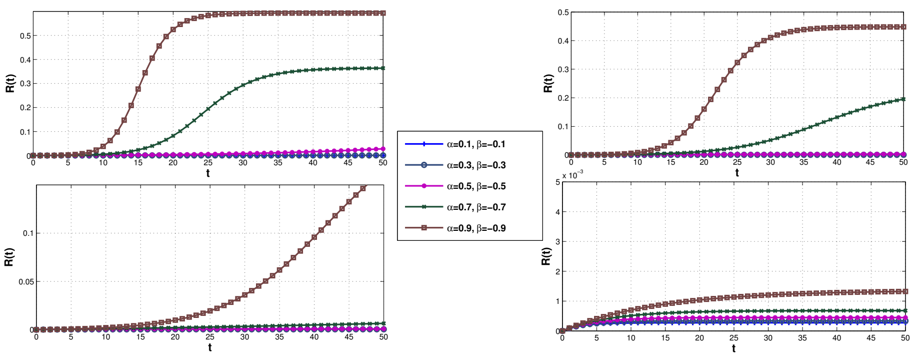


Fig. 10. $R(t)$ *versus* t with $\alpha + \beta = 0$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).

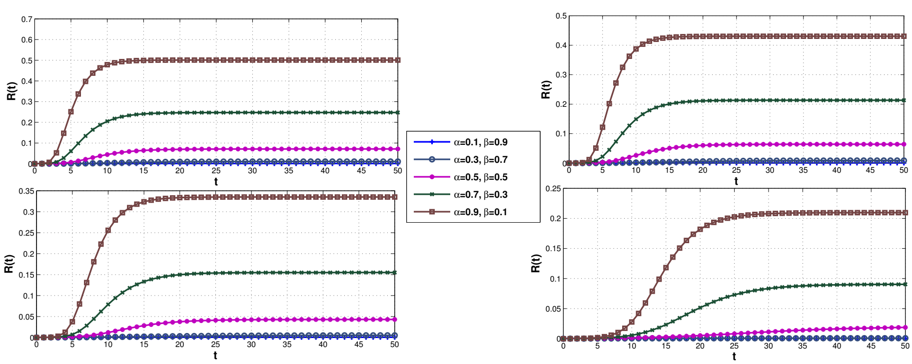


Fig. 11. $R(t)$ *versus* t with $\alpha + \beta = 1$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).

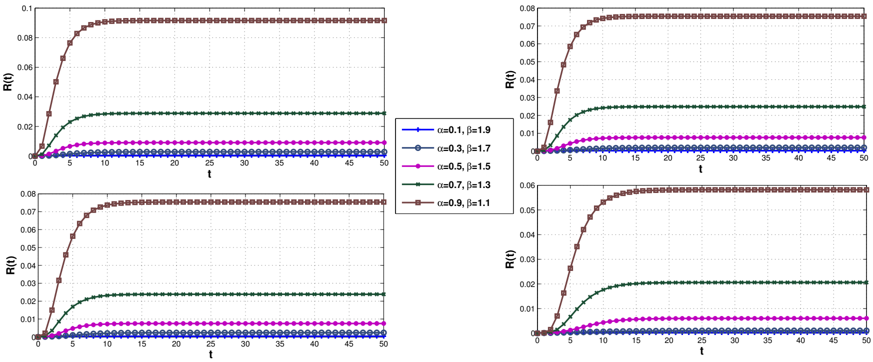


Fig. 12. $R(t)$ *versus* t with $\alpha + \beta = 2$, $\lambda = 0.6$ in random inoculation for $g = 0.1$, 0.3 (upper) 0.5, 0.7 (lower).

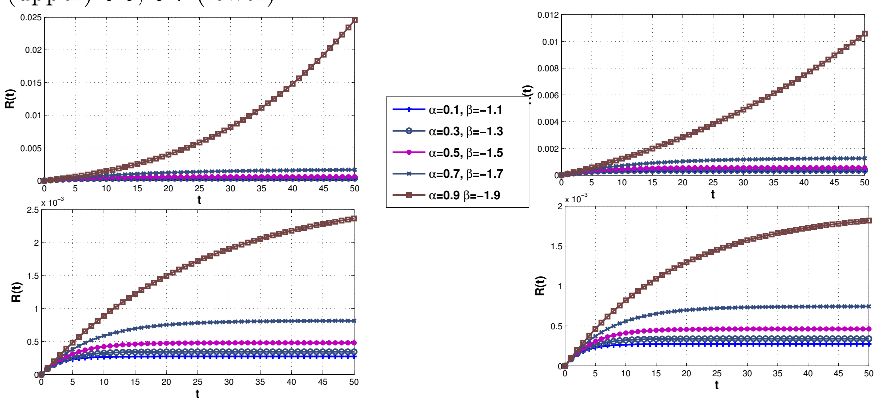


Fig. 13. $R(t)$ *versus* t with $\alpha + \beta = -1$, $\lambda = 0.6$ in targeted inoculation for $g = 0.05$, 0.1 (upper) 0.15, 0.2 (lower).

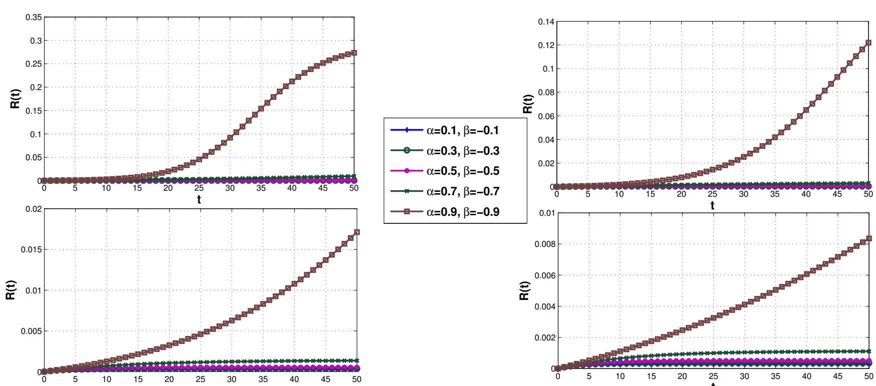


Fig. 14. $R(t)$ *versus* t with $\alpha + \beta = 0$, $\lambda = 0.6$ in targeted inoculation for $g = 0.05$, 0.1 (upper) 0.15, 0.2 (lower).

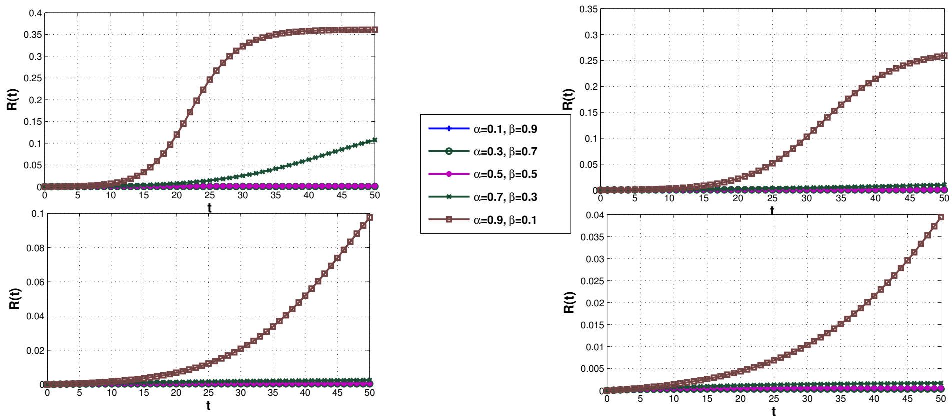


Fig. 15. $R(t)$ *versus* t with $\alpha + \beta = 1$, $\lambda = 0.6$ in targeted inoculation for $g = 0.05$, 0.1 (upper) 0.15, 0.2 (lower).

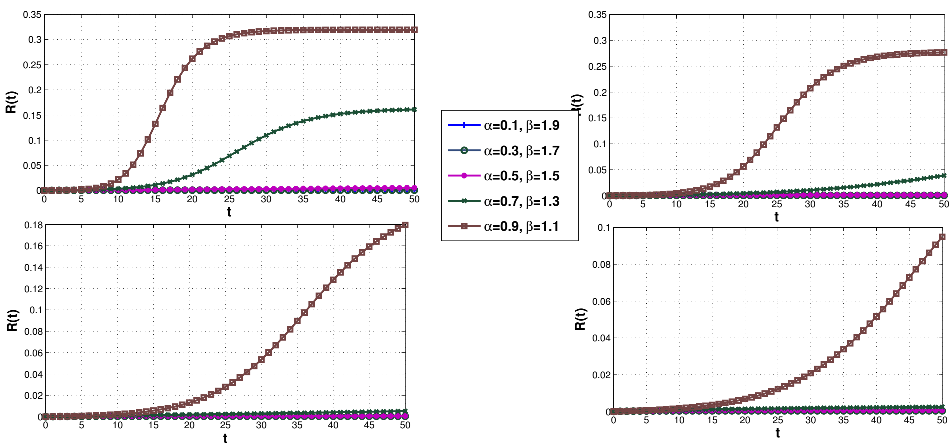


Fig. 16. $R(t)$ *versus* t with $\alpha + \beta = 2$, $\lambda = 0.6$ in targeted inoculation for $g = 0.05$, 0.1 (upper) 0.15, 0.2 (lower).

8. Conclusion

In presented study, the modified SIR model has been proposed by considering standard SIR rumor spreading model with degree dependent tie strength of nodes and nonlinear spread of rumor. The two parameters nonlinear exponent α and degree dependent tie strength exponent β have been introduced. In modified rumor spreading model, finite rumor spreading threshold has been found for finite scale free networks while fixed rumor threshold has been found for any size of network when $\alpha + \beta + 2 < \gamma$. Random and targeted inoculation schemes have been introduced in the proposed modified model. Rumor threshold in targeted inoculation scheme is found to be higher than the random inoculation. On the other hand, the rumor threshold in random inoculation is higher than the modified model without inoculation. It has also been observed that for scale free networks targeted inoculation scheme is successful in suppressing the rumor spreading in the network, since it requires to inoculate less number of nodes than random inoculation. Further, α is found to be more sensitive than β, as it affects more to rumor threshold. Finally, it is seen that in real world networks finite rumor threshold can be achieved by considering more realistic parameters (degree dependent tie strength of nodes and nonlinear spread of rumor). The targeted inoculation scheme can be successfully applied to suppress the rumor spreading over scale free networks.

REFERENCES

[1] R. Albert, A.L. Barabási, *Rev. Mod. Phys.* **74**, 47 (2002).

[2] A.L. Barabási, R. Albert, *Science* **286**, 509 (1999).

[3] Y. Moreno, M. Nekovee, A.F. Pacheco, *Phys. Rev.* **E69**, 066130 (2004).

[4] M. Nekovee, Y. Moreno, G. Bianconi, M. Marsili, *Physika A* **374**, 457 (2007).

[5] M.E.J. Newman, *Phys. Rev.* **E66**, 016128 (2002).

[6] M.E.J. Newman, *Siam Rev.* **45**, 167 (2003).

[7] R. Pastor-Satorras, A. Vespignani, *Phys. Rev.* **E65**, 036104 (2002).

[8] D.J. Watts, S.H. Strogatz, *Nature* **393**, 440 (1998).

[9] Y. Moreno, R. Pastor-Satorras, A. Vespignani, *Eur. Phys. J.* **B26**, 521 (2002).

[10] R. Pastor-Satorras, A. Vespignani, in: *Handbook of Graphs and Networks: From the Genome to the Internet*, eds. S. Bornholdt, H.G. Schuster, Wiley-VCH, Berlin 2002, p. 113.

[11] M. Boguñá, R. Pastor-Satorras, A. Vespignani, *Phys. Rev. Lett.* **90**, 028701 (2003).

[12] R. Pastor-Satorras, A. Vespignani, *Phys. Rev. Lett.* **86**, 3200 (2001).

[13] A. Sudbury, *J. Appl. Probab.* **22**, 443 (1985).

[14] D.J. Daley, J. Gani, J.M. Gani, *Epidemic Modelling: An Introduction*, Cambridge University Press, Cambridge, UK, 2001.

[15] D.P. Maki, M. Thompson, *Mathematical Models and Applications: With Emphasis on the Social, Life, and Management Sciences*, Prentice-Hall, NJ, 1973.

[16] D.H. Zanette, *Phys. Rev.* **E64**, 050901 (2001).

[17] D.H. Zanette, *Phys. Rev.* **E65**, 041908 (2002).

[18] Z.H. Liu, Y.C. Lai, N. Ye, *Phys. Rev.* **E67**, 031911 (2003).

[19] R. Pasor-Satorras, A. Vespignani, *Evolution and Structures of the Internet: A Statistical Physics Approach*, Cambridge University Press, Cambridge, UK, 2004.

[20] A. Barrat, M. Barthélemy, R. Pastor-Satorras, A. Vespignani, *Proc. Natl. Acad. Sci. USA* **101**, 3747 (2004).

[21] R. Pastor-Satorras, A. Vespignani, *Phys. Rev.* **E65**, 035108 (2002).

[22] G. Yan, Z.-Q. Fu, G. Chen, *Eur. Phys. J.* **B65**, 591 (2008).

[23] Wen Jie Bai, Tao Zhou, Bing Hong Wang, *Physica A* **384**, 656 (2007).

[24] Tao Zhou *et al.*, *Phys. Rev.* **E74**, 056109 (2006).

[25] Xinchu Fu, M. Small, D.M. Walker, Haifeng Zhang, *Phys. Rev.* **E77**, 036113 (2008).

[26] A.L. Barabasi, Hawoong Jeong, *Nature* **406**, 378 (2000).

[27] R. Cohen, K. Erez, D. ben Avraham, S. Havlin, *Phys. Rev. Lett.* **85**, 4626 (2000).

[28] R. Cohen, K. Erez, D. ben Avraham, S. Havlin, *Phys. Rev. Lett.* **86**, 3682 (2001).

[29] C. Kuhlman *et al.*, *Lect. Notes Comput. Sci.* **6589**, 188 (2011).

[30] A. Singh, Y.N. Singh, in: Proceedings of the 21st International Conference Companion on World Wide Web, WWW'12 Companion, ACM, New York, USA, 2012, p. 675.

[31] A. Singh, R. Kumar, Y.N. Singh, *Signal Image Technology and Internet Based Systems (SITIS)*, Eighth International Conference on "Rumor Dynamics with Acceptability Factor and Inoculation of Nodes in Scale Free Networks", 2012, pp. 798–804.

[32] N. Madar *et al.*, *Eur. Phys. J.* **B38**, 269 (2004).

[33] D. Chen *et al.*, *Physica A* **391**, 1777 (2012).

[34] M. Kitsak *et al.*, *Nature Physics* **6**, 888 (2010).

[35] B. Macdonald, P. Shakarian, N. Howard, G. Moores, `arXiv:1208.4269 [cs.SI]`.

[36] R.M. May, A.L. Lloyd, *Phys. Rev.* **E64**, 066112 (2001).

[37] J.P. Onnela *et al.*, *Proc. Natl. Acad. Sci. USA* **104**, 7332 (2007).