

# QUANTUM GEOMETRY OF SPACE IN SPINFOAM GRAVITY\*

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In these lectures we describe how a quantum geometry of space arises from the composition of angular momenta providing a realization of Penrose's spin-geometry.

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## 1. Introduction

Spins in quantum mechanics and the action of general relativity share a simple and surprising relation. This relation is at the roots of spinfoam gravity [1, 2]. In these lectures we describe its simplest realization focusing on the quantum geometry of an atom of space and its relation to the geometry of a classical polyhedron.

The Wigner  $6j$  symbol is an elementary object that appears in the theory of ‘composition of angular momenta’ in quantum mechanics. It is the simplest non-trivial invariant under rotations that can be built from Clebsch–Gordan coefficients only [3]. It turns out that this familiar quantity is related to the action of general relativity in 3 space-time dimensions. In the limit of large spins  $j_i \gg 1$ , the following asymptotic formula holds [4]

$$\left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \approx \frac{\hbar^{3/2}}{\sqrt{-6\pi i V}} e^{+\frac{i}{\hbar} S} + c.c. \quad (1)$$

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Here,  $S$  is a function obtained from the 3d Einstein–Hilbert action for a compact region  $\mathcal{R}$  (in units  $8\pi G = 1$ )

$$I_{\mathcal{R}}[g_{\mu\nu}] = \frac{1}{2} \int_{\mathcal{R}} d^3x \sqrt{g} R + \int_{\partial\mathcal{R}} d^2x \sqrt{h} K \quad (2)$$

as follows: the action is evaluated on the flat Euclidean metric,  $S = I_{\mathcal{R}}[\eta_{\mu\nu}]$ , and the region  $\mathcal{R}$  is chosen so that its induced geometry is the one of a flat Euclidean tetrahedron. Under these conditions, the Einstein–Hilbert action determines the building-block of the so-called Regge action  $S$ , [5].  $S$  depends only on a finite number of variables, specifically the lengths  $\ell_1, \dots, \ell_6$  of the six edges of the tetrahedron. The quantity  $V = \int_{\mathcal{R}} d^3x \sqrt{g}$  in (1) is the volume of the tetrahedron expressed as a function of the edge-lengths. The relation between the *spins*  $j_i$  and the edge-lengths  $\ell_i$  is

$$\ell_i = (j_i + 1/2) \hbar. \quad (3)$$

The asymptotic formula (1) holds in the classically allowed region in which a tetrahedron with edges of lengths  $\ell_i$  exists. Large spins  $j_i \gg 1$  correspond to a classical limit  $\hbar \rightarrow 0$  with the edge-lengths  $\ell_i$  fixed. Figure 1 shows how accurate formula (1) is.

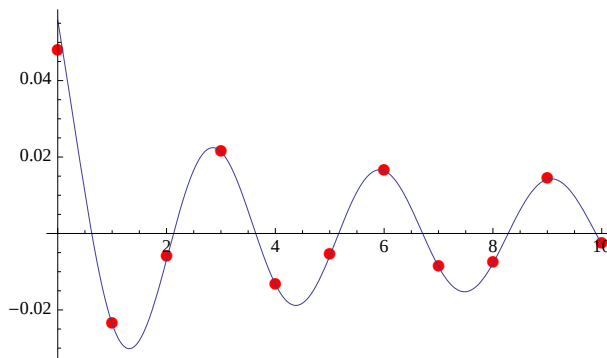


Fig. 1. The  $6j$  symbol  $\left\{ \begin{smallmatrix} 10 & 10 & 10 \\ 10 & 10 & j \end{smallmatrix} \right\}$  as a function of  $j$  (red dots) and the Ponzano–Regge approximation (continuous line).

This surprising relation discovered by Ponzano and Regge in 1968 provides the simplest and oldest example of Spinfoam Model for quantum gravity, a realization of the path-integral over spacetime geometries

$$Z = \int \mathcal{D}g_{\mu\nu} e^{\frac{i}{\hbar} S[g_{\mu\nu}]} \quad (4)$$

in terms of a sum over spins. The analogous quantity for Lorentzian General Relativity in 4 space-time dimensions has long been searched and found only recently [6–9].

In these lectures we describe how a quantum geometry of space arises from the composition of angular momenta.

## 2. The classical geometry of a Euclidean tetrahedron

A tetrahedron is the convex hull of four points in 3-dimensional Euclidean space  $\mathbb{R}^3$ . Its geometry can be described using a *triad* of edge-vectors  $\vec{e}_i$  ( $i = 1, 2, 3$ ). For instance, the volume of the tetrahedron is given by  $V = \frac{1}{3!} |\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3)|$ . From the triad we can compute the normal to the plane supporting a face of the tetrahedron, for instance  $\vec{E}_3 = \frac{1}{2} \vec{e}_1 \times \vec{e}_2$  as in Fig. 2. The normals  $\vec{E}_a$ , with  $a = 1, 2, 3, 4$ , are normalized to the area  $A_a$  of the associated face and can be chosen to be outward-pointing. Notice that they sum up to zero, as it happens for any closed surface.

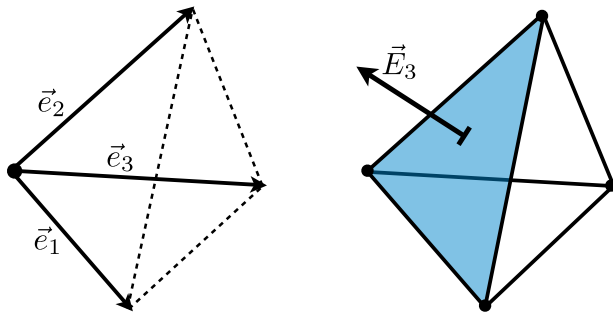


Fig. 2. Edge-vectors and normals to the faces of a tetrahedron.

A remarkable property of the face-normals is that they can be used as fundamental variables: a set of four vectors  $\vec{E}_a$  satisfying the closure condition

$$\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 = 0 \quad (5)$$

completely describes the geometry of a tetrahedron<sup>1</sup>. The norm of the vector  $\vec{E}_a$  is the area of the face  $a$  of the tetrahedron, so that we can write

$$\vec{E}_a = A_a \vec{n}_a, \quad (6)$$

<sup>1</sup> A counting of the number of independent variables up to rotations is in order: we have  $4 \times 3$  vector components,  $-3$  components from the closure condition,  $-3$  rotations, equals 6 independent variables. This number matches the number of edge-lengths that one can use to describe a tetrahedron.

where  $\vec{n}_a$  is the unit outward-pointing normal to the face  $a$ . The scalar product

$$\vec{E}_a \cdot \vec{E}_b = A_a A_b \cos \theta_{ab} \quad (7)$$

measures the dihedral angle  $\theta_{ab}$  between two faces of the tetrahedron. Similarly, the triple product of any three normals is related to the volume of the tetrahedron by the formula

$$V = \frac{\sqrt{2}}{3} \sqrt{\left| \vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3) \right|}. \quad (8)$$

The space of tetrahedra with faces of fixed areas  $A_a$  has a remarkable property: it has the structure of a phase space with respect to a natural choice of rotationally-invariant Poisson brackets induced by the canonical brackets  $\{n^i, n^j\} = \varepsilon^{ij}_k n^k$  for unit vectors on the sphere<sup>2</sup>. Consider two functions  $f(\vec{E}_a)$  and  $g(\vec{E}_a)$  on the space of shapes of tetrahedra with faces of fixed areas  $A_a$ . The Poisson brackets

$$\left\{ f(\vec{E}_a), g(\vec{E}_a) \right\} = \sum_{a=1}^4 \vec{E}_a \cdot \left( \frac{\partial f}{\partial \vec{E}_a} \times \frac{\partial g}{\partial \vec{E}_a} \right) \quad (9)$$

make this space into a phase space. The phase space of a tetrahedron with fixed areas is two dimensional and a set of canonical variables  $\{q, p\} = 1$  is given by

$$q = \text{angle between } \vec{E}_1 \times \vec{E}_2 \text{ and } \vec{E}_3 \times \vec{E}_4, \quad (10)$$

$$p = \left| \vec{E}_1 + \vec{E}_2 \right|. \quad (11)$$

Every geometric property of the tetrahedron can be understood as a function of  $q$  and  $p$ , for instance the volume of a tetrahedron is given by

$$\begin{aligned} V(q, p) = & \frac{1}{2} \sqrt{|\sin q|} \left( \left( 1 - \frac{(A_1 - A_2)^2}{p^2} \right) \left( 1 - \frac{(A_3 - A_4)^2}{p^2} \right) \right. \\ & \left. \times \left( 1 - \frac{(A_1 + A_2)^2}{p^2} \right) \left( 1 - \frac{(A_3 + A_4)^2}{p^2} \right) \right)^{1/4} p^{3/2} \end{aligned} \quad (12)$$

as shown in Fig. 3.

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<sup>2</sup> Equivalently, in terms of symplectic structure on the sphere  $S^2$ ,  $\omega = \varepsilon_{ijk} n^i dn^j \wedge dn^k$ .

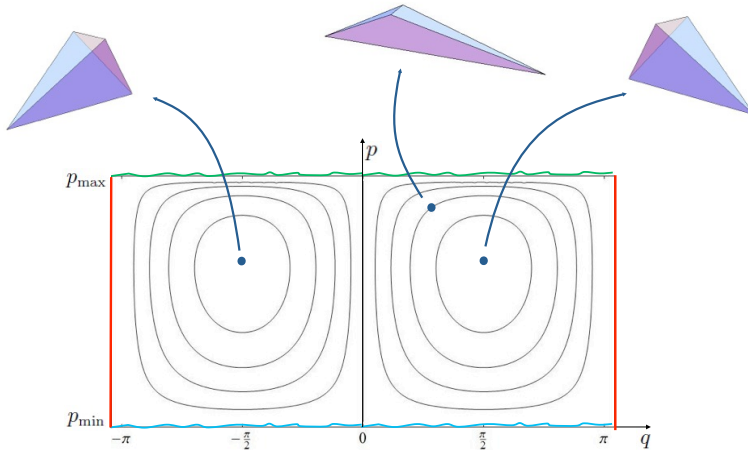


Fig. 3. The phase space of tetrahedra of fixed face-areas. The closed curves are orbits of constant volume.

### 3. Minkowski theorem and the phase space of polyhedra

There are two elegant mathematical results that allow to extend the previous construction from tetrahedra to convex polyhedra in three-dimensional Euclidean space. The first result is a theorem of Minkowski's [11] that states that the areas  $A_a$  and the unit-normals  $\vec{n}_a$  to the faces of the polyhedron fully characterize its shape<sup>3</sup>, see Fig. 4. We define the vectors  $\vec{E}_a = A_a \vec{n}_a$  and call  $\mathcal{P}_N$  the *space of shapes of polyhedra* with  $N$  faces of given areas  $A_a$

$$\mathcal{P}_N = \left\{ \vec{E}_a, a = 1 \dots N \mid \vec{E}_1 + \dots + \vec{E}_N = 0, \|\vec{E}_a\| = A_a \right\} / \text{SO}(3). \quad (13)$$

The second is a result of Kapovich and Millson's that states that the set  $\mathcal{P}_N$  has naturally the structure of a *phase space*, [12]. The Poisson brackets between two functions  $f(\vec{E}_a)$  and  $g(\vec{E}_a)$  on  $\mathcal{P}_N$  are

$$\{f, g\} = \sum_{a=1}^N \vec{E}_a \cdot \left( \frac{\partial f}{\partial \vec{E}_a} \times \frac{\partial g}{\partial \vec{E}_a} \right). \quad (14)$$

As in the case of the tetrahedron, these brackets arise (via symplectic reduction) from the rotationally-invariant Poisson brackets between functions  $f(\vec{E}_a)$  on  $(S^2)^N$ . Thus, we have that convex polyhedra with  $N$  faces of given areas form a  $2(N-3)$  dimensional phase space [10].

<sup>3</sup> More precisely, given a set of  $N$  positive numbers  $A_a$ , and  $N$  unit-vectors  $\vec{n}_a$  satisfying the condition  $\sum_a A_a \vec{n}_a = 0$ , there always exists a convex polyhedron having these data as areas and normals to its faces. Moreover, up to rotations  $\text{SO}(3)$ , the polyhedron is unique.

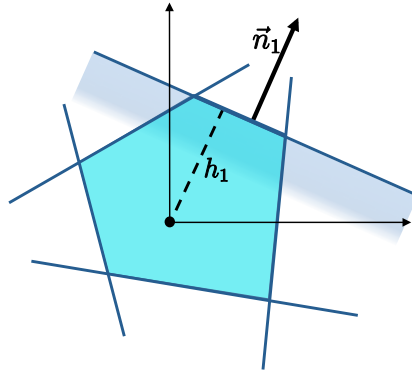


Fig. 4. A convex polyhedron can be obtained starting from  $N$  planes passing through the origin of 3d Euclidean space. Moving each plane in the direction of its normal  $\vec{n}_a$  defines a convex hull, a polyhedron. Adjusting the distance  $h_a$  of a plane from the origin, changes the areas  $A_a$  of the faces of the polyhedron. Can we use this procedure to build a polyhedron with faces of given areas  $A_a$  satisfying the closure condition  $\sum_a A_a \vec{n}_a = 0$ ? The Minkowski theorem states that such polyhedron exists and, up to rotations, is unique. A variational algorithm to reconstruct the polyhedron can be found in [10].

Canonical variables on this phase space can be chosen as follows: consider the set of vectors  $\vec{p}_i = \sum_{a=1}^{i+1} \vec{E}_a$ , where  $i = 1, \dots, N-3$ ; we define the coordinate  $q_i$  as the angle between the vectors  $\vec{p}_i \times \vec{E}_{i+1}$  and  $\vec{p}_i \times \vec{E}_{i+2}$ , and the momentum variable  $p_i = \|\vec{p}_i\|$  as the norm of the vector  $\vec{p}_i$ . From (14), it follows that these are canonically conjugate variables,  $\{q_i, p_j\} = \delta_{ij}$ .

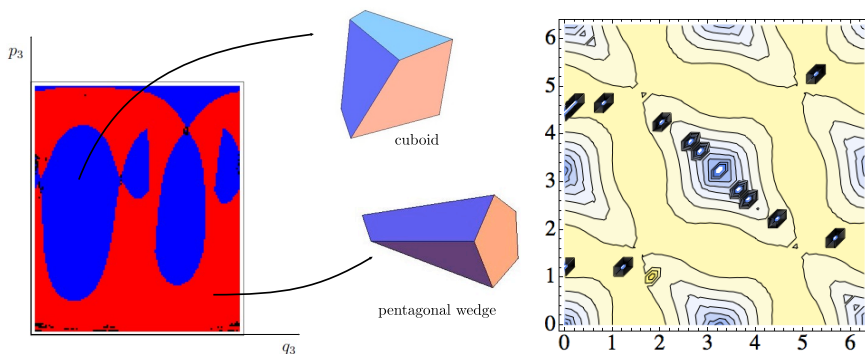


Fig. 5. Left panel: a portion of the phase space of polyhedra with  $N = 6$  faces. The black (blue) region corresponds to cuboids, the dark grey (red) region to pentagonal wedges [10]. Right panel: a portion of the phase space of polyhedra with  $N = 5$  faces with volume orbits showing a chaotic behavior [16].

Every geometric quantity, *e.g.* the length of an edge or the volume of the polyhedron [13], is a function of the canonical variables  $(q_i, p_i)$ . The problem of determining this function is well-defined but not immediate to solve. The reason is that we have to reconstruct first the shape of the polyhedron from the normals to its faces, or equivalently from the point in phase space [10]. In general, the problem can be solved numerically using Lasserre's algorithm [14] as shown in Fig. 5 (a). In the case of a pentahedron  $N = 5$  the problem has been solved analytically, and an expression of the volume  $V(q_1, q_2, p_1, p_2)$  as a function in phase space is available [15]. It is interesting to notice that the classical dynamics of this system is strongly chaotic [15, 16], Fig. 5 (b).

#### 4. Atoms of space, spin-geometry, and quantum polyhedra

In quantum mechanics a spin system identifies a quantum direction in space. A remarkable idea proposed by Penrose in 1971 is that the angles between these quantum directions define a geometry, Penrose's 'spin-geometry', and can provide the elementary building-block of quantum space [17–20]. This model for an *atom of space* has later been shown to coincide with the notion of 'quantum polyhedron', the quantization of the classical system described in the previous section [10, 21].

The simple quantum system that plays the role of the atom of space consists of  $N$  spins  $j_1 \otimes \cdots \otimes j_N$  in a collective state  $|i\rangle$  that is rotationally invariant<sup>4</sup>

$$|i\rangle = \sum_{m_1 \cdots m_N} i^{m_1 \cdots m_N} |j_1 m_1\rangle \cdots |j_N m_N\rangle. \quad (15)$$

The observables of the system are the rotational invariant operators that can be built from the angular momenta  $\vec{L}_a$  only. We will use dimensionful quantities  $\vec{E}_a$  defined as<sup>5</sup>

$$\vec{E}_a = 8\pi G\hbar\gamma \vec{L}_a, \quad (16)$$

where  $\gamma > 0$  is a dimensionless constant to be identified with the Immirzi parameter. As the state  $|i\rangle$  of our system is rotationally invariant<sup>6</sup>, we have

$$(\vec{E}_1 + \cdots + \vec{E}_N) |i\rangle = 0 \quad (17)$$

<sup>4</sup> The space of rotationally invariant states can be understood as the ground state with  $H = 0$  of the Hamiltonian  $H = \vec{L}_{\text{tot}}^2$ , where  $\vec{L}_{\text{tot}} = \vec{L}_1 + \cdots + \vec{L}_N$  is the total spin of the system. The ground state of  $H$  is, in general, degenerate and, when  $H = 0$ , the associated eigenspace is the Hilbert space of intertwiners  $\text{Inv}(j_1 \otimes \cdots \otimes j_N)$ .

<sup>5</sup> Here we use units  $c = 1$ , so that  $\sqrt{G\hbar}$  has the dimensions of length, the Planck length.

<sup>6</sup> The proof is immediate: a finite rotation of the system is generated by the unitary operator  $U(\vec{\alpha}) = \exp(i\vec{\alpha} \cdot \vec{L}_{\text{tot}})$ ; the invariance of the state under rotations is  $U(\vec{\alpha})|i\rangle = |i\rangle$  for all rotation parameters  $\vec{\alpha}$ . Expanding at the linear order in small  $\vec{\alpha}$  one recovers the closure condition.

a quantum closure condition analogous to Eq. (5). The geometric interpretation of the observables  $\vec{E}_a$  comes from identifying them with the normals to planes passing through a point in three-dimensional Euclidean space. In particular, the Penrose ‘metric’ operator  $\hat{g}_{ab}$  defined as

$$\hat{g}_{ab} = \vec{E}_a \cdot \vec{E}_b \quad (18)$$

measures the angle  $\theta_{ab}$  between two planes  $a$  and  $b$ <sup>7</sup>. A basis of intertwiner states  $|i_{k_1 \dots k_{N-3}}\rangle$  is a basis of eigenstates of a maximal commuting set of operators  $\hat{g}_{ab}$ . For instance, the spectrum of  $\hat{g}_{12}$  is given by

$$\hat{g}_{12} |i_{k_1 \dots k_{N-3}}\rangle = \frac{k_1(k_1 + 1) - j_1(j_1 + 1) - j_2(j_2 + 1)}{2} |i_{k_1 \dots k_{N-3}}\rangle. \quad (19)$$

As explained in the previous section, the point of intersection of the  $N$  planes can be inflated into a polyhedron by moving the planes away from the origin. This defines a polygon on each plane, *i.e.* a face of the polyhedron. The norm of the operator  $\vec{E}_a$  measures the area  $A_a$  of this face

$$A_a = \sqrt{\vec{E}_a \cdot \vec{E}_a}. \quad (20)$$

Its eigenvalues are immediate to compute and every state  $|i\rangle$  in our Hilbert space is an eigenstates of the area operator

$$A_a |i\rangle = 8\pi G\hbar\gamma \sqrt{j_a(j_a + 1)} |i\rangle. \quad (21)$$

The spectrum of the area is discrete and gapped, with a Planck scale gap  $a_0 = 8\pi G\hbar\gamma\sqrt{3}/2$  corresponding to the minimum non-trivial spin  $j_a = 1/2$ .

The system that we have described plays the role of atom of space in spinfoam gravity. It can be understood as a quantum polyhedron as it can be obtained by quantizing a classical dynamical system: a convex polyhedron with canonical Poisson brackets. This is analogous to the case of the hydrogen atom, a purely quantum system that can be defined via the quantization of a classical particle in a Keplerian orbit. In the next section, we discuss coherent states and the semiclassical behavior of the quantum system.

## 5. Heisenberg uncertainty relations for quantum geometry

Different components of the angular momentum do not commute  $[L^i, L^j] = i\epsilon^{ij}_k L^k$ . As a result the dispersions  $\Delta L^i$  on any spin state satisfy the uncertainty relations<sup>8</sup>

$$\Delta L^i \Delta L^j \geq \frac{1}{2} \left| \epsilon^{ij}_k \langle L^k \rangle \right|. \quad (22)$$

<sup>7</sup> The angle operator is defined as  $\theta_{ab} = \arccos(\hat{g}_{ab}/\sqrt{\hat{g}_{aa}\hat{g}_{bb}})$ , [20].

<sup>8</sup> As usual  $\Delta A \equiv \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ , where  $\langle A \rangle \equiv \langle s|A|s \rangle$  is the expectation value of the operator  $A$  on the state  $|s\rangle$ .



This is also the behavior of the quantum directions  $\vec{E}_a$ : the atom of space has a non-commutative quantum geometry. The phenomenon is most clearly illustrated in terms of the Penrose metric operator  $\hat{g}_{ab}$ . Consider three quantum planes  $a$ ,  $b$ ,  $c$ , and the angle between  $a, b$  and  $a, c$ . The associated Penrose operators do not commute

$$[\hat{g}_{ab}, \hat{g}_{ac}] = i 8\pi G \hbar \gamma \vec{E}_a \cdot (\vec{E}_b \times \vec{E}_c) \quad (23)$$

and the commutator measures the linear independence of the three quantum planes. The Heisenberg uncertainty relation for the quantum geometry reads

$$\Delta \hat{g}_{ab} \Delta \hat{g}_{ac} \geq \frac{1}{2} 8\pi G \hbar \gamma \left| \left\langle \vec{E}_a \cdot (\vec{E}_b \times \vec{E}_c) \right\rangle \right|. \quad (24)$$

As a result, the shape of a quantum polyhedron is fuzzy: if we try to determine with precision the angle between the planes  $a$  and  $b$ , then we lose control of the angle between the planes  $a$  and  $c$  unless the three are coplanar.

At the classical level we saw that  $q_i$  and  $p_i$  are canonical variables on the phase space of polyhedra. At the quantum level they correspond to operators with canonical commutation relations,  $[\hat{q}_i, \hat{p}_j] = i 8\pi G \hbar \gamma \delta_{ij}$ . Their dispersions satisfy uncertainty relations that are simpler than the ones we have seen above

$$\Delta \hat{q}_i \Delta \hat{p}_i \geq 4\pi G \hbar \gamma. \quad (25)$$

The geometric interpretation is particularly clear in the case of the tetrahedron ( $N = 4$ ): the states  $|i_k\rangle$  that we use as a basis of the Hilbert space have definite angle between two faces of the tetrahedron, *i.e.*  $\Delta \hat{p}_i = 0$ ; as a result the angle  $\hat{q}_i$  between two opposite edges of the tetrahedron has maximal dispersion.

Coherent states provide an over-complete basis of the Hilbert space of the quantum polyhedron such that the uncertainty relations (25) are saturated [22, 23]. The simplest way to introduce them is to start from Bloch coherent states for a spin system, *i.e.* states that saturate the uncertainty relation (22), [24, 25]. The state  $|j, j\rangle$  pointing in the  $z$  direction does it<sup>9</sup>, and all the others can be obtained by simply rotating this state in the direction  $\vec{n}$

$$|j, \vec{n}\rangle \equiv U(R)|j, j\rangle = \sum_{m=-j}^{+j} \phi_m(\vec{n}) |j, m\rangle, \quad (26)$$

where  $R$  is a rotation from the  $z$  direction to  $\vec{n}$  and  $\phi_m(\vec{n}) = \langle j, m | U(R) | j, j \rangle$ . These states point in the direction  $\vec{n}$

$$\langle \vec{L} \rangle = j \vec{n} \quad (27)$$

<sup>9</sup> The state  $|j, j\rangle$  has  $\langle L_z \rangle = j$ ,  $\langle L_x \rangle = \langle L_y \rangle = 0$ , and  $\Delta L_z = 0$ ,  $\Delta L_x = \Delta L_y = \sqrt{j/2}$ . As a result it saturates the uncertainty relation  $\Delta L_x \Delta L_y = \frac{1}{2} \langle L_z \rangle$ .

and have dispersion  $\Delta(\vec{L} \cdot \vec{n}) = \sqrt{\frac{1-(\vec{n} \cdot \vec{m})^2}{2}} j$ . As a result, the relative dispersion vanishes in the limit of large spin

$$\frac{\Delta(\vec{L} \cdot \vec{n})}{|\langle \vec{L} \rangle|} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (28)$$

Moreover, they provide a resolution of the identity in  $V^{(j)}$

$$\mathbb{1}_j = \frac{2j+1}{4\pi} \int_{S^2} d\vec{n} |j, \vec{n}\rangle \langle j, \vec{n}|. \quad (29)$$

Coherent states for quantum polyhedra  $|i(\vec{n}_a)\rangle$ , also called coherent intertwiners, are defined as the rotational invariant projection of  $N$  coherent spins  $|j_a, \vec{n}_a\rangle$  satisfying the closure constraint  $\sum_a j_a \vec{n}_a = 0$ . Their explicit expression is<sup>10</sup>

$$|i(\vec{n}_a)\rangle = \sum_m \Phi_{m_1 \dots m_N}(\vec{n}_a) |j_1, m_1\rangle \dots |j_N, m_N\rangle, \quad (30)$$

with

$$\Phi_{m_1 \dots m_N}(\vec{n}_a) = \int_{\text{SU}(2)} dh \prod_{a=1}^N \langle j_a, m_a | U(h) | j_a, \vec{n}_a \rangle. \quad (31)$$

These states are peaked on the polyhedron with normals  $\vec{n}_a$ , and the relative dispersion of geometric observables vanish in the limit  $j_a \rightarrow \infty$ : the classical limit arises at large quantum numbers, *i.e.* large spins. This regime corresponds to a size of the polyhedron that is large compared to the Planck scale, for instance the area of a face being much larger than the area gap  $a_0 = 8\pi G\hbar\gamma\sqrt{3}/2$ . Formally large  $j$  corresponds to the limit  $8\pi G\hbar\gamma \rightarrow 0$ . In this limit the Heisenberg uncertainty relations (25) become trivial.

The shape of a classical polyhedron is completely coded in the canonical variables  $(q_i, p_i)$  on phase space, in particular the normals  $\vec{n}_a$  can be computed from them. It is useful to write the coherent states as functions on phase space  $|i(q_i, p_i)\rangle \equiv |i(\vec{n}_a(q_i, p_i))\rangle$ . The resolution of the identity on the Hilbert space of quantum polyhedra can then be written as an integral on phase space as [22]

$$\mathbb{1} = \int_{\mathcal{P}_N} d\mu(q_i, p_i) |i(q_i, p_i)\rangle \langle i(q_i, p_i)|. \quad (32)$$

<sup>10</sup> This formula is obtained from the definition  $|i(\vec{n}_a)\rangle \equiv P |j_1, \vec{n}_1\rangle \dots |j_N, \vec{n}_N\rangle$ , by writing the projector to the rotationally-invariant space as an integral over the group  $\text{SU}(2)$ ,  $P = \int d\mu(\vec{\alpha}) \exp(i\vec{\alpha} \cdot \vec{L}_{\text{tot}}) = \int_{\text{SU}(2)} dh U(h)$ .

This formula shows that we can write any quantum state of an atom of space as a superposition of coherent quantum polyhedra.

It is interesting to connect these recent developments with the original idea proposed by Penrose [17, 18]. The spin-geometry theorem states that there exist collective spin states such that, in the classical limit, the expectation value of the Penrose metric reproduces the scalar products of a set of  $N$  vectors  $\vec{v}_a$  in 3d Euclidean space,  $\langle \hat{g}_{ab} \rangle = \vec{v}_a \cdot \vec{v}_b$ , and the relative dispersions vanish. The coherent states for an atom of space discussed above provide a concrete example of such states and play a central role in spinfoam gravity.

## 6. Quantum volume of an atom of space

At the classical level, a convex polyhedron has a well-defined volume. In the simplest case of a tetrahedron, the expression of the volume is given by Eq. (33). The associated operator in the quantum theory is immediate to define, it is given by

$$\hat{V} = \frac{\sqrt{2}}{3} \sqrt{|\vec{E}_1 \cdot (\vec{E}_2 \times \vec{E}_3)|}, \quad (33)$$

where now  $\vec{E}_a = 8\pi G\hbar\gamma \vec{L}_a$  are quantum normals. To compute the spectrum of the volume of a quantum tetrahedron it is useful to introduce the operator  $Q$

$$W = \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3). \quad (34)$$

The quantum volume and the operator  $W$  share the same eigenvectors  $|w_\alpha\rangle$ , and the eigenvalues are simply related by  $v_\alpha = (8\pi G\hbar\gamma)^{3/2} \frac{\sqrt{2}}{3} \sqrt{|w_\alpha|}$ . We compute the spectrum in the simplest non-trivial cases: four spins with their minimum non-zero value, *i.e.*  $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}$ . In this case, the Hilbert space is two dimensional and the matrix elements of  $W$  in the  $|i_k\rangle$  basis are immediate to compute using standard relations for Pauli matrices<sup>11</sup>

$$W_i^j = \langle i | \vec{L}_1 \cdot (\vec{L}_2 \times \vec{L}_3) | j \rangle = \begin{pmatrix} 0 & i\frac{\sqrt{3}}{4} \\ -i\frac{\sqrt{3}}{4} & 0 \end{pmatrix}. \quad (35)$$

The eigenvalues and the eigenvectors,  $W|w_\pm\rangle = w_\pm|w_\pm\rangle$ , are given by

$$w_\pm = \pm \frac{\sqrt{3}}{4}, \quad |w_\pm\rangle = \frac{|0\rangle \pm i|1\rangle}{\sqrt{2}}. \quad (36)$$

<sup>11</sup> We use the notation  $|0\rangle \equiv |i_{k=0}\rangle$ ,  $|1\rangle \equiv |i_{k=1}\rangle$ , and  $i, j = 0, 1$ .

As a result the eigenvalues of the volume are twice degenerate and given by

$$v_{\pm} = (8\pi G\hbar\gamma)^{3/2} \frac{\sqrt{2}}{3} \sqrt{\frac{\sqrt{3}}{4}} \approx (8\pi G\hbar\gamma)^{3/2} \times 0.310. \quad (37)$$

For larger spins the matrix elements of  $W$  can be computed in a closed form using the standard methods of composition of angular momenta<sup>12</sup>. In general, one has to resort to numerical methods to diagonalize the matrix  $W_i^j$ . Large eigenvalues can be derived via the WKB approximation or simply by applying the Bohr–Sommerfeld quantization to the classical phase space of a tetrahedron. Some eigenvalues are shown in Fig. 6.

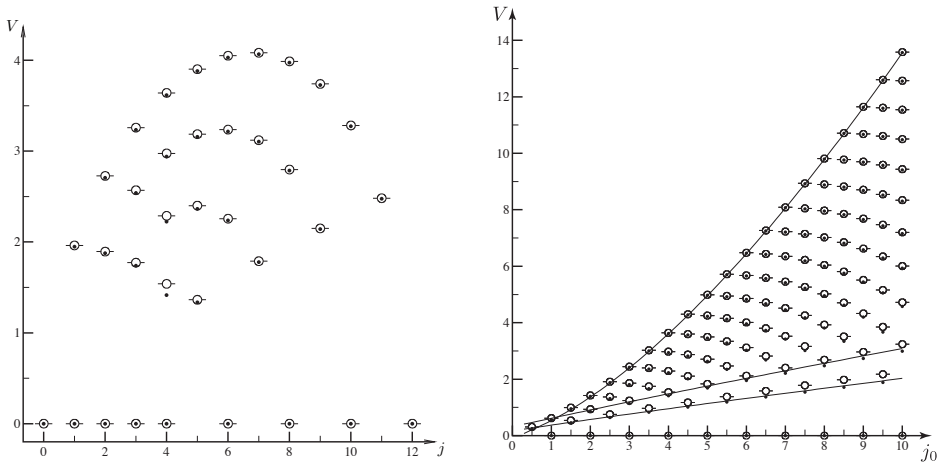


Fig. 6. Volume spectrum in units  $8\pi G\hbar\gamma = 1$ . On the left, spins  $4 \otimes 4 \otimes 4 \otimes j$ , on the right  $j \otimes j \otimes j \otimes j$ . The circles are volume eigenvalues computed numerically, the dots are volume eigenvalues computed via a WKB approximation [26].

The case  $N = 4$  of a quantum tetrahedron is the simplest non-trivial one<sup>13</sup>. For an atom of space with larger  $N$  the volume can be similarly defined starting from the expression of the volume of the classical polyhedron and quantizing the normals  $\vec{E}_a$ . In the process a choice of operator ordering is needed. A simple choice is to use the over-complete basis of coherent polyhedra to define the operator starting from its classical expression  $V = V(q_i, p_i)$  as a function on phase space

$$\hat{V} = \int_{\mathcal{P}_N} d\mu(q_i, p_i) V(q_i, p_i) |i(q_i, p_i)\rangle \langle i(q_i, p_i)|. \quad (38)$$

<sup>12</sup> See for instance App. A of [26] for an elementary derivation.

<sup>13</sup> For  $N = 3$  and 2 the volume vanishes at the classical and at the quantum level [21].

Notice that, in general, an eigenvector of the volume corresponds to a superposition of coherent polyhedra with very different shape; for instance in the case  $N = 6$ , a superposition of cuboids and pentagonal wedges (see Fig. 5). The operator has a discrete spectrum. Moreover, as the classical volume has a chaotic behavior, the quantum volume shows the phenomenon of ‘level repulsion’ and Wigner surmise. This phenomenon is expected to produce a gap in the spectrum of the volume<sup>14</sup>[15].

## 7. Spin-network states and quantum space

Consider a network  $\Gamma$  consisting of  $N$  nodes connected by  $L$  links. To each link  $\ell$  we associate a state  $|j_\ell, m_\ell\rangle|j_\ell, m'_\ell\rangle$  of two spins with the same  $j_\ell$ . Each of the two spins lives at an endpoint of the link. The Hilbert space of the system is simply given by a tensor product over the links of factors  $V^{(j_\ell)} \otimes V^{(j_\ell)}$ . Equivalently, we can organize the spins in groups sitting at nodes  $n$  of the network,  $\bigotimes_{\ell|n \in \partial\ell} V^{(j_\ell)}$ . We are interested in configurations of this system such that the spins sitting at each node are in a rotationally-invariant state. These states form a Hilbert space

$$\mathcal{H}_{\Gamma, j_\ell} = \bigotimes_{n \in \Gamma} \mathcal{H}_n \quad (39)$$

with  $\mathcal{H}_n = \text{Inv}(\bigotimes_{\ell|n \in \partial\ell} V^{(j_\ell)})$ : the system consists of an atom of space at each node of the network. We can describe the state of the system in terms of the states  $i_n$  of each atom of space at a node  $n$ , with the network  $\Gamma$  coding which nodes are connected by a link and thus share the same spin  $j_\ell$ . The state  $|\Gamma, j_\ell, i_n\rangle$  that we have described is a spin-network state.

Spin-network states play a central role in Loop Quantum Gravity and in Spin Foams [27–29]. The picture that arises is of a collection of neighboring quantum polyhedra that make quantum space (see Fig. 7). The simplest non-trivial example of a spin-network state is the following. Consider a triangulation  $\Delta_3$  of a 3-manifold; the dual network  $\Gamma = \Delta_3^*$  has a node per tetrahedron in  $\Delta_3$  and two nodes connected by a link if two tetrahedra share a face. The state that we consider has all spins equal to the lowest non-trivial spin,  $j_\ell = 1/2$ . The state of the atom of space at each node is still to be chosen. The associated Hilbert space  $\mathcal{H}_2 = \text{Inv}(\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2})$  is 2-dimensional with basis  $|0\rangle \equiv |i_{k=0}\rangle$ ,  $|1\rangle \equiv |i_{k=1}\rangle$ . Therefore, the system

<sup>14</sup> At fixed  $N$  and fixed largest spin  $j$  the Hilbert space of the system is finite dimensional, the spectrum of the volume is discrete and necessarily has a gap, *i.e.* a smallest non-vanishing eigenvalue of the volume. The question of the presence of a gap arises only in the limit in which the dimension  $d = \dim \text{Inv}(\bigotimes_{a=1}^N V^{(j_a)})$  of the Hilbert space diverges; for instance for large spins  $j \rightarrow \infty$  at fixed  $N$ , or for large  $N$ .

consists of  $N$  qubits sitting at the nodes of a network

$$|\Gamma = \Delta_3^*, j_\ell = 1/2, i_n\rangle \in \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2. \quad (40)$$

A region  $R$  is defined as a set of connected nodes in the network, and the boundary  $\partial R$  of the region as the set of links that connect nodes in  $R$  with nodes outside. The quantum volume of the region  $R$  is simply given by the sum of the volumes associated to each atom of space in the region<sup>15</sup>. The quantum area of the boundary  $\partial R$  of the region is given by the sum of area operators for each spin on a link that connects a node in  $R$  with a node outside  $R$ . Notice that the state that we have just described consists of a collection of atoms of space that are completely uncorrelated. The dynamics introduces correlations between the different nodes, and, in general, the physically relevant states are entangled [30]. These quantum correlations result in non-trivial two-point correlations functions for the Penrose metric operator  $\hat{g}_{ab}$ , and in general propagating degrees of freedom [31, 32].

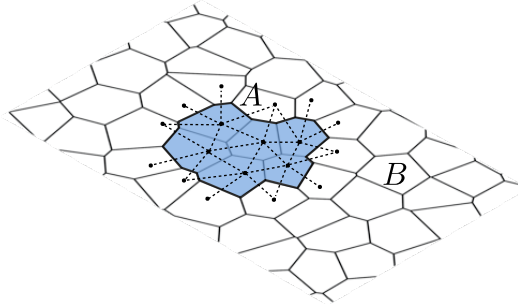


Fig. 7. A cellular decomposition of space with its dual spin-network graph.

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## REFERENCES

- [1] C. Rovelli, *PoS QGQGS2011*, 003 (2011) [[arXiv:1102.3660 \[gr-qc\]](#)].
- [2] A. Perez, *Living Rev. Rel.* **16**, 3 (2013) [[arXiv:1205.2019 \[gr-qc\]](#)].
- [3] L. Landau, E. Lifshits, *Quantum Mechanics Non-Relativistic Theory*, vol. 3, first ed., Pergamon, UK, 1958.

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<sup>15</sup> In this simple case, as the volume in  $\mathcal{H}_2 = \text{Inv}(\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2})$  has a degenerate spectrum with  $v_+ = v_-$ , the volume is simply given by  $v_+$  times the number of nodes in the region.

- [4] G. Ponzano, T. Regge, Semiclassical limit of Racah coefficients, in: *Spectroscopic and Group Theoretical Methods in Physics*, ed. F. Block, North Holland, Amsterdam 1968.
- [5] T. Regge, *Nuovo Cim.* **19**, 558 (1961).
- [6] J. Engle, R. Pereira, C. Rovelli, *Phys. Rev. Lett.* **99**, 161301 (2007) [arXiv:0705.2388 [gr-qc]].
- [7] J. Engle, E. Livine, R. Pereira, C. Rovelli, *Nucl. Phys.* **B799**, 136 (2008) [arXiv:0711.0146 [gr-qc]].
- [8] L. Freidel, K. Krasnov, *Class. Quantum Grav.* **25**, 125018 (2008) [arXiv:0708.1595 [gr-qc]].
- [9] E.R. Livine, S. Speziale, *Phys. Rev.* **D76**, 084028 (2007) [arXiv:0705.0674 [gr-qc]].
- [10] E. Bianchi, P. Dona, S. Speziale, *Phys. Rev.* **D83**, 044035 (2011) [arXiv:1009.3402 [gr-qc]].
- [11] H. Minkowski, *Allgemeine Lehrsätze über die Konvexe Polyeder*, Nachr. Ges. Wiss., Göttingen 1897, pp. 198–219.
- [12] M. Kapovich, J.J. Millson, *J. Differential Geom.* **44**, 479 (1996).
- [13] E. Bianchi, *Nucl. Phys.* **B807**, 591 (2009) [arXiv:0806.4710 [gr-qc]].
- [14] J.B. Lasserre, *J. Optim. Theor. Appl.* **39**, 363 (1983).
- [15] H.M. Haggard, *Phys. Rev.* **D87**, 044020 (2013) [arXiv:1211.7311 [gr-qc]].
- [16] C. Coleman-Smith, B. Müller, *Phys. Rev.* **D87**, 044047 (2013) [arXiv:1212.1930 [gr-qc]].
- [17] R. Penrose, Angular Momentum: an Approach to Combinatorial Space-time, in: *Quantum Theory and Beyond*, ed. T. Bastin, Cambridge University Press, Cambridge, 1971, pp. 151–180.
- [18] R. Penrose, On the Nature of Quantum Geometry, in: *Magic Without Magic: J.A. Wheeler Festschrift*, ed. J.R. Klauder, W.H. Freeman & Co Ltd, 1973.
- [19] C. Rovelli, L. Smolin, *Nucl. Phys.* **B442**, 593 (1995) [arXiv:gr-qc/9411005].
- [20] S.A. Major, *Class. Quantum Grav.* **16**, 3859 (1999) [arXiv:gr-qc/9905019].
- [21] A. Barbieri, *Nucl. Phys.* **B518**, 714 (1998) [arXiv:gr-qc/9707010].
- [22] F. Conrady, L. Freidel, *J. Math. Phys.* **50**, 123510 (2009) [arXiv:0902.0351 [gr-qc]].
- [23] L. Freidel, E.R. Livine, *J. Math. Phys.* **51**, 082502 (2010) [arXiv:0911.3553 [gr-qc]].
- [24] B.S.S. John R. Klauder, *Coherent States: Applications in Physics and Mathematical Physics*, World Scientific, 1985.
- [25] A.M. Perelomov, *Generalized Coherent States and Their Applications*, Springer-Verlag, 1986.
- [26] E. Bianchi, H.M. Haggard, *Phys. Rev.* **D86**, 124010 (2012) [arXiv:1208.2228 [gr-qc]].

- [27] C. Rovelli, *Quantum Gravity*, Cambridge, UK, Univ. Pr., 2004, p. 455.
- [28] A. Ashtekar, J. Lewandowski, *Class. Quantum Grav.* **21**, R53 (2004) [[arXiv:gr-qc/0404018](#)].
- [29] T. Thiemann, *Modern Canonical Quantum General Relativity*, Cambridge University Press 2007.
- [30] E. Bianchi, R.C. Myers, *On the Architecture of Spacetime Geometry*, [[arXiv:1212.5183](#) [[hep-th](#)]].
- [31] E. Bianchi, L. Modesto, C. Rovelli, S. Speziale, *Class. Quantum Grav.* **23**, 6989 (2006) [[arXiv:gr-qc/0604044](#)].
- [32] E. Bianchi, E. Magliaro, C. Perini, *Nucl. Phys.* **B822**, 245 (2009) [[arXiv:0905.4082](#) [[gr-qc](#)]].