NEW METHOD FOR MAPPING OF EXACT ANALYTIC S-WAVE SOLUTIONS FOR REGENERATED CENTRAL HYPERBOLIC POTENTIALS

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We present a new method in the framework of non-relativistic quantum mechanics for mapping of exact analytic s-wave solutions for hyperbolic central potentials from the angular wave functions of already known quantum systems with exactly solvable ring-shaped potentials. The method is based on a coordinate redesignation and a coordinate transformation supplemented by a functional transformation. Invocation of plausible ansatz is indispensable to (re)generate hyperbolic central potentials, and the radial wave functions for the generated central potentials are shown to be normalizable elegantly.

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1. Introduction

The Schrödinger stationary-state wave equation in non-relativistic quantum mechanics can be solved exactly only for a very limited number of physical quantum systems (QSs). Therefore, the approximation schemes, *e.g.* the WKB analytical method, perturbation theory, variational technique, *etc.* have been used to procure information on the physical QSs with non-exactly solvable potentials (non-ESPs). However, the effectiveness of an approximation scheme in gathering information from a physical QS depends largely on the solutions of ESP which is found to be the potential of the QS having some sort of perturbation. This warrants the construction/generation of more and more number of ESPs for the successful implementation of the

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approximate methods. The study of exactly solvable QSs again provokes new mathematical techniques and/or physical ideas to quantum mechanics. Many authors have reported various types of non-ESPs with appended conditions, *e.g.* quasi-ESPs [1, 2], conditionally ESPs [3, 4], conditionally quasi-ESPs [5], *etc.*

Again, the extended transformation (ET) [6] is applied successfully by Ahmed *et al.* and others for the generation of exactly solvable central potentials (ESCPs) in an Euclidean space of any desired dimension from already known ESCPs (power law and non-power law) [7–12]. The ET includes a coordinate transformation (CT) required to modify the spatial character of an already known ESCP to generate a new ESCP and a functional transformation (FT) for manipulation of the dimensionality of the space to which the known QS gets transformed. Invocation of plausible ansatz is an integral part of the method to accomplish the energy eigenvalues of the generated QS. Here, we report on a new mapping method based on the ET [6] entailing a coordinate redesignation mechanism for the (re)generation of ESCPs (hyperbolic) in the three dimensional Euclidean space from already known QSs with exactly solvable ring-shaped potentials (ESRPs). The polar angle in the Schrödinger angular equation with a known ESRP is redesignated exclusively in the mathematical sense by radial coordinate to obtain a second-order radial differential equation. The ET is then performed on the second-order radial differential equation to retrieve the Schrödinger radial equation form and by invoking plausible ansatz, an ESCP (hyperbolic) is (re)generated. The quantum numbers, different parameters *etc.* related to the original potential have lost their usual meaning in the transformation and new parameters, quantum numbers, *etc.* have been reformulated by reshuffling those for the original potential so as to have a (new) physical central potential. The striking point in the present method is that the solutions of the Schrödinger radial equation, *i.e.* the radial wave functions for the generated potentials are not gathered by solving any differential equations, but they are mapped in a very straightforward manner from those for the already known ESRPs in the spirit similar to that of the supersymmetric (SUSY) approach [13–15], and the transformed radial wave functions are shown to be normalizable, solving the pertinent issue of normalizability elegantly.

The arrangement of the paper is as follows — Sec. 2 is devoted to describe the formalism of the method, the normalizability property of the transformed wave functions is discussed in Sec. 3, the method is demonstrated with examples in Sec. 4 and lastly, concluding remarks are included in Sec. 5.

2. Formalism

The Schrödinger stationary-state wave equation in natural units $\hbar = 2m = 1$ for a QS with a non-central potential $V(r,\theta) = V(r) + \frac{1}{r^2}V(\theta)$, where $V(r)$ is central potential and $V(\theta)$ is ring-shaped potential, is given by

$$\nabla^2\Psi(\vec{r}) + [E_n - V(r,\theta)]\Psi(\vec{r}) = 0\,. \tag{1}$$

We take a solution of the second-order differential equation (1) as

$$\Psi(\vec{r}) = R(r)\,\chi(\theta)\,\Phi(\phi) \tag{2}$$

and the separation variable method is applied in equation (1) to have

$$R(r)'' + \frac{2}{r}R(r)' + \left[E_n - V(r) - \frac{\lambda}{r^2}\right]R(r) = 0\,, \tag{3}$$

(Schrödinger radial equation)

$$\chi(\theta)'' + \cot\theta\chi(\theta)' + \left[\lambda - \frac{m^2}{\sin^2\theta} - V(\theta)\right]\chi(\theta) = 0\,, \tag{4}$$

(Schrödinger θ equation)

$$\Phi(\phi)'' + m^2\Phi(\phi) = 0\,. \tag{5}$$

(Schrödinger ϕ equation)

The prime symbolizes differentiation with respect to the argument. λ and m appear in the above equations (3)– (5) from the imposition of separation variable method in equation (1). In the case of central potential, $\lambda = l(l+1)$ and the admissible values for orbital quantum number l and magnetic quantum number m are $l = 0, 1, 2, 3, \ldots, (n-1)$ and $m = 0, \pm1, \pm2, \pm3, \ldots, \pm l$, if $R(r)$ is to be the wave function for a physical system. However, in the presence of the ring-shaped potential $V(\theta)$, l needs to be redefined [16], but m is fixed through the solution of equation (5), *i.e.* $\Phi(\phi) = \exp(\pm\, im\,\phi)$ and the periodicity condition $\Phi(\phi) = \Phi(\phi + 2\pi)$. The boundary conditions demand that radial wave function $R(r)$ vanishes as $r \to \infty$ and if $R(r) = r^{-1}U(r)$, then boundary conditions should be $U(0) = 0 = U(\propto)$ [17]. The angular wave function $\chi(\theta)$ at $\theta = 0$ and $\theta = \pi$ are finite, but it does not require any periodicity condition.

Here, we start with the Schrödinger θ equation (4) for a known physical QS with an ESRP $V_{\mathrm{ES}}(\theta)$, henceforth called parent potential, where the polar angle (θ) is redesignated by the radial coordinate (r) exclusively in the mathematical sense to construct a second-order radial differential equation as

$$\chi(r)'' + \cot r\, \chi(r)' + \left[\lambda - \frac{m^2}{\sin^2 r} - V_{\mathrm{ES}}(\theta = r)\right]\chi(r) = 0\,. \tag{6}$$

The above equation does not represent the standard Schrödinger radial equation (3) and at present keeping aside its physical meaning, if there is any in the deep root level, the equation is treated simply as a second-order homogeneous differential equation of the radial coordinate r.

The ET is followed to mold the constructed radial equation (6) to the Schrödinger radial equation form in an Euclidean space of any desired dimension. The ET consists of a CT as

$$r \to g(r) \tag{7}$$

and a FT as

$$R(r) = f(r)^{-1}\chi[g(r)]\,, \tag{8}$$

where the transformation function $g(r)$, required for generation of potential by modifying the spatial character of the parent potential, is a differentiable function of at least class C^2 and $f(r)^{-1}$ is the modulating function that plays a role in dealing with the dimensionality of the space to which the parent system gets transformed. Application of the CT on equation (6) yields an intermediate auxiliary differential equation as

$$\frac{d^2}{dg^2}\chi(g) + \cot g \frac{d}{dg}\chi(g) + \left[\lambda - \frac{m^2}{\sin^2 g} - V_{\mathrm{ES}}(\theta = g)\right]\chi(g) = 0\,.$$

Using the relations $\frac{d}{dg} \equiv \frac{1}{g'}\frac{d}{dr}$ and $\frac{d^2}{dg^2} \equiv \frac{1}{g'^2}\frac{d^2}{dr^2} - \frac{g''}{g'^3}\frac{d}{dr}$ in the intermediate equation and then subjecting it to the FT, we have

$$R(r)'' + \left(\frac{d}{dr}\ln\frac{f^2 \sin g}{g'}\right)R(r)' + \left\{\left(\frac{d}{dr}\ln f\right)\left(\frac{d}{dr}\ln\frac{f' \sin g}{g'}\right) + g'^2\left[\lambda - \frac{m^2}{\sin^2 g} - V_{\mathrm{ES}}(\theta = g(r))\right]\right\}R(r) = 0\,. \tag{9}$$

We require the coefficient of the first derivative equal to $\frac{D-1}{r}$ to cast the above equation (9) to the form of the standard Schrödinger radial equation in D-dimensional Euclidean space [6], *i.e.*

$$\frac{d}{dr}\ln\frac{f^2 \sin g}{g'} = \frac{D-1}{r}\,,$$

fixing the functional form of $f(r)$ as

$$f(r) = \text{constant} \times r^{\frac{D-1}{2}} g'^{\frac{1}{2}} \sin^{-\frac{1}{2}} g\,, \tag{10}$$

and changes equation (9) to

$$R(r)'' + \frac{D-1}{r} R(r)' + \left\{ \frac{1}{2}\{g,r\} + \left(\frac{D-1}{2}\right)\left(\frac{D-3}{2}\right)\frac{1}{r^2} + g'^2 \times \left[\left(\lambda + \frac{1}{4}\right) - \frac{m^2 - \frac{1}{4}}{\sin^2 g} - V_{\mathrm{ES}}\left(\theta = g(r)\right) \right] \right\} R(r) = 0\,, \tag{11}$$

where the Schwartzian derivative symbol $\{g,r\} = \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2$.

To generate a central potential in the above equation, we invoke the following ansatz

$$g'^2 \left(\frac{m^2 - \frac{1}{4}}{\sin^2 g} \right) = -\varepsilon\,, \tag{12}$$

which is the integral part of the mapping method and where ε is a constant, fixing the functional form of the transformation function $g(r)$ as

$$g(r) = 2 \arctan\left(C \exp \eta r\right)\,, \tag{13}$$

where

$$\eta = \pm\sqrt{\frac{-\varepsilon}{m^2 - \frac{1}{4}}}\,. \tag{14}$$

Invocation of the ansatz is the only approach to achieve a suitable transformation function $g(r)$, which will work in the generation of potential and an educative guess only helps us to identify an expedient ansatz. The square root of r.h.s. of the ansatz must be integrable to obtain $g(r)$ and at present, the r.h.s. is chosen as a constant so as to have a transformation function of simplest form, which will simplify the subsequent mathematical exercise.

Applying expression (13) for the transformation function $g(r)$ with the selection $C = 1$ to satisfy the local property as $g(0) = \frac{\pi}{2}$ required for checking normalizability of the transformed radial wave functions for the (re)generated central potential, the equation (11) becomes

$$R(r)'' + \frac{D-1}{r} R(r)' + \left[\left(\varepsilon - \frac{1}{4}\eta^2\right) - \left\{ \frac{\eta^2}{\cosh^2 \eta r} V_{\mathrm{ES}}\left[\theta = g(r)\right] - \frac{\lambda \eta^2}{\cosh^2 \eta r} \right\} + \left(\frac{D-1}{2}\right)\left(\frac{D-3}{2}\right)\frac{1}{r^2} \right] R(r) = 0\,, \tag{15}$$

and to retrieve the standard Schrödinger radial equation for the s-wave in $D = 3$ dimensional Euclidean space, the following identity is applied

$$E_n - V(r) = \left(\varepsilon - \frac{1}{4}\eta^2\right) - \frac{\eta^2}{\cosh^2 \eta r}\left\{V_{\mathrm{ES}}\left[\theta = g(r)\right] - \lambda\right\}. \tag{16}$$

Putting the expressions for the known parent potential $V_{\mathrm{ES}}(\theta)$ and transformation function $g(r)$ given by equation (13) in the above identity (16), the constant terms altogether in the right-hand side of the above identity are assembled to procure the energy eigenvalues E_n, while the remaining r-dependent terms are clubbed together to generate the central potential $V(r)$. Using equations (10) and (13) in (8), the radial wave function for the generated potential becomes

$$R(r) \approx r^{-1}\chi(g) \tag{17}$$

and is known, since angular wave function $\chi(\theta)$ for the parent potential and the transformation function $g(r)$ are known.

3. Normalizability of the transformed radial wave functions

Since the transformation method maps the generated central potentials from the already known ESRPs, the transformed radial wave functions given by (16) are always normalizable. The normalization condition for radial wave function $R(r)$ in D dimensional Euclidean space is

$$I\left[r = 0, r = \infty\right] = \int_0^\infty \left[R(r)\right]^2 r^{D-1}\, dr = \text{finite}. \tag{18}$$

Under equations (13) and (17), the normalization integral becomes

$$I\left[r = 0, r = \infty\right] = \int_{\frac{\pi}{2}}^{\pi} \left(\frac{1}{\sin^2\theta}\right) \mid \chi(\theta) \mid^2 \sin\theta\, d\theta = I\left[\theta = \frac{\pi}{2}, \theta = \pi\right].$$

If $\chi(\theta)$ is the normalized angular wave function for the parent ring-shaped potential $V_{\mathrm{ES}}(\theta)$ containing a term like $\frac{1}{\sin^2\theta}$, the quantity $I\left[\theta = 0, \theta = \pi\right] = \langle\frac{1}{\sin^2\theta}\rangle$ necessarily exits. Since $\left[\frac{\pi}{2}, \pi\right] \subseteq \left[0, \pi\right], I\left[\frac{\pi}{2}, \pi\right]$ will be finite and hence the transformed radial wave function $R(r)$ will be normalizable.

4. Applications

4.1. Mapping of exact analytic solutions for regenerated hyperbolic central potentials in the Pöschl–Teller and Scarf family

For the implementation of the method, we consider exactly solved θ-dependent ring-shaped potential from the Hartmann non-central potential [18] as the parent potential, which is

$$V_{\mathrm{ES}}\left(\theta\right) = \frac{A}{\sin^2\theta} \,. \tag{19}$$

For this ring-shaped potential $l = n \pm \sqrt{m^2 + A}$, where $n, \mid m \mid = 0, 1, 2, 3, \ldots$ and angular wave functions are

$$\chi_l^m\left(\theta\right) = \sin^\delta \theta P_n^{(\delta,\delta)}\left(\cos\theta\right) \,, \tag{20}$$

where $\delta = \sqrt{m^2 + A}$ and $P_n^{(a,b)}$ represents the Jacobi polynomial.

Using equations (13) and (19) in the identity (16) and selecting $\alpha = -\left(n+\delta\right)\left(n+\delta+1\right)\eta^2$, the central potential in the three dimensional Euclidean space ($D = 3$) is generated as

$$V\left(r\right) = \frac{\alpha}{\cosh^2\eta r} \,, \tag{21}$$

which is the potential in hyperbolic Pöschl–Teller family [19] and its energy eigenvalues are found to be

$$E_n = -\delta^2\eta^2 = -\left[\left(n+\frac{1}{2}\right)\eta - \sqrt{\frac{\eta^2}{4} - \alpha}\,\right]^2 \,, \tag{22}$$

where $n = 0, 1, 2, 3, \ldots$ and $\eta^2 \geq 4\alpha$.

Using equations (13) and (20) in (17), the radial wave functions for the generated Pöschl–Teller potential for s-wave is found to be

$$R_n\left(r\right) = r^{-1}\cosh^{-\sigma}\eta r \, P_n^{(\sigma,\sigma)}\left(\tanh\eta r\right) \,, \tag{23}$$

where $\sigma = -\left(n+\frac{1}{2}\right) \pm \sqrt{\frac{1}{4} - \frac{\alpha}{\eta^2}}$.

The generated central potential can also be reshaped to the form of potential of the hyperbolic Scarf family [19] as

$$V\left(r\right) = \alpha\frac{\sinh^2\eta r}{\cosh^2\eta r} \,, \tag{24}$$

for which the energy eigenvalues and radial wave functions for s-wave in the three dimensional Euclidean space become

$$E_n = -\left[\left(n+\frac{1}{2}\right)\eta - \sqrt{\frac{\eta^2}{4}+\alpha}\,\right]^2 + \alpha\,; \qquad n = 0,1,2,3,\ldots \tag{25}$$

using the identity (16) and

$$R_n\left(r\right) = r^{-1}\cosh^{-\rho}\eta r\, P_n^{(\rho,\rho)}\left(\tanh\eta r\right)\,, \tag{26}$$

using equations (13) and (20) in equation (17), where $\rho = -\left(n+\frac{1}{2}\right) \pm \sqrt{\frac{1}{4}+\frac{\alpha}{\eta^2}}$.

4.2. Mapping of exact analytic solutions of the regenerated hyperbolic central Rosen–Morse potential

The θ-dependent ring-shaped potential in the Makarov non-central potential [15, 16] is given by

$$V_{\mathrm{ES}}\left(\theta\right) = \frac{A}{\sin^2\theta} + \frac{B\cos\theta}{\sin^2\theta}\,, \tag{27}$$

for which

$$l = n \pm \sqrt{\frac{1}{2}\left(m^2 + A + \sqrt{\left(m^2+A\right)^2 - B^2}\right)}; \qquad n, \mid m \mid = 0,1,2,3,\ldots$$

and angular wave functions are

$$\chi_l^m\left(\theta\right) = \cos^{2p}\frac{\theta}{2}\sin^{2q}\frac{\theta}{2}\quad {}_2F_1\left(-n, -n+2l+1; 2p+1; \cos^2\frac{\theta}{2}\right)\,, \tag{28}$$

where $p = \frac{1}{2}\sqrt{m^2+A-B}$ and $q = \frac{1}{2}\sqrt{m^2+A+B}$, and ${}_2F_1\left(a,b;c;x\right)$ represents hypergeometric function.

Using equations (13) and (27), and choosing $\alpha = \eta^2 B$ and $\beta = -\lambda\eta^2$ in identity (16), the generated central potential ($D = 3$) emerges as

$$V\left(r\right) = \frac{\alpha\sinh\eta r}{\cosh\eta r} + \frac{\beta}{\cosh^2\eta r}\,, \tag{29}$$

which is the hyperbolic Rosen–Morse potential [19] and the energy eigenvalues with the selection $\delta = \eta\sqrt{m^2+A}$ come out as

$$E_n = -\delta^2 = -\left[\frac{\alpha^2}{4\eta^2\left(n+\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{\beta}{\eta^2}}\right)^2} + \eta^2\left(n+\frac{1}{2}\pm\sqrt{\frac{1}{4}-\frac{\beta}{\eta^2}}\right)^2\right]\,, \tag{30}$$

where $\eta^2 \geq 4\beta$ and $n = 0, 1, 2, 3, \ldots$ Again, by using equations (13) and (28) in (17), the radial wave functions ($l = 0$) for the generated hyperbolic potential are found as

$$R_n\left(r\right) = r^{-1}\left(1 - \tanh\eta r\right)^p \left(1 + \tanh\eta r\right)^q \, {}_2F_1\left(a, b; c; \tfrac{1}{2}\left(1 - \tanh\eta r\right)\right) , \tag{31}$$

where

$$p = \frac{1}{2\eta}\sqrt{\delta^2 + \alpha} \, , \qquad q = \frac{1}{2\eta}\sqrt{\delta^2 - \alpha}$$

and

$$\begin{aligned} a &= -n = p + q + \frac{1}{2} \mp \sqrt{\frac{1}{4} - \frac{\beta}{\eta^2}} \, , \\ b &= p + q + \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\beta}{\eta^2}} \, , \\ c &= 2p + 1 \, . \end{aligned}$$

The radial wavefunctions in equation (23) for hyperbolic potential in the Pöshl–Teller family, (26) for hyperbolic potential in the Scarf family and (31) for hyperbolic Rosen–Morse potentials are normalizable as per Sec. 3.

4.3. Mapping of exact analytic solutions for the regenerated hyperbolic central Scarf potential

We consider the ESRP [20]

$$V_{\mathrm{ES}}\left(\theta\right) = \frac{A\cos\theta}{\sin\theta} \, , \tag{32}$$

of which $A = -\lambda\left(2\lambda + 1\right)$ and the redefined azimuthal quantum number $l = -\frac{1}{2} + \left(\frac{1}{16} - \frac{A}{2}\right)^{1/4}$. Its angular wave functions are

$$\chi_l^m\left(\theta\right) = \left(1 + \omega^2\right)^{-\lambda/2} \exp\left(-\lambda\arctan\omega\right) R_n^{\left(\lambda + \frac{1}{2}, -2\lambda\right)}\left(\omega\right) \, , \tag{33}$$

where $\omega = -\cot\theta$, $n = \lambda - m$ and the symbol $R_n^{(a,b)}$ represents the Romanovski polynomial.

Inserting equations (13) and (32) into identity (16), and selecting $\alpha = -A\eta^2$ and $\beta = -\lambda\eta^2$, we generate the following central Scarf potential

$$V\left(r\right) = \frac{\alpha\sinh\eta r}{\cosh^2\eta r} + \frac{\beta}{\cosh^2\eta r} \, . \tag{34}$$

The characteristic constants of this potential satisfy the following constraint relation

$$\alpha = \beta - 2\left(\frac{\beta}{\eta}\right)^2 . \tag{35}$$

The energy eigenvalues for the central potential are

$$E_n = -\left(\frac{\beta}{\eta} - n\eta\right)^2 \tag{36}$$

and using equations (13) and (33) in (17), the radial wave functions for the s-wave are found to be

$$R_n(r) = r^{-1}\left(1+\nu^2\right)^{-\lambda/2} \exp\left(-\lambda \arctan \nu\right) R_n^{\left(-\frac{\beta}{\eta^2}+\frac{1}{2}, 2\frac{\beta}{\eta^2}\right)}(\nu) , \tag{37}$$

where $\nu = \sinh \eta r$.

Since the normalization integral for the wave function is equivalent to the expectation value of a term in the square of the parent potential in equation (32) in the range $\left[\frac{\pi}{2}, \pi\right]$, the integration must yield a finite value, indicating that the wave function is normalizable.

4.4. Mapping of exact analytic solutions for the regenerated hyperbolic central Pöschl–Teller potential

We now consider an ESRP of the following form [21]

$$V_{\mathrm{ES}}(\theta) = \frac{A + B\cos^2\theta + C\cos^4\theta}{\sin^2\theta \cos^2\theta} \tag{38}$$

and

$$\lambda = l(l+1) = (1+2n+p)(1+2n+p+q) + A - C , \tag{39}$$

where $p = \sqrt{m^2 + A + B + C}$ and $q = \sqrt{1+4A}$, and the angular wave functions are

$$\chi_l^m(\theta) = (\sin\theta)^p (\cos\theta)^{(1+q)/2} \, {}_2F_1\left(a, b; c; \cos^2\theta\right) , \tag{40}$$

where $a = -n$, $b = n + p + \frac{q}{2} + 1$ and $c = \frac{q}{2} + 1$.

Using equations (13) and (38) in the identity (16) and selecting $\alpha = A\eta^2$ and $\beta = -(C+\lambda)\eta^2$, the following hyperbolic central Pöschl–Teller potential is generated

$$V(r) = \frac{\alpha}{\sinh^2 \eta r} + \frac{\beta}{\cosh^2 \eta r} \tag{41}$$

and the energy eigenvalues are

$$E_n = -\delta^2 = -\left[\left(1 + 2n + \sqrt{\frac{1}{4} + \frac{\alpha}{\eta^2}}\right)\eta \pm \sqrt{\frac{\eta^2}{4} - \beta}\right]^2 . \tag{42}$$

Using equations (13) and (40) in (17), the radial wave functions for s-wave of the generated potential are found to be

$$R_n(r) = r^{-1} (\sinh \eta r)^{(1+q)/2} (\cosh \eta r)^{-p-(1+q)/2} \, {}_2F_1\left(a, b; c; \tanh^2 \eta r\right) , \tag{43}$$

where $p = \frac{\delta}{\eta}$ and $q = \sqrt{1 + \frac{4\alpha}{\eta^2}}$.

The normalization integral for the above radial wave functions is equivalent to the expectation value of a term in the parent potential in equation (38) in the range $\left[\frac{\pi}{2}, \pi\right]$, so, the integration must yield a finite value, indicating that the radial wave function for the generated potential in (41) is normalizable.

5. Conclusions

We present a new mapping method, in the spirit similar to that of the SUSY approach, to generate exact analytic s-wave solutions for ESCPs (hyperbolic) from the angular wave functions of the already known ESRPs. The mapping method includes a redesignation of coordinate required to transform the Schrödinger angular equation with an ESRP to a second-order radial differential equation on which a CT followed by a FT is performed to retrieve the Schrödinger radial equation form. A plausible ansatz is invoked to generate central potential and its energy eigenvalues and instead of solving the Schrödinger radial equation for the generated potentials, the solutions are mapped from those for the original ring-shaped potentials. The method is exemplified by mapping the solutions of the Schrödinger radial equation for the regenerated well-known hyperbolic central Pöschl–Teller, Rosen–Morse and Scarf potentials and potentials in the Pöschl–Teller and Scarf family from those for ESRPs. The wave functions of the generated potentials are normalizable and verified analytically. Since the functional form of the transformation function in the present method is specified through the initiation of an ansatz resulting in tangent inverse of an exponential function, the method always maps a ring-shaped potential no other than to a hyperbolic central potential. Presently, we are trying to devise mapping methods to generate exact analytic solutions of new central power law and trigonometric potentials from those of the exactly solvable ring-shaped potentials.

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