NEXT-TO-LEADING ORDER PERTURBATIVE CONTRIBUTIONS IN THE QCD SUM RULES FOR MESONIC TWO-POINT CORRELATION FUNCTIONS WITH UNEQUAL QUARK MASSES

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(Received August 5, 2013; revised version received September 11, 2013; final version received September 27, 2013)

In this article, we assume that two quarks have unequal masses and calculate the next-to-leading order contributions to the spectral densities of the mesonic two-point correlation functions of the vector, axialvector, scalar and pseudoscalar currents. We take dimensional regularization to regularize both the ultraviolet and infrared divergences, and use optical theorem to obtain the spectral densities directly, furthermore, we present some necessary technical details for readers convenience. The analytical expressions are applicable in many phenomenological analysis besides the QCD sum rules.

DOI:10.5506/APhysPolB.44.1971
PACS numbers: 12.38.Bx, 12.38.Lg

1. Introduction

The QCD sum rules is a powerful theoretical tool in studying the ground state mesons [1, 2] and has given many successful descriptions of the meson properties. In the case of the interpolating currents consisting of quarks with equal masses or zero masses, the leading order and next-to-leading order perturbative contributions to the spectral densities of the mesonic two-point correlation functions are known [2, 3]. There are few works on the next-to-leading order perturbative contributions to the spectral densities of the interpolating currents for unequal quark masses, as the calculations are very difficult.

In Refs. [4, 5], Reinders, Rubinstein and Yazaki calculate the next-to-leading order contributions to the spectral densities of the mesonic two-point correlation functions for the pseudoscalar and vector currents in the unequal mass case. Some technical details are presented in Ref. [2]. In the famous review (see Ref. [2]), Reinders, Rubinstein and Yazaki illustrate how to use optical theorem to calculate the next-to-leading order contributions to spectral densities of the mesonic two-point correlation functions (for the scalar, pseudoscalar, vector and axialvector currents) directly, and regularize the divergences in the massive-gluon scheme. In Ref. [6], Schilcher, Tran and Nasrallah also resort to the massive-gluon scheme to regularize the infrared divergence, and use optical theorem to calculate the next-to-leading order contributions to the longitudinal and transverse spectral densities of the mesonic two-point correlation functions directly for arbitrary conserved or non-conserved vector or axialvector currents of the massive quarks, and obtain analytical expressions. Although those works are all performed in the massive-gluon scheme, the resulting (infrared) divergent terms are different from each other, the expressions also differ from each other.

In Ref. [7], Generalis calculates the two-loop Feynman diagrams directly using the dimensional regularization, and obtains the next-to-leading order contributions to the mesonic two-point correlation functions for the vector and axialvector currents with unequal quark masses. We have to obtain the spectral densities through dispersion relation from the cumbersome expressions, it is a difficult task.

In this article, we resort to optical theorem (or Cutkosky's rule) to calculate the next-to-leading order contributions to the spectral densities of the mesonic two-point correlation functions for the scalar, pseudoscalar, vector and axialvector currents in the case of unequal quark masses. In calculations, we regularize both the ultraviolet and infrared divergences in the scheme of dimensional regularization, and present the necessary technical details for the readers convenience, *i.e.* one can check the calculations. There are two routines in application of optical theorem. We resort to the routine used in Refs. [2, 6], not the one used in Ref. [8]. After the present work was finished and submitted to the net `http://arxiv.org/`, some friends kindly draw my attention to Refs. [9, 10]. In Ref. [9], Djouadi and Gambino study the QCD corrections to the electro-weak gauge boson self-energies and Higgs self-energies for the arbitrary momentum transfer and for different internal quark masses, and obtain explicit expressions for both the real and imaginary parts of the self-energies. They calculate the two-loop Feynman diagrams directly using the dimensional regularization, the tedious work is difficult to follow. On the other hand, if we calculate the imaginary parts of the mesonic two-point correlation functions (or the self-energies) directly via optical theorem, the calculations are greatly simplified and easier to follow, furthermore, we can separate the contributions come from the real and virtual gluon emissions explicitly, which have direct applications in studying the decays of a polarized W boson into massive quark–antiquark pairs, such as $t \to bW^+ \to bq_1\bar{q}_2$, $H \to W^-W^{*+} \to W^-q_1\bar{q}_2$, H denotes the Higgs boson [10].

The article is arranged as follows: we calculate the next-to-leading order contributions to spectral densities of the mesonic two-point correlation functions in Sect. 2; in Sect. 3, we present the final analytical expressions; Sect. 4 is reserved for our conclusions.

2. Explicit calculations of the spectral densities at the next-to-leading order

In the following, we write down the mesonic two-point correlation functions in the QCD sum rules

$$
\begin{aligned}
\Pi_{\mu\nu}(p) &= i\int d^4x e^{ip\cdot x}\langle 0|T\left\{J_\mu(x)J_\nu^\dagger(0)\right\}|0\rangle\,,\\
&= \Pi_{\rm V/A}(p)\left(-g_{\mu\nu}+\frac{p^\mu p^\nu}{p^2}\right)+\Pi_{\rm S/P}(p)\frac{p^\mu p^\nu}{p^2}\,,\\
\Pi_{\rm S/P}(p) &= \frac{(m_1\mp m_2)^2}{p^2}\widetilde{\Pi}_{\rm S/P}(p)\,,\\
\widetilde{\Pi}_{\rm S/P}(p) &= i\int d^4x e^{ip\cdot x}\langle 0|T\left\{J(x)J^\dagger(0)\right\}|0\rangle\,,
\end{aligned}
\tag{1}
$$

where $J^\mu(x)=J^\mu_{\rm V}(x), J^\mu_{\rm A}(x)$ and $J(x)=J_{\rm S}(x), J_{\rm P}(x)$. The lower indexes denote the scalar (S), pseudoscalar (P), vector (V) and axialvector (A) currents, respectively. The correlation functions can be expressed in the following form through dispersion relation

$$
\Pi_i\left(p^2\right) = \frac{1}{\pi}\int_{\Delta^2}^{\infty} ds\frac{{\rm Im}\Pi_i(s)}{s-p^2}\,,
\tag{2}
$$

where $i={\rm S,P,V,A}$, the $\Delta^2=(m_1+m_2)^2$ is the threshold parameter, and

$$
\frac{{\rm Im}\Pi_i(s)}{\pi} = \rho_i^0(s)+\rho_i^1(s)+\rho_i^2(s)+\cdots\,,
\tag{3}
$$

the $\rho_i^0(s)$, $\rho_i^1(s)$, $\rho_i^2(s)$, ... are the spectral densities of the leading order, next-to-leading order, and next-to-next-to-leading order, ... At the leading order, the perturbative spectral densities are rather simple

$$\rho^0_{\mathrm{V/A}}(s) \;=\; \frac{3}{8\pi^2}\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s}\left\{s-(m_1\mp m_2)^2-\frac{\lambda\left(s,m_1^2,m_2^2\right)}{3s}\right\}\,,$$

$$\rho^0_{\mathrm{S/P}}(s) \;=\; \frac{(m_1\mp m_2)^2}{s}\frac{3}{8\pi^2}\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s}\left\{s-(m_1\pm m_2)^2\right\}\,, \tag{4}$$

where

$$\lambda\left(s,m_1^2,m_2^2\right) \;=\; s^2+m_1^4+m_2^4-2sm_1^2-2sm_2^2-2m_1^2m_2^2\,. \tag{5}$$

At the next-to-leading order, there are three Feynman diagrams contributing to the correlation functions, see Fig. 1. We calculate the imaginary parts (or the spectral densities) using Cutkosky's rule or optical theorem — the two approaches lead to the same results — then use the dispersion relation to obtain the correlation functions. There are ten possible cuts, six cuts attributed to virtual gluon emissions, see Fig. 2, and four cuts attributed to real gluon emissions, see Fig. 3.

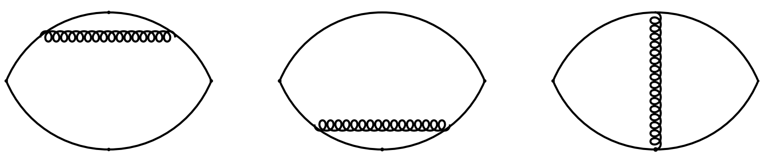

Fig. 1. The next-to-leading order contributions to the correlation functions.

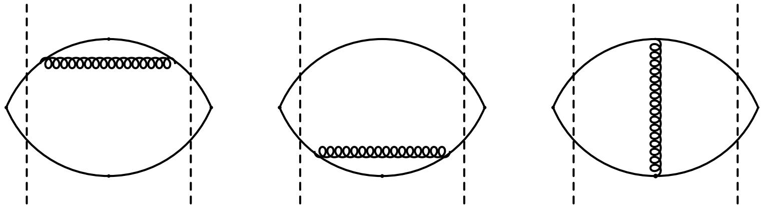

Fig. 2. Six possible cuts correspond to virtual gluon emissions.

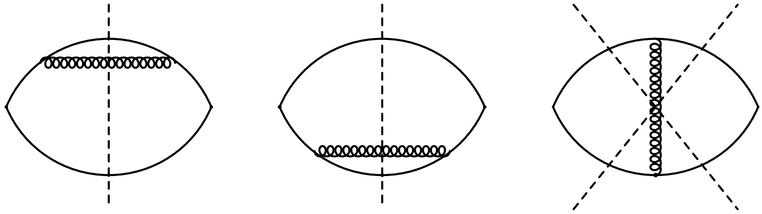

Fig. 3. Four possible cuts correspond to real gluon emissions.

2.1. Contributions of the virtual gluon emissions

The six cuts shown in Fig. 2 are attributed to virtual gluon emissions and correspond to the self-energy corrections and vertex corrections, respectively. The first four cuts corresponding to one-loop fermion's self-energy corrections, we calculate the Feynman diagrams directly using the dimensional regularization and choose the on-shell renormalization scheme to subtract the divergences so that to implement the wave-function renormalization and mass renormalization. Then, we can take into account all contributions coming from the six cuts shown in Fig. 2 by the following simple replacement for each vertex in the interpolating currents[1]

$$\begin{aligned}\bar{u}(p_1)\gamma_\mu u(p_2) &\to \bar{u}(p_1)\gamma_\mu u(p_2) + \bar{u}(p_1)\widetilde{\Gamma}_\mu u(p_2)\\ &= \sqrt{Z_1}\sqrt{Z_2}\bar{u}(p_1)\gamma_\mu u(p_2) + \bar{u}(p_1)\Gamma_\mu u(p_2)\\ &= \bar{u}(p_1)\gamma_\mu u(p_2)\left(1+\tfrac{1}{2}\delta Z_1+\tfrac{1}{2}\delta Z_2\right) + \bar{u}(p_1)\Gamma_\mu u(p_2)\,, \end{aligned} \tag{6}$$

where

$$Z_i = 1+\delta Z_i = 1+\frac{4}{3}\frac{\alpha_s}{\pi}\left(-\frac{1}{4\varepsilon_{\rm UV}}+\frac{1}{2\varepsilon_{\rm IR}}+\frac{3}{4}\log\frac{m_i^2}{4\pi\mu^2}+\frac{3}{4}\gamma-1\right)\,, \tag{7}$$

is the ith quark's wave-function renormalization constant due to the self-energy correction, see Fig. 4, and

$$\begin{aligned}\Gamma_\mu =& \gamma_\mu\frac{4}{3}g_s^2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^Dk_{\rm E}}{(2\pi)^D}\frac{\Gamma(3)}{\left[k_{\rm E}^2+(xp_1+yp_2)^2\right]^3}\left\{k_{\rm E}^2\left(1-\frac{3}{2}\varepsilon_{\rm UV}\right)\right.\\ &+2(1-x)(1-y)\left(s-m_1^2-m_2^2\right)+2(x+y)m_1m_2+2x(1-x)m_1^2\\ &\left.+2y(1-y)m_2^2\right\}+\frac{4}{3}g_s^2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^Dk_{\rm E}}{(2\pi)^D}\\ &\times\frac{4\Gamma(3)\left\{\left[x^2m_1-y(1-x)m_2\right]p_{1\mu}+\left[y^2m_2-x(1-y)m_1\right]p_{2\mu}\right\}}{\left[k_{\rm E}^2+(xp_1+yp_2)^2\right]^3}\end{aligned} \tag{8}$$

is the vertex correction after performing the Wick rotation, see Fig. 5. Here, γ is the Euler constant, μ^2 is the renormalization scale, and the Euclidean momentum $k_{\rm E} = (k_1, k_2, k_3, k_4)$. In this article, we take the dimension $D = 4-2\varepsilon_{\rm UV} = 4+2\varepsilon_{\rm IR}$ to regularize the ultraviolet and infrared divergences

[1] Here, we use the vector current to illustrate the procedure. Other currents can be treated analogously.

respectively, and add the renormalization scale factors $\mu^{2\varepsilon_{\rm UV}}$ or $\mu^{-2\varepsilon_{\rm IR}}$ when necessary. In the limit $m_1 = m_2 = m$, the Γ_μ can be reduced as

$$\begin{aligned}\Gamma_\mu &= \gamma_\mu \frac{4}{3} g_s^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_{\rm E}}{(2\pi)^D} \frac{\Gamma(3)}{\left[k_{\rm E}^2 + (xp_1 + yp_2)^2\right]^3} \left\{ k_{\rm E}^2 \left(1 - \frac{3}{2}\varepsilon_{\rm UV}\right) \right. \\ &\left. +2(1-(x+y)+xy)\left(s-2m^2\right) + 4(x+y)m^2 - 2(x+y)^2 m^2 + 4xym^2 \right\} \\ &+ \frac{4}{3} g_s^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_{\rm E}}{(2\pi)^D} \frac{2\Gamma(3) m \left[(x+y)^2 - (x+y)\right](p_{1\mu} + p_{2\mu})}{\left[k_{\rm E}^2 + (xp_1 + yp_2)^2\right]^3} \end{aligned} \quad (9)$$

then, the resulting expression is greatly simplified.

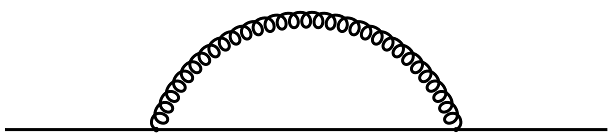

Fig. 4. The quark self-energy correction.

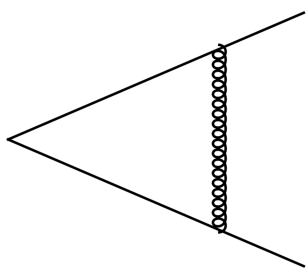

Fig. 5. The vertex correction.

We carry out the integral over the variables x, y and $k_{\rm E}$, and observe that the ultraviolet divergent terms $\frac{1}{\varepsilon_{\rm UV}}$ in the Γ_μ, δZ_1 and δZ_2 are canceled out with each other, which is a consequence of the Ward identity. The net contributions have no ultraviolet divergence

$$\widetilde{\Gamma}_\mu = \frac{1}{3}\frac{\alpha_{\rm s}}{\pi}\gamma_\mu f(s) + \frac{1}{3}\frac{\alpha_{\rm s}}{\pi} f_1(s) p_{1\mu} + \frac{1}{3}\frac{\alpha_{\rm s}}{\pi} f_2(s) p_{2\mu}\,, \quad (10)$$

where

$$
\begin{aligned}
f(s) &= \overline{f}(s) + \frac{2}{\varepsilon_{\rm IR}} + \log\frac{4\pi\mu^2}{s} + 2\gamma - 3 + 3\log\frac{m_1 m_2}{4\pi\mu^2} - \frac{2\left(s-m_1^2-m_2^2\right)}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}} \\
&\quad \times \log\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{1}{\varepsilon_{\rm IR}} + \log\frac{s}{4\pi\mu^2} + \gamma\right), \\
\overline{f}_{\pm}(s) &= \overline{V}(s) + 2\left(s-m_1^2-m_2^2\right)\left[\overline{V}_{00}(s) - V_{10}(s) - V_{01}(s) + V_{11}(s)\right] \\
&\quad \pm 2m_1 m_2\left[V_{10}(s) + V_{01}(s)\right] + 2m_1^2\left[V_{10}(s) - V_{20}(s)\right] \\
&\quad + 2m_2^2\left[V_{01}(s) - V_{02}(s)\right], \\
f_{1\pm}(s) &= 4m_1 V_{20}(s) \mp 4m_2 V_{01}(s) \pm 4m_2 V_{11}(s), \\
f_{2\pm}(s) &= \pm 4m_2 V_{02}(s) - 4m_1 V_{10}(s) + 4m_1 V_{11}(s), \\
\omega &= \sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}},
\end{aligned}
\tag{11}
$$

and $s = p^2$, the $\overline{V}(s)$, $\overline{V}_{00}(s)$ and $V_{ij}(s)$ with $i,j = 0,1,2$ are given explicitly in Appendix. Here, we introduce the lower-indexes $\pm$ to denote the corresponding expressions of the vector $(+)$ and axialvector $(-)$ currents respectively. The infrared divergence $\frac{1}{\varepsilon_{\rm IR}}$ comes from the wave-function renormalization constants Z_i in Eq. (7), while the infrared divergence $\log\left(\frac{1+\omega}{1-\omega}\right)\frac{1}{\varepsilon_{\rm IR}}$ comes from the term

$$
\gamma_\mu \frac{4}{3} g_s^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_{\rm E}}{(2\pi)^D} \frac{2\Gamma(3)\left(s-m_1^2-m_2^2\right)}{\left[k_{\rm E}^2 + (xp_1+yp_2)^2\right]^3} \tag{12}
$$

in Eq. (8). The infrared divergent terms obtained in the present work are consistent with that obtained by Groote, Körner and Tuvike in Ref. [10]. If we regularize the infrared divergences with a supposed gluon mass m_g in the quantum field theory, the infrared divergent terms appear as $\log\frac{m_g^2}{\Lambda^2}$ instead of $\frac{1}{\varepsilon_{\rm IR}}$, where the Λ has a dimension of mass. So in the massive-gluon scheme, we expect that the infrared divergences $\frac{1}{\varepsilon_{\rm IR}}$ and $\log\left(\frac{1+\omega}{1-\omega}\right)\frac{1}{\varepsilon_{\rm IR}}$ appear in the forms $\log\frac{m_g^2}{m_1 m_2}$ and $\log\left(\frac{1+\omega}{1-\omega}\right)\log\frac{m_g^2}{m_1 m_2}$, respectively. In Ref. [2], there only exists the infrared divergence $\log\frac{m_g^2}{m_1 m_2}$; while in Ref. [6], there exist the infrared divergences $\log\left(\frac{1+\omega}{1-\omega}\right)\log\frac{m_g^2}{m_1 m_2}$ and $\log\frac{m_1}{m_2}\log\frac{m_g^2}{m_1 m_2}$. In this article, we calculate the vertex corrections using the conventional Feynman parameters, while in Ref. [6], Schilcher, Tran and Nasrallah calculate the vertex corrections using the optical theorem and dispersion relation.

The total contributions of the virtual gluon emissions (see Fig. 2) to imaginary parts of the correlation functions can be expressed in the following form

$$\frac{\mathrm{Im}\Pi^{\mathrm{V}}_{\mu\nu}(s)}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}\frac{3}{\pi}\int \frac{d^{D-1}\vec{p}_1}{(2\pi)^{D-1}2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{(2\pi)^{D-1}2E_{p_2}}(2\pi)^D\delta^D(p-p_1-p_2)$$
$$\times\left\{ f(s)\left[p_{1\mu}p_{2\nu}+p_{2\mu}p_{1\nu}-\frac{s-(m_1-m_2)^2}{2}g_{\mu\nu}\right]\right.$$
$$\left.+f_1(s)(m_1p_{1\mu}p_{2\nu}-m_2p_{1\mu}p_{1\nu})+f_2(s)(m_2p_{2\mu}p_{1\nu}-m_1p_{2\mu}p_{2\nu})\right\}. \tag{13}$$

In this article, the upper indexes V and R denote the contributions coming from the virtual and real gluon emissions respectively. We carry out the integrals in Eq. (13) directly in $D = 4 + 2\varepsilon_{\mathrm{IR}}$ dimension as there is no ultraviolet divergence, and obtain the following results

$$\frac{\mathrm{Im}\Pi^{\mathrm{V}}_{\mathrm{V/A}}(s)}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}\rho^0_{\mathrm{V/A}}(s)\left\{\frac{1}{\varepsilon_{\mathrm{IR}}}-2\log 4\pi+2\gamma-\frac{7}{2}\right.$$
$$+\frac{1}{2}\log\frac{\lambda^2\left(s,m_1^2,m_2^2\right)m_1^3m_2^3}{\mu^8s^3}+\frac{1}{2}\overline{f}_{\pm}(s)-\frac{s-m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)$$
$$\left.\times\left[\frac{1}{\varepsilon_{\mathrm{IR}}}-2\log 4\pi+2\gamma-2+\log\frac{\lambda\left(s,m_1^2,m_2^2\right)}{\mu^4}\right]\right\}+\frac{4}{3}\frac{\alpha_s}{\pi}\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{s^2}$$
$$\times\left\{\frac{1}{12\pi^2}\left[1-\frac{s-m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)\right]-\frac{(m_1\pm m_2)\left[f_{1\pm}(s)+f_{2\pm}(s)\right]}{32\pi^2}\right\}, \tag{14}$$

$$\frac{\mathrm{Im}\Pi^{\mathrm{V}}_{\mathrm{S/P}}(s)}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}\rho^0_{\mathrm{S/P}}(s)\left\{\frac{1}{\varepsilon_{\mathrm{IR}}}-2\log 4\pi+2\gamma-\frac{7}{2}\right.$$
$$+\frac{1}{2}\log\frac{\lambda^2\left(s,m_1^2,m_2^2\right)m_1^3m_2^3}{\mu^8s^3}+\frac{1}{2}\overline{f}_{\pm}(s)-\frac{s-m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)$$
$$\left.\left[\frac{1}{\varepsilon_{\mathrm{IR}}}-2\log 4\pi+2\gamma-2+\log\frac{\lambda\left(s,m_1^2,m_2^2\right)}{\mu^4}\right]\right\}+\frac{4}{3}\frac{\alpha_s}{\pi}\frac{3}{32\pi^2}\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s^2}$$

$$\left\{ \left(s+m_1^2-m_2^2\right)\left(s+m_2^2-m_1^2\right)\left[f_{1\pm}(s)m_1 \pm f_{2\pm}(s)m_2\right] \right.$$
$$\left. \mp f_{1\pm}(s)m_2\left(s+m_1^2-m_2^2\right)^2 - f_{2\pm}(s)m_1\left(s+m_2^2-m_1^2\right)^2 \right\}, \tag{15}$$

where we have used the γ_5 symmetry of the interpolating currents to obtain the spectral densities $\frac{\mathrm{Im}\Pi_A^V(s)}{\pi}$ and $\frac{\mathrm{Im}\Pi_P^V(s)}{\pi}$.

2.2. Contributions of the real gluon emissions

The four cuts shown in Fig. 3 correspond to real gluon emissions. The scattering amplitudes for the real gluon emissions are shown explicitly in Fig. 6. From the figure, we can write down the scattering amplitude $T^a_{\mu\alpha}(p)$

$$T^a_{\mu\alpha}(p) = \bar{u}(p_1)\left\{ ig_s\frac{\lambda^a}{2}\gamma_\alpha \frac{i}{\not{p}_1+\not{k}-m_1}\gamma_\mu + \gamma_\mu \frac{i}{-\not{p}_2-\not{k}-m_2} ig_s\frac{\lambda^a}{2}\gamma_\alpha \right\} v(p_2)\,, \tag{16}$$

where λ^a is the Gell-Mann matrix. Then, we obtain the corresponding contributions $\mathrm{Im}\Pi_V^R(s)$ and $\mathrm{Im}\Pi_S^R(s)$ to the imaginary parts of the correlation function with the optical theorem

$$\begin{aligned}
\frac{\mathrm{Im}\Pi_V^R(s)}{\pi} &= -\frac{1}{2\pi(D-1)}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}2E_k}\frac{d^{D-1}\vec{p}_1}{(2\pi)^{D-1}2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{(2\pi)^{D-1}2E_{p_2}} \\
&\times (2\pi)^D\delta^D(p-k-p_1-p_2)\mathrm{Tr}\left\{T^a_{\mu\alpha}(p)T^{a\dagger}_{\nu\beta}(p)\right\} g^{\alpha\beta}\left(-g^{\mu\nu}+\frac{p^\mu p^\nu}{p^2}\right) \\
&= -\frac{2g_s^2}{\pi(D-1)}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}2E_k}\frac{d^{D-1}\vec{p}_1}{(2\pi)^{D-1}2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{(2\pi)^{D-1}2E_{p_2}} \\
&\times (2\pi)^D\delta^D(p-k-p_1-p_2)\left\{ 6\left[\frac{m_1^2}{(k\cdot p_1)^2}+\frac{m_2^2}{(k\cdot p_2)^2}-\frac{s-m_1^2-m_2^2}{k\cdot p_1 k\cdot p_2}\right.\right. \\
&\left. +\frac{s-K^2}{k\cdot p_1 k\cdot p_2}\right]\left[s-(m_1-m_2)^2-\frac{\lambda\left(s,m_1^2,m_2^2\right)}{3s}\right] + 4\varepsilon_{\mathrm{IR}}\left[s-(m_1-m_2)^2\right] \\
&\times\left[\frac{m_1^2}{(k\cdot p_1)^2}+\frac{m_2^2}{(k\cdot p_2)^2}-\frac{s-m_1^2-m_2^2}{k\cdot p_1 k\cdot p_2}\right]-\frac{\left(s-K^2\right)^2}{k\cdot p_1 k\cdot p_2}\left[2+\frac{(m_1-m_2)^2}{s}\right]\left.+16\right\},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{\mathrm{Im}\Pi_S^R(s)}{\pi} &= -\frac{1}{2\pi}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}2E_k}\frac{d^{D-1}\vec{p}_1}{(2\pi)^{D-1}2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{(2\pi)^{D-1}2E_{p_2}} \\
&\times (2\pi)^D\delta^D(p-k-p_1-p_2)\mathrm{Tr}\left\{T^a_{\mu\alpha}(p)T^{a\dagger}_{\nu\beta}(p)\right\} g^{\alpha\beta}\frac{p^\mu p^\nu}{p^2}
\end{aligned}$$

$$= -\frac{2g_s^2}{\pi}\int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}2E_k}\frac{d^{D-1}\vec{p}_1}{(2\pi)^{D-1}2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{(2\pi)^{D-1}2E_{p_2}}(2\pi)^D\delta^D(p-k-p_1-p_2)$$
$$\times\frac{(m_1-m_2)^2}{s}\left\{2\left[s-(m_1+m_2)^2\right]\left[\frac{m_1^2}{(k\cdot p_1)^2}+\frac{m_2^2}{(k\cdot p_2)^2}\right.\right.$$
$$\left.\left.-\frac{s-m_1^2-m_2^2}{k\cdot p_1 k\cdot p_2}+\frac{s-K^2}{k\cdot p_1 k\cdot p_2}\right]-\frac{\left(s-K^2\right)^2}{k\cdot p_1 k\cdot p_2}\right\}, \tag{18}$$

where we have used the identities $\sum u(p_1)\bar{u}(p_1) = \not{p}_1 + m_1$ and $\sum v(p_2)\bar{v}(p_2) = \not{p}_2 - m_2$ for the particle and antiparticle respectively, and take the notation $K^2 = (p_1+p_2)^2$. We carry out the integrals in Eqs. (17) and (18) in $D = 4+2\varepsilon_{\rm IR}$ dimension as there is only infrared divergence, and obtain the contributions of the real gluon emissions (see Fig. 3) to spectral densities

$$\frac{{\rm Im}\Pi^{\rm R}_{\rm V/A}(s)}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}\rho^0_{\rm V/A}(s)\left\{-\frac{1}{\varepsilon_{\rm IR}}+2\log 4\pi-2\gamma+\frac{8}{3}-\log\frac{\lambda^3\left(s,m_1^2,m_2^2\right)}{m_1^2m_2^2s^2\mu^4}\right.$$
$$+\left(s-m_1^2-m_2^2\right)\overline{R}_{12}(s)-\overline{R}_{11}(s)-\overline{R}_{22}(s)-\frac{2\left(s-m_1^2-m_2^2\right)}{3\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)$$
$$-R^1_{12}(s)+\frac{s-m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)\left[\frac{1}{\varepsilon_{\rm IR}}-2\log 4\pi+2\gamma-2\right.$$
$$\left.\left.+\log\frac{\lambda^3\left(s,m_1^2,m_2^2\right)}{m_1^2m_2^2s^2\mu^4}\right]\right\}+\frac{4}{3}\frac{\alpha_s}{\pi}\left\{\frac{s-(m_1\mp m_2)^2}{4s\pi^2}\left[\log\left(\frac{1+\omega}{1-\omega}\right)\left(s-m_1^2-m_2^2\right)\right.\right.$$
$$\left.\left.-\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}\right]+\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{16s\pi^2}R^2_{12}(s)\left[2+\frac{(m_1\mp m_2)^2}{s}\right]-\frac{1}{\pi^2}R_0(s)\right\}, \tag{19}$$

$$\frac{{\rm Im}\Pi^{\rm R}_{\rm S/P}(s)}{\pi} = \frac{4}{3}\frac{\alpha_s}{\pi}\rho^0_{\rm S/P}(s)\left\{-\frac{1}{\varepsilon_{\rm IR}}+2\log 4\pi-2\gamma+2-\log\frac{\lambda^3\left(s,m_1^2,m_2^2\right)}{m_1^2m_2^2s^2\mu^4}\right.$$
$$+\left(s-m_1^2-m_2^2\right)\overline{R}_{12}(s)-\overline{R}_{11}(s)-\overline{R}_{22}(s)-R^1_{12}(s)+\frac{R^2_{12}}{2}\frac{1}{s-(m_1\pm m_2)^2}$$
$$\left.+\frac{s-m_1^2-m_2^2}{\sqrt{\lambda(s,m_1^2,m_2^2)}}\log\left(\frac{1+\omega}{1-\omega}\right)\left[\frac{1}{\varepsilon_{\rm IR}}-2\log 4\pi+2\gamma-2+\log\frac{\lambda^3\left(s,m_1^2,m_2^2\right)}{m_1^2m_2^2s^2\mu^4}\right]\right\}, \tag{20}$$

the expressions of the $\overline{R}_{11}(s)$, $\overline{R}_{22}(s)$, $\overline{R}_{12}(s)$, $R_{12}^1(s)$ and $R_{12}^2(s)$ are given explicitly in Appendix. We have used the γ_5 symmetry of the interpolating currents to obtain the spectral densities $\frac{\mathrm{Im}\Pi_{\mathrm{A}}^{\mathrm{R}}(s)}{\pi}$ and $\frac{\mathrm{Im}\Pi_{\mathrm{P}}^{\mathrm{R}}(s)}{\pi}$.

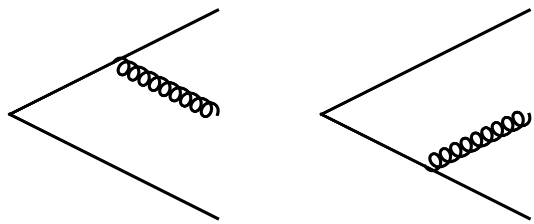

Fig. 6. The amplitudes for the real gluon emissions.

3. Analytical expressions of the total contributions

We add the contributions of the virtual and real gluon emissions together, and obtain the total perturbative spectral densities at the next-to-leading order

$$\begin{aligned}
\rho_{\mathrm{V/A}}^1(s) &= \frac{4}{3}\frac{\alpha_{\mathrm{s}}}{\pi}\rho_{\mathrm{V/A}}^0(s)\left\{\frac{1}{2}\overline{f}_{\pm}(s) - \overline{R}_{11}(s) - \overline{R}_{22}(s) + \left(s-m_1^2-m_2^2\right)\overline{R}_{12}(s)\right. \\
&-\frac{5}{6} + 2\log\frac{\sqrt[4]{m_1^7 m_2^7 s}}{\lambda\left(s,m_1^2,m_2^2\right)} + \frac{2\left(s-m_1^2-m_2^2\right)}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right) \\
&\left.\log\frac{\lambda\left(s,m_1^2,m_2^2\right)}{m_1 m_2 s} - \frac{2\left(s-m_1^2-m_2^2\right)}{3\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right) - R_{12}^1(s)\right\} \\
&+\frac{4}{3}\frac{\alpha_{\mathrm{s}}}{\pi}\left\{\frac{s-(m_1\mp m_2)^2}{4s\pi^2}\left[\log\left(\frac{1+\omega}{1-\omega}\right)\left(s-m_1^2-m_2^2\right) - \sqrt{\lambda\left(s,m_1^2,m_2^2\right)}\right]\right. \\
&+\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{16s\pi^2}R_{12}^2(s)\left[2+\frac{(m_1\mp m_2)^2}{s}\right] - \frac{1}{\pi^2}R_0(s) + \frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{s^2} \\
&\left.\left[\frac{1}{12\pi^2}\left(1-\frac{s-m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)\right) - \frac{(m_1\pm m_2)\left(f_{1\pm}(s)+f_{2\pm}(s)\right)}{32\pi^2}\right]\right\},
\end{aligned} \tag{21}$$

$$
\begin{aligned}
\rho^1_{\rm S/P}(s) &= \frac{4}{3}\frac{\alpha_{\rm s}}{\pi}\rho^0_{\rm S/P}(s)\left\{\frac{1}{2}\overline{f}_{\pm}(s) - \overline{R}_{11}(s) - \overline{R}_{22}(s) + \left(s - m_1^2 - m_2^2\right)\overline{R}_{12}(s)\right. \\
&-\frac{3}{2} + 2\log\frac{\sqrt[4]{m_1^7 m_2^7 s}}{\lambda\left(s, m_1^2, m_2^2\right)} + \frac{2\left(s - m_1^2 - m_2^2\right)}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)\log\frac{\lambda\left(s, m_1^2, m_2^2\right)}{m_1 m_2 s} \\
&\left.-R^1_{12}(s) + \frac{R^2_{12}(s)}{2}\frac{1}{s - (m_1 \pm m_2)^2}\right\} + \frac{4}{3}\frac{\alpha_{\rm s}}{\pi}\frac{3}{32\pi^2}\frac{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}}{s^2} \\
&\times\left\{\left(s + m_1^2 - m_2^2\right)\left(s + m_2^2 - m_1^2\right)\left[f_{1\pm}(s)m_1 \pm f_{2\pm}(s)m_2\right]\right. \\
&\left.\mp f_{1\pm}(s)m_2\left(s + m_1^2 - m_2^2\right)^2 - f_{2\pm}(s)m_1\left(s + m_2^2 - m_1^2\right)^2\right\}. \qquad (22)
\end{aligned}
$$

The infrared divergences $\frac{1}{\varepsilon_{\rm IR}}$, $\log\left(\frac{1+\omega}{1-\omega}\right)\frac{1}{\varepsilon_{\rm IR}}$ from the virtual and real gluon emissions are canceled out with each other, which is guaranteed by the Lee–Nauenberg theorem [11]. The spectral densities at the next-to-leading order do not have renormalization scale dependence, although the strong coupling constant $\alpha_{\rm s}(\mu)$ is energy scale dependent quantity. In Appendix, we present some necessary technical details for readers convenience, one can follow the necessary steps to check the calculations if one does not feel confident with the final expressions.

We compare the spectral densities of the order of $\mathcal{O}(\alpha_{\rm s})$ numerically to other results in Refs. [9, 10] by taking the energy scale $\mu = 5$ GeV, pole masses $m_1 = m_b = 4.8$ GeV, $m_2 = m_c = 1.5$ GeV, and observe that there are differences that cannot be neglected. We can use the present expressions to study the masses and decay constants of the scalar, pseudoscalar, vector and axialvector B_c mesons with the QCD sum rules in a systematic way.

4. Conclusion

In this article, we calculate the next-to-leading order perturbative contributions to the spectral densities of the mesonic two-point correlation functions of the vector, axialvector, scalar and pseudoscalar currents. In calculations, we assume that the quarks have unequal masses and use optical theorem to obtain the spectral densities directly, furthermore, we take the scheme of the dimensional regularization to regularize both the ultraviolet and infrared divergences. The ultraviolet and infrared divergent terms are canceled out with each other separately, the net spectral densities are free of divergences. The analytical expressions are applicable in many phenomenological analysis besides the QCD sum rules.

This work is supported by the National Natural Science Foundation, Grant Numbers 11075053, 11375063 and the Fundamental Research Funds for the Central Universities.

Appendix

Firstly, we write down the fundamental integrals in calculating the vertex corrections

$$
\begin{aligned}
V_{ab}(s) &= 16\pi^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_{\rm E}}{(2\pi)^D} \frac{x^a y^b \Gamma(3)}{\left[k_{\rm E}^2 + (xp_1 + yp_2)^2\right]^3}\,, \\
V(s) &= 16\pi^2 \left(1 - \frac{3}{2}\varepsilon_{\rm UV}\right) \int_0^1 dx \int_0^{1-x} dy \int \frac{d^D k_{\rm E}}{(2\pi)^D} \frac{k_{\rm E}^2 \Gamma(3)}{\left[k_{\rm E}^2 + (xp_1 + yp_2)^2\right]^3}\,,
\end{aligned} \tag{23}
$$

and carry out the integrals to obtain the following analytical expressions

$$
\begin{aligned}
V_{00}(s) &= \frac{1}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}} \Bigg\{ -\log\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{1}{\varepsilon_{\rm IR}} + \log\frac{s}{4\pi\mu^2} + \gamma\right) \\
&\quad + \frac{\log^2\left(1-\omega_1^2\right)}{4} - \log^2(1+\omega_1) + \frac{\log^2\left(1-\omega_2^2\right)}{4} - \log^2(1+\omega_2) \\
&\quad + 2\log(\omega_1+\omega_2)\log\left(\frac{1+\omega}{1-\omega}\right) - \log\omega_1 \log\left(\frac{1+\omega_2}{1-\omega_2}\right) \\
&\quad - \log\omega_2 \log\left(\frac{1+\omega_1}{1-\omega_1}\right) - {\rm Li}_2\left(\frac{2\omega_1}{1+\omega_1}\right) - {\rm Li}_2\left(\frac{2\omega_2}{1+\omega_2}\right) + \pi^2 \Bigg\}\,, \\
&= \overline{V}_{00}(s) - \frac{1}{\sqrt{\lambda\left(s, m_1^2, m_2^2\right)}} \log\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{1}{\varepsilon_{\rm IR}} + \log\frac{s}{4\pi\mu^2} + \gamma\right)\,, \\
V_{10}(s) &= \frac{1}{s}\left\{\frac{1}{2}\log\left(\frac{1-\omega_1^2}{1-\omega_2^2}\right) - \frac{1}{\omega_2}\log\left(\frac{1+\omega}{1-\omega}\right) + \log\frac{\omega_2}{\omega_1}\right\}\,, \\
V_{01}(s) &= V_{10}(s)|_{\omega_1 \leftrightarrow \omega_2}\,, \\
V_{20}(s) &= \frac{1}{2s}\Bigg\{ -\frac{\omega_1\omega_2}{\omega_1+\omega_2}\log\left(\frac{1+\omega}{1-\omega}\right) - \frac{\omega_1}{\omega_2(\omega_1+\omega_2)}\log\left(\frac{1+\omega}{1-\omega}\right) \\
&\quad + \frac{\omega_1}{\omega_1+\omega_2}\log\left(\frac{1-\omega_1^2}{1-\omega_2^2}\right) + \frac{2\omega_1}{\omega_1+\omega_2}\log\frac{\omega_2}{\omega_1} + 1 \Bigg\}\,,
\end{aligned}
$$

$$
\begin{aligned}
V_{02}(s) &= V_{20}(s)|_{\omega_1\leftrightarrow\omega_2}\,, \\
V_{11}(s) &= \frac{1}{2s}\left\{\frac{\omega_1\omega_2}{\omega_1+\omega_2}\log\left(\frac{1+\omega}{1-\omega}\right)-\frac{\omega_1-\omega_2}{2(\omega_1+\omega_2)}\log\left(\frac{1-\omega_1^2}{1-\omega_2^2}\right)\right. \\
&\left.-\frac{1}{\omega_1+\omega_2}\log\left(\frac{1+\omega}{1-\omega}\right)+\frac{\omega_1}{\omega_1+\omega_2}\log\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1+\omega_2}\log\frac{\omega_2}{\omega_1}-1\right\}, \\
V(s) &= \frac{1}{\varepsilon_{\rm UV}}+\log\frac{4\pi\mu^2}{s}-\gamma+1-\frac{2\omega_1\omega_2}{\omega_1+\omega_2}\log\left(\frac{1+\omega}{1-\omega}\right) \\
&-\frac{\omega_2}{\omega_1+\omega_2}\log\left(1-\omega_1^2\right)-\frac{\omega_1}{\omega_1+\omega_2}\log\left(1-\omega_2^2\right) \\
&-2\frac{\omega_1\log\omega_1+\omega_2\log\omega_2}{\omega_1+\omega_2}+2\log(\omega_1+\omega_2)\,, \\
&= \overline{V}(s)+\frac{1}{\varepsilon_{\rm UV}}+\log\frac{4\pi\mu^2}{s}-\gamma+1\,,
\end{aligned}
\tag{24}
$$

where

$$
\begin{aligned}
\omega_1 &= \frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s+m_1^2-m_2^2}\,, \\
\omega_2 &= \frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}{s+m_2^2-m_1^2}\,, \\
M &= \frac{m_1+m_2}{m_1-m_2}\,, \\
{\rm Li}_2(x) &= -\int_0^x dt\frac{\log(1-t)}{t}\,.
\end{aligned}
\tag{25}
$$

Secondly, we take the notation

$$
\int dps = \int \frac{d^{D-1}\vec{k}}{2E_k}\frac{d^{D-1}\vec{p}_1}{2E_{p_1}}\frac{d^{D-1}\vec{p}_2}{2E_{p_2}}\delta^D(p-k-p_1-p_2)
$$

for simplicity, and write down the analytical expressions of the three-body phase-space integrals

$$
\begin{aligned}
R_0(s) = \frac{1}{\pi^2}\int dps = \frac{m_1^2-m_2^2}{4}\log\left(\frac{M+\omega}{M-\omega}\right)-\frac{sm_1^2+sm_2^2-2m_1^2m_2^2}{4s} \\
\times\log\left(\frac{1+\omega}{1-\omega}\right)+\frac{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}\left(s+m_1^2+m_2^2\right)}{8s}\,,
\end{aligned}
$$

$$R_{11}(s) = \frac{sm_1^2}{\pi^2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}(2\pi)^{-4\varepsilon_{\rm IR}}\mu^{-2\varepsilon_{\rm IR}}\int dps\frac{1}{(k\cdot p_1)^2}$$

$$= \frac{1}{2\varepsilon_{\rm IR}} - \log 4\pi + \gamma - 1 + \log\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{m_1m_2s\mu^2} - \frac{s+m_1^2-m_2^2}{2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}$$

$$\times\log\left(\frac{1+\omega_1}{1-\omega_1}\right) - \frac{m_1^2-m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega_1}{1-\omega_1}\right) - \frac{s-m_1^2+m_2^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)$$

$$= \overline{R}_{11}(s) + \frac{1}{2\varepsilon_{\rm IR}} - \log 4\pi + \gamma - 1 + \log\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{m_1m_2s\mu^2}\,,$$

$$R_{22}(s) = R_{11}(s)|_{m_1\leftrightarrow m_2}\,,$$

$$R_{12}(s) = \frac{s}{\pi^2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}(2\pi)^{-4\varepsilon_{\rm IR}}\mu^{-2\varepsilon_{\rm IR}}\int dps\frac{1}{k\cdot p_1 k\cdot p_2}$$

$$= \frac{1}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\left\{\log\left(\frac{1+\omega}{1-\omega}\right)\left[\frac{1}{\varepsilon_{\rm IR}} - 2\log 4\pi + 2\gamma - 2\right.\right.$$

$$\left.+2\log\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{m_1m_2s\mu^2}\right] - 2\log\frac{m_1}{m_2}\log\left(\frac{M+\omega}{M-\omega}\right) - \log^2\left(\frac{1+\omega}{1-\omega}\right)$$

$$+2\log\frac{s}{\bar{s}}\log\left(\frac{1+\omega}{1-\omega}\right) - 4{\rm Li}_2\left(\frac{2\omega}{1+\omega}\right) + 2{\rm Li}_2\left(\frac{\omega-1}{\omega-M}\right) + 2{\rm Li}_2\left(\frac{\omega-1}{\omega+M}\right)$$

$$-2{\rm Li}_2\left(\frac{\omega+1}{\omega-M}\right) - 2{\rm Li}_2\left(\frac{\omega+1}{\omega+M}\right) - \frac{1}{2}{\rm Li}_2\left(\frac{1+\omega_1}{2}\right) - \frac{1}{2}{\rm Li}_2\left(\frac{1+\omega_2}{2}\right)$$

$$\left.-{\rm Li}_2\left(\omega_1\right) - {\rm Li}_2\left(\omega_2\right) + \frac{\log 2\log\left[(1+\omega_1)(1+\omega_2)\right]}{2} - \frac{\log^2 2}{2} + \frac{\pi^2}{12}\right\},$$

$$= \overline{R}_{12}(s) + \frac{1}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\log\left(\frac{1+\omega}{1-\omega}\right)\left[\frac{1}{\varepsilon_{\rm IR}} - 2\log 4\pi + 2\gamma - 2\right.$$

$$\left.+2\log\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{m_1m_2s\mu^2}\right]\,,$$

$$R_{12}^1(s) = \frac{s}{\pi^2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\int dps\frac{s-K^2}{k\cdot p_1 k\cdot p_2}$$

$$= \frac{s}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\left\{\log^2(1-\omega) - \log^2(1+\omega) + 2\log\frac{2s}{\bar{s}}\log\left(\frac{1+\omega}{1-\omega}\right)\right.$$

$$+2\mathrm{Li}_2\left(\frac{1-\omega}{2}\right)-2\mathrm{Li}_2\left(\frac{1+\omega}{2}\right)+2\mathrm{Li}_2\left(\frac{1+\omega}{1+M}\right)+2\mathrm{Li}_2\left(\frac{1+\omega}{1-M}\right)$$
$$-2\mathrm{Li}_2\left(\frac{1-\omega}{1-M}\right)-2\mathrm{Li}_2\left(\frac{1-\omega}{1+M}\right)\Bigg\},$$

$$R^2_{12}(s)=\frac{s}{\pi^2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\int dps\frac{\left(s-K^2\right)^2}{k\cdot p_1 k\cdot p_2}$$
$$=\frac{s^2}{\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\Bigg\{\log^2(1-\omega)-\log^2(1+\omega)+2\log\frac{4s}{\bar{s}}\log\left(\frac{1+\omega}{1-\omega}\right)$$
$$+2\mathrm{Li}_2\left(\frac{1-\omega}{2}\right)-2\mathrm{Li}_2\left(\frac{1+\omega}{2}\right)+2\mathrm{Li}_2\left(\frac{1+\omega}{1+M}\right)+2\mathrm{Li}_2\left(\frac{1+\omega}{1-M}\right)$$
$$-2\mathrm{Li}_2\left(\frac{1-\omega}{1-M}\right)-2\mathrm{Li}_2\left(\frac{1-\omega}{1+M}\right)+\frac{2\omega\bar{s}}{s}-\frac{\bar{s}}{s}(1+\omega^2)\log\left(\frac{1+\omega}{1-\omega}\right)\Bigg\}, \quad (26)$$

where $\bar{s}=s-(m_1-m_2)^2$.

In the following, we present some necessary technical details in calculating the integrals for both the virtual and real gluon emissions, one can check the calculations; here, we smear the lower index IR in $\varepsilon_{\mathrm{IR}}$ for simplicity

$$\int_0^1 dx\int_0^{1-x}dy\int\frac{d^Dk_{\mathrm{E}}}{(2\pi)^D}\frac{\Gamma(3)}{\left[k_{\mathrm{E}}^2+(xp_1+yp_2)^2\right]^3}=\int_0^1 dx\int_0^{1-x}dy\frac{(4\pi)^{-\varepsilon}\Gamma(1-\varepsilon)}{16\pi^2(xp_1+yp_2)^{2-2\varepsilon}}$$
$$=\int_0^1 dz\int_0^1 dtt\frac{(4\pi)^{-\varepsilon}\Gamma(1-\varepsilon)}{16\pi^2\left\{t^2s\left[z-\frac{\omega_1(1+\omega_2)}{\omega_1+\omega_2}\right]\left[z-\frac{\omega_1(1-\omega_2)}{\omega_1+\omega_2}\right]\right\}^{1-\varepsilon}}$$
$$=\frac{1}{16\pi^2\sqrt{\lambda\left(s,m_1^2,m_2^2\right)}}\Bigg\{-\log\left(\frac{1+\omega}{1-\omega}\right)\left(\frac{1}{\varepsilon}+\log\frac{s}{4\pi\mu^2}+\gamma\right)+\frac{\log^2\left(1-\omega_1^2\right)}{4}$$
$$-\log^2(1+\omega_1)+\frac{\log^2\left(1-\omega_2^2\right)}{4}-\log^2\left(1+\omega_2\right)+2\log(\omega_1+\omega_2)\log\left(\frac{1+\omega}{1-\omega}\right)$$
$$-\log\omega_1\log\left(\frac{1+\omega_2}{1-\omega_2}\right)-\log\omega_2\log\left(\frac{1+\omega_1}{1-\omega_1}\right)-\mathrm{Li}_2\left(\frac{2\omega_1}{1+\omega_1}\right)$$
$$-\mathrm{Li}_2\left(\frac{2\omega_2}{1+\omega_2}\right)+\pi^2\Bigg\}, \quad (27)$$

$$
\begin{aligned}
&\int dps \frac{\mu^{-2\varepsilon}}{k\cdot p_1 k\cdot p_2} = \frac{\pi^{\frac{5}{2}+2\varepsilon}\mu^{-2\varepsilon}}{\Gamma(1+\varepsilon)\Gamma\left(\frac{3}{2}+\varepsilon\right)} \int\limits_{(m_1+m_2)^2}^{s} dK^2 \frac{\left(s-K^2\right)^{1+2\varepsilon}}{(4s)^{1+\varepsilon}} \\
&\times \frac{\lambda^{1/2+\varepsilon}\left(K^2,m_1^2,m_2^2\right)}{(4K^2)^{1+\varepsilon}} \int\limits_{-1}^{1} d\cos\theta \\
&\times \frac{\left(1-\cos^2\theta\right)^{\varepsilon}}{\left[\frac{s-K^2}{2\sqrt{s}}\right]^2 \left[\frac{K^2+m_1^2-m_2^2}{2K} - \frac{\sqrt{\lambda\left(K^2,m_1^2,m_2^2\right)}}{2K}\cos\theta\right]\left[\frac{K^2+m_2^2-m_1^2}{2K} + \frac{\sqrt{\lambda\left(K^2,m_1^2,m_2^2\right)}}{2K}\cos\theta\right]} \\
&= \frac{\pi^{3+2\varepsilon}\mu^{-2\varepsilon}}{\Gamma\left(\frac{3}{2}+\varepsilon\right)\Gamma\left(\frac{3}{2}+\varepsilon\right)} \int\limits_{(m_1+m_2)^2}^{s} dK^2 \frac{1}{4K^2\left(s-K^2\right)^{1-2\varepsilon}} \left[1+\varepsilon\log\frac{\lambda\left(K^2,m_1^2,m_2^2\right)}{16sK^2}\right] \\
&\left\{2\log\left(\frac{1+u}{1-u}\right) - \varepsilon\left[\mathrm{Li}_2\left(\frac{1+u_1}{2}\right) + \mathrm{Li}_2\left(\frac{1+u_2}{2}\right) + 2\mathrm{Li}_2(u_1) + 2\mathrm{Li}_2(u_2)\right.\right. \\
&\left.\left. -4\log\left(\frac{1+u}{1-u}\right) - \log 2\log\left[(1+u_1)(1+u_2)\right] + \log^2 2 - \frac{\pi^2}{6}\right]\right\} \\
&= \frac{\pi^{3+2\varepsilon}\mu^{-2\varepsilon}}{\Gamma\left(\frac{3}{2}+\varepsilon\right)\Gamma\left(\frac{3}{2}+\varepsilon\right)} \frac{2m_1m_2}{(m_1-m_2)^2\left[s-(m_1-m_2)^2\right]^{1-2\varepsilon}} \\
&\times \int\limits_0^{\omega} du \frac{u}{\left(M^2-u^2\right)\left(\omega^2-u^2\right)^{1-2\varepsilon}} \left[1+\varepsilon\log\frac{\lambda\left(K^2,m_1^2,m_2^2\right)}{16sK^2} - 2\varepsilon\log\left(1-u^2\right)\right] \\
&\times \left\{2\int\limits_{-1}^{1} dx \frac{u}{u^2-x^2} - \varepsilon\left[\mathrm{Li}_2\left(\frac{1+u_1}{2}\right) + \mathrm{Li}_2\left(\frac{1+u_2}{2}\right) + 2\mathrm{Li}_2(u_1) + 2\mathrm{Li}_2(u_2)\right.\right. \\
&\left.\left. -4\log\left(\frac{1+u}{1-u}\right) - \log 2\log\left[(1+u_1)(1+u_2)\right] + \log^2 2 - \frac{\pi^2}{6}\right]\right\} \\
&= \frac{\pi^2}{s}\left\{\log\left(\frac{1+\omega}{1-\omega}\right)\left[\frac{1}{\varepsilon} + 2\log\pi + 2\gamma - 2 + 2\log\frac{\lambda^{3/2}\left(s,m_1^2,m_2^2\right)}{m_1m_2s\mu^2}\right]\right. \\
&-2\log\frac{m_1}{m_2}\log\left(\frac{M+\omega}{M-\omega}\right) - \log^2\left(\frac{1+\omega}{1-\omega}\right) + 2\log\frac{s}{\bar{s}}\log\left(\frac{1+\omega}{1-\omega}\right) - 4\mathrm{Li}_2\left(\frac{2\omega}{1+\omega}\right) \\
&+2\mathrm{Li}_2\left(\frac{\omega-1}{\omega-M}\right) + 2\mathrm{Li}_2\left(\frac{\omega-1}{\omega+M}\right) - 2\mathrm{Li}_2\left(\frac{\omega+1}{\omega-M}\right) - 2\mathrm{Li}_2\left(\frac{\omega+1}{\omega+M}\right) \\
&-\frac{1}{2}\mathrm{Li}_2\left(\frac{1+\omega_1}{2}\right) - \frac{1}{2}\mathrm{Li}_2\left(\frac{1+\omega_2}{2}\right) - \mathrm{Li}_2\left(\omega_1\right) - \mathrm{Li}_2\left(\omega_2\right) \\
&\left. +\frac{\log 2\log\left[(1+\omega_1)(1+\omega_2)\right]}{2} - \frac{\log^2 2}{2} + \frac{\pi^2}{12}\right\}, \qquad (28)
\end{aligned}
$$

where

$$
\begin{aligned}
u_1 &= \frac{\sqrt{\lambda\left(K^2, m_1^2, m_2^2\right)}}{K^2 + m_1^2 - m_2^2}\,, \\
u_2 &= \frac{\sqrt{\lambda\left(K^2, m_1^2, m_2^2\right)}}{K^2 + m_2^2 - m_1^2}\,, \\
u &= \sqrt{\frac{K^2 - (m_1 + m_2)^2}{K^2 - (m_1 - m_2)^2}}\,,
\end{aligned} \tag{29}
$$

and we have neglected the imaginary terms. The infrared divergent terms of the form $\log\left(\frac{1+\omega}{1-\omega}\right)\frac{1}{\varepsilon}$ come from the diagrams of the virtual gluon emissions (vertex corrections) and real gluon emissions are canceled out with each other.

In calculating the divergent integrals, we have used the following trick

$$
\begin{aligned}
&\int_0^\omega dx \frac{f(x,\varepsilon)}{(\omega - x)^{1-\varepsilon}}\left[1 + \varepsilon g(x)\right] = \int_0^\omega dx \frac{f(x,\varepsilon)}{(\omega - x)^{1-\varepsilon}} \\
&+\varepsilon \int_0^\omega dx \frac{f(x,\varepsilon)\left[g(x) - g(\omega)\right]}{(\omega - x)^{1-\varepsilon}} + \varepsilon \int_0^\omega dx \frac{f(x,\varepsilon) g(\omega)}{(\omega - x)^{1-\varepsilon}} \\
&= \int_0^\omega dx \frac{f(x,\varepsilon)}{(\omega - x)^{1-\varepsilon}} + \varepsilon \int_0^\omega dx \frac{f(x)\left[g(x) - g(\omega)\right]}{\omega - x} + \varepsilon \int_0^\omega dx \frac{f(x,\varepsilon) g(\omega)}{(\omega - x)^{1-\varepsilon}} \\
&= \int_0^\omega dx \frac{f(x,\varepsilon)}{(\omega - x)^{1-\varepsilon}} + \varepsilon \int_0^\omega dx \frac{f(x,\varepsilon) g(\omega)}{(\omega - x)^{1-\varepsilon}}\,,
\end{aligned} \tag{30}
$$

the functions $f(x,\varepsilon)$ and $g(x)$ have no poles in the range $x = 0 - \omega$ and $\varepsilon \to 0^+$.

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