

# RESIDUAL SYMMETRIES IN THE LEPTON MASS MATRICES\*

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*(Received October 21, 2013)*

It has been suggested that residual symmetries in the charged-lepton and neutrino mass matrices can possibly reveal the flavour symmetry group of the lepton sector. We review the basic ideas of this purely group-theoretical approach and discuss some of its results. Finally, we also list its shortcomings.

DOI:10.5506/APhysPolB.44.2331

PACS numbers: 11.30.Hv, 14.60.Pq

## 1. Introduction

Recent measurements of reactor neutrinos have clearly demonstrated that the mixing angle  $\theta_{13}$  of the lepton mixing matrix  $U = (U_{\alpha j})$  ( $\alpha = e, \mu, \tau$ ,  $j = 1, 2, 3$ ) and, therefore, the element  $|U_{e3}| \equiv s_{13}$  is non-zero. Consequently, the tri-bimaximal mixing (TBM) [1] is not viable anymore. Writing the mixing matrix in terms of its column vectors  $u_j$  ( $j = 1, 2, 3$ ), *i.e.*

$$U = (U_{\alpha j}) = (u_1, u_2, u_3), \quad (1)$$

we can say that  $u_3$  cannot have the TBM form. However, it could still be that  $u_1$  or  $u_2$  agrees with the corresponding column in TBM. These cases are called  $TM_1$  and  $TM_2$ , respectively, in [2]:

$$TM_1 : \quad u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \quad TM_2 : \quad u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \quad (2)$$

Let us compare the predictions of these two cases. In both cases,  $s_{12}^2$  and the product  $\cos \delta \tan 2\theta_{23}$ , where  $\delta$  is the CKM-type phase in  $U$ , are determined

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\* Presented at the XXXVII International Conference of Theoretical Physics "Matter to the Deepest" Ustroń, Poland, September 1–6, 2013.

by  $s_{13}$ . In the first case, these quantities are given by [2]

$$\text{TM}_1: \quad s_{12}^2 = 1 - \frac{2}{3c_{13}^2} < \frac{1}{3}, \quad \cos \delta \tan 2\theta_{23} \simeq -\frac{1}{2\sqrt{2}s_{13}} \left( 1 - \frac{7}{2}s_{13}^2 \right), \quad (3)$$

whereas in the second case, one obtains [2, 3]

$$\text{TM}_2: \quad s_{12}^2 = \frac{1}{3c_{13}^2} > \frac{1}{3}, \quad \cos \delta \tan 2\theta_{23} \simeq \frac{1}{\sqrt{2}s_{13}} \left( 1 - \frac{5}{4}s_{13}^2 \right). \quad (4)$$

In these formulas, we use the customary abbreviations  $c_{ij}^2 \equiv \cos^2 \theta_{ij}$  and  $s_{ij}^2 \equiv \sin^2 \theta_{ij}$ . Moreover, in the formulas for  $\cos \delta \tan 2\theta_{23}$ , small terms of the order of  $s_{13}^4$  have been neglected for simplicity. It should be noted that the best-fit values of  $s_{12}^2$  are smaller than  $1/3$  [4], but at any rate,  $1/3$  is in the  $3\sigma$  range of  $s_{12}^2$ . In that sense,  $\text{TM}_1$  fits the data slightly better than  $\text{TM}_2$ . Unfortunately, present data do not allow to determine the quantity  $\cos \delta \tan 2\theta_{23}$ .

Of course, it could also be that the “true” mixing matrix is not related to the tri-bimaximal mixing at all; in this case, none of the columns of the TBM matrix is a sensible approximation to one of the columns in the “true” mixing matrix.

Recently, a purely group-theoretical attempt to track down the flavour group  $G$  of the lepton sector has been put forward which postulates that residual symmetries in the charged-lepton and neutrino mass matrices are reflections of  $G$  and that the full flavour group can be determined by assembling the residual symmetries from both mass matrices [5, 6]. This approach is based on the fact that at low energies, the Standard Model gives the correct gauge structure of any extension trying to explain mass and mixing phenomena. Therefore, the left-handed charged and neutral lepton fields are not only in the same gauge doublets but also in the same multiplets with respect to  $G$ , a fact which is hidden by the spontaneous symmetry breaking. In this approach, it is possible that the flavour group determines one column of  $U$  in terms of numbers of a purely group-theoretical origin. It is also possible that two columns are determined, which means that, due to unitarity, the *full* mixing matrix  $U$  has a group-theoretical origin. In this context, we will see that  $\text{TM}_2$  plays a role.

The plan of this report is as follows. In Sec. 2, we discuss the idea of residual symmetries. In Sec. 3, we briefly review the results of [7] based on a computer-algebraic group scan. Then, we point out the connection between residual symmetries and roots of unity in Sec. 4. After applying this in Sec. 5 to find all flavour groups which enforce  $\text{TM}_1$ , we conclude in Sec. 6 by discussing some caveats of the method of residual symmetries.

## 2. Residual symmetries and lepton mixing

Though the method of residual symmetries can also be applied to Dirac fermions, here we confine ourselves to Majorana neutrinos. Then the mass Lagrangian in the lepton sector is given by

$$\mathcal{L}_{\text{mass}} = -\bar{\ell}_L M_\ell \ell_R + \frac{1}{2} \nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \text{H.c.}, \quad (5)$$

where  $C$  is the charge-conjugation matrix. The mass matrices of the charged leptons and neutrinos are diagonalized as

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \text{diag}(m_e^2, m_\mu^2, m_\tau^2) \quad \text{and} \quad U_\nu^T \mathcal{M}_\nu U_\nu = \text{diag}(m_1, m_2, m_3), \quad (6)$$

respectively, leading to the mixing matrix  $U = U_\ell^\dagger U_\nu$ . The fact that these matrices are diagonalizable can be reformulated as

$$V_\ell(\alpha)^\dagger M_\ell M_\ell^\dagger V_\ell(\alpha) = M_\ell M_\ell^\dagger, \quad V_\nu(\epsilon)^T \mathcal{M}_\nu V_\nu(\epsilon) = \mathcal{M}_\nu, \quad (7)$$

with unitary matrices  $V_\ell(\alpha)$ ,  $V_\nu(\epsilon)$  defined as

$$V_\ell(\alpha) = U_\ell \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) U_\ell^\dagger, \quad (8)$$

$$V_\nu(\epsilon) = U_\nu \text{diag}(\epsilon_1, \epsilon_2, \epsilon_3) U_\nu^\dagger. \quad (9)$$

Equation (7) holds for arbitrary  $\alpha_j$  and arbitrary  $\epsilon_j = \pm 1$ . In group-theoretical terms, the mass matrices are invariant under

$$V_\ell(\alpha) \in U(1) \times U(1) \times U(1), \quad V_\nu(\epsilon) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (10)$$

Obviously,  $V_\ell(\alpha)$  and  $V_\nu(\epsilon)$  depend on the vacuum expectation values and Yukawa coupling constants, and equation (7) contains no information beyond diagonalizability.

The idea of residual symmetries is the following. In a weak basis, the fields  $\ell_L$ ,  $\nu_L$  are in the same multiplet of the flavour group  $G$  under which the Lagrangian is invariant. The flavour group  $G$  is broken to different subgroups  $G_\ell$  and  $G_\nu$  in the charged-lepton and neutrino sector, respectively. From equation (7), we know that

$$G_\ell \subseteq U(1) \times U(1) \times U(1), \quad G_\nu \subseteq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (11)$$

For simplicity, we assume that there is one generator  $T$  of  $G_\ell$  and one generator  $S$  of  $G_\nu$ . Therefore, we have

$$T^\dagger M_\ell M_\ell^\dagger T = M_\ell M_\ell^\dagger, \quad S^T \mathcal{M}_\nu S = \mathcal{M}_\nu. \quad (12)$$

We furthermore assume that  $T$  has three different eigenvalues. Then  $T$  and  $S$  determine one column of  $U$ , as we will argue now.

Due to equation (11), we have

$$S^2 = \mathbb{1} \quad \Rightarrow \quad S = \pm \left( 2uu^\dagger - \mathbb{1} \right) \quad (13)$$

with a unit vector  $u$  and  $Su = \pm u$ . Since  $T$  commutes with  $M_\ell M_\ell^\dagger$  and has three different eigenvalues, we know that  $U_\ell^\dagger T U_\ell = \tilde{T}$  is diagonal. Therefore,  $U_\ell$  is determined by  $T$  and is thus independent of any parameters of the Lagrangian. For the rest of the argument, we use the following theorem.

**Theorem 1.** *If  $S^T \mathcal{M}_\nu S = \mathcal{M}_\nu$  with  $S = \pm(2uu^\dagger - \mathbb{1})$ , then  $\mathcal{M}_\nu u \propto u^*$ .*

Thus,  $u$  is, apart from a phase, one of the columns of  $U_\nu$  and, therefore,  $U_\ell^\dagger u$  is a column in the mixing matrix  $U$ . Because  $U_\ell^\dagger u$  is determined by the group, it does not contain parameters of the model.

If there are two matrices  $S_1, S_2$  with  $S_j^T \mathcal{M}_\nu S_j = \mathcal{M}_\nu$  and  $S_1 S_2 = S_2 S_1$ , two columns of  $U$  are determined and thus the complete mixing matrix.

There are two ways to tackle the mathematical problem of residual symmetries for the determination of possible flavour symmetry groups:

1. scanning classes of finite groups,
2. solving relations involving roots of unity.

### 3. Group scans

Scans of groups have, for instance, been performed in [7, 8] using GAP [9] and the small groups library [10]. This library contains all finite groups with the order up to 2 000, with the exception of the order of 1024. Here we want to discuss the results of [7]. The authors of this paper have assumed that  $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$ , i.e. there are two matrices  $S_j$  in  $G_\nu$ , and that the group produces mixing parameters  $s_{ij}^2$  within the  $3\sigma$  range of the fit results of [4]. The authors have performed two scans. In the first one, they allowed for ord  $G < 1536$ , with the exception of one group whose order is just 1536, and assumed that  $G_\ell$  is generated by  $\tilde{T} = \text{diag}(1, \omega, \omega^2)$  with  $\omega = e^{2\pi i/3}$ . It is amazing that only three groups, namely  $\Delta(6 \times 10^2)$ ,  $(\mathbb{Z}_{18} \times \mathbb{Z}_6) \rtimes S_3$  and  $\Delta(6 \times 16^2)$ , lead to acceptable mixing patterns. All three groups have  $\text{TM}_2$  and a trivial CKM-type phase. Therefore,  $s_{12}^2$  is given by equation (4). One can show [11] that, in the case of the three viable groups, for every  $s_{13}^2$  there are two solutions of  $s_{23}^2$  given by

$$s_{23}^2 = \frac{1}{2} \left( 1 \pm \frac{\sqrt{2s_{13}^2 - 3s_{13}^4}}{c_{13}^2} \right). \quad (14)$$

With this formula, the numbers in the third column in Table 3 of [7] are reproduced.

In the second scan, the authors of [7] have assumed that the group order is smaller than 512 but allowed for  $G_\ell$  all Abelian finite groups. In this case, no candidates were found.

#### 4. Residual symmetries and roots of unity

Now, we come to the second way of treating residual symmetries. Let us assume for simplicity that

- $G_\ell$  is generated by  $T$  and  $G_\nu$  by  $S$ , respectively,
- $\det S = 1$  and thus  $S = 2uu^\dagger - \mathbb{1}$ .

Then finiteness of the group  $G$  requires the existence of positive integers  $m, n$  such that

$$T^m = S^2 = (ST)^n = \mathbb{1}. \quad (15)$$

We denote the eigenvalues of  $T$  by  $e^{i\phi_\alpha}$ . If  $ST$  has eigenvalues  $\lambda_j$ , then  $\lambda_j^n = 1$ . If the unit vector  $u$  in  $S$  coincides with the  $i$ th column of  $U = (U_{\alpha j})$ , then the trace and determinant of  $ST$  give [6]

$$\sum_{\alpha=e,\mu,\tau} \left(2|U_{\alpha i}|^2 - 1\right) e^{i\phi_\alpha} = \lambda_1 + \lambda_2 + \lambda_3 \quad \text{and} \quad \prod_{\alpha} e^{i\phi_\alpha} = \lambda_1 \lambda_2 \lambda_3, \quad (16)$$

respectively. Thus we have a vanishing sum of six roots of unity plus the condition for the determinant. Equation (16) can be used in two ways. Departing from elements  $S$  and  $T$  of a known group  $G$ , one can search for suitable  $|U_{\alpha i}|^2$  ( $i = 1, 2, 3$ ). On the other hand, for a given column in  $U$ , one can try to find a suitable group  $G$ .

#### 5. $TM_1$ and roots of unity

As an application of equation (16), we want to investigate which groups can enforce  $TM_1$  [12]. In this case, the coefficients in equation (16) are  $2|U_{e1}|^2 - 1 = 1/3$  and  $2|U_{\mu 1}|^2 - 1 = 2|U_{\tau 1}|^2 - 1 = -2/3$ . Therefore, equation (16) leads to the vanishing sum

$$-e^{i\phi_e} + 2e^{i\phi_\mu} + 2e^{i\phi_\tau} + 3\lambda_1 + 3\lambda_2 + 3\lambda_3 = 0. \quad (17)$$

In order to solve this equation, one can apply a theorem of Conway and Jones [13], referring to all possible vanishing sums of roots of unity up to nine roots<sup>1</sup>. Amazingly, the solution of equation (17) allows very little freedom:

$$e^{i\phi_e} = \eta, \quad e^{i\phi_\mu} = \eta\omega, \quad e^{i\phi_\tau} = \eta\omega^2, \quad \lambda_1 = \epsilon, \quad \lambda_2 = -\epsilon, \quad \lambda_3 = \eta, \quad (18)$$

<sup>1</sup> Actually, theorem 1 in [12], quoted from the book of Miller, Blichfeldt and Dickson, is wrong and one has to use the theorem of Conway and Jones instead, in order to find the solution of equation (17). We thank R. Fonseca for pointing this out to us.

where  $\eta$  is an arbitrary root of unity and  $\epsilon = \pm i\eta$ . In the basis where the charged-lepton mass matrix is diagonal, which we indicate by a tilde on  $T$  and  $S$ , this solution leads to

$$\tilde{T} = \eta \operatorname{diag} (1, \omega, \omega^2) , \quad \tilde{S} = 2u_1 u_1^\dagger - \mathbb{1} \quad (19)$$

with  $u_1$  of equation (2). Clearly,  $\tilde{S}$  must have this form because we departed from  $\text{TM}_1$ . One can show that  $\tilde{T}$  and  $\tilde{S}$  generate the group  $\mathbb{Z}_q \times S_4$  with  $\eta$  being a primitive root of the order of  $q$  [12]. Thus with the method of residual symmetries, one finds an almost unique group for  $\text{TM}_1$ . For the generators of  $S_4$  in its three-dimensional irreducible representations, see *e.g.* [14].

It is instructive to go into another basis where  $S$  has a simpler form. Performing a similarity transformation with the matrix  $U_\omega$  defined for instance in [12], we obtain

$$S = U_\omega \tilde{S} U_\omega^\dagger = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad T = U_\omega \tilde{T} U_\omega^\dagger = \eta \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} . \quad (20)$$

In that basis,  $M_\ell M_\ell^\dagger$  is invariant under cyclic permutations and the eigenvector  $u$  of  $S$  with eigenvalue 1 is given by  $u = (0, 1, 1)^T / \sqrt{2}$ . Consequently, up to a basis transformation, the mechanism for  $\text{TM}_1$  developed here boils down to  $U_\omega^\dagger u = u_1$ , where  $u_1$  is the first column of TBM [12, 15].

## 6. Residual symmetries and caveats

Up to now, we have operated under the premises that the residual symmetries in  $M_\ell M_\ell^\dagger$  and  $\mathcal{M}_\nu$  really determine the flavour group  $G$  as a symmetry group of the Lagrangian. Let us be more precise now and denote by  $\bar{G}$  the group generated by the residual symmetries. What is the possible relationship between  $G$  and  $\bar{G}$ ? Clearly,  $\bar{G} \subset U(3)$  due to three families of fermions. Since the method is purely group-theoretical and uses only information contained in the mass matrices,  $\bar{G}$  can at most yield  $D(G)$ , the representation of  $G$  on the lepton gauge doublets. Moreover, there are models with accidental symmetries in the mass matrices, in which case  $\bar{G}$  not even a subgroup of  $D(G)$ . Finally, there are predictive models with total breaking of  $G$ . Clearly, there the method of residual symmetries is not applicable.

The author thanks the Organizers for their hospitality and the pleasurable atmosphere at the conference.

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