

EXOTIC GEOMETRICAL SYMMETRIES IN NUCLEI: FROM GROUP THEORY TO EXPERIMENTS*

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We report on the progress in conceptual and mathematical solutions of the theory of nuclear geometrical symmetries — more precisely: nuclear point-group symmetries — obtained within the TetraNuc Collaboration in recent years. We shortly summarise the basic concepts, the strategic lines of the solutions as well as the link between theoretical and experimental considerations.

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1. Introduction

Interactions governing the nuclear many-body systems are known to be among the most complex existing in nature. They combine the features of non-central and non-local character together with a non-negligible presence of the three-body forces. These interactions are invariant under a number of fundamental symmetries one of them being rotational symmetry implying that the corresponding nuclear Hamiltonians must be scalar *i.e.* insensitive

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to the orientation in space. And yet, the notion of the nuclear shapes and surfaces is so frequently used in the literature that one is seldom asking why scalar Hamiltonians define an orientation in space?

The answers have to do with two *fundamental properties* of the nuclear interactions: The first one is related with the short-range of the underlying forces which allows to introduce the notion of closed nuclear surfaces outside of which the interactions die-off exponentially. The second one has to do with the notion of spontaneous symmetry breaking — a mechanism allowing for the ground-state solutions to have symmetries different from those of the original Hamiltonian. Both of these mechanisms combined lead to the possible existence of non-spherical shapes of atomic nuclei and bring us to the possibility of applying point-group symmetries and related theory of the group representations to study in mathematically rigorous terms the solutions of the implied equations of motion. In the present context, we distinguish two levels of analysis: (a) The single-nucleonic degrees of freedom implying certain specific features such as possibly unusual degeneracies of levels together with the accompanying ‘magic’, energy gaps, as well as: (b) Collective degrees of freedom in the form of rotations and vibrations.

The interest in the exotic point-group symmetries announced in Ref. [1], following the early pilot project in Ref. [2], resulted in a number of dedicated experiments several of which are discussed in Ref. [3]. In this article, we are going to overview the past evolution focussing in particular on one of the point group symmetries: the tetrahedral one.

2. Theory of geometrical symmetries: progress report

The two fundamental mechanisms mentioned above provide back-ground for the mean-field theory realisations, either self-consistent (constrained) Hartree–Fock or a phenomenological approach of Strutinsky — both very well suited to model nuclear shapes and thus collective rotations and associated symmetries. Since the mean-field theory of nuclear structure and theory of group representations are among the most powerful tools in advanced nuclear quantum mechanics, combination of the two methods is probably the best, one can do at present in the context of geometrical nuclear symmetries.

2.1. Mean-field nuclear theories and nuclear shapes in the 3D space

To describe nuclear shapes, one may conveniently employ the basis of spherical harmonics, $\{Y_{\lambda\mu}(\vartheta, \varphi)\}$, allowing to describe an arbitrary nuclear surface Σ parametrized as usual as $R(\vartheta, \varphi) \sim [1 + \sum_{\lambda} \sum_{\mu} \alpha_{\lambda\mu}^* Y_{\lambda\mu}(\vartheta, \varphi)]$. Having expressed the nuclear surface, we may define implied phenomenological mean-fields for instance in the form of deformed Woods–Saxon central

and spin-orbit potentials. Alternatively, applying the constrained Hartree–Fock approach, one would employ the multipole moment operators defined by $Q_{\lambda\mu} \sim r^\lambda Y_{\lambda\mu}(\vartheta, \varphi)$, whose expectation values are used to control nuclear shapes through constraints. Both approaches use essentially the same tool: The spherical harmonic basis — and this tool will be used below.

2.2. Constructing mean-field Hamiltonians with a predefined symmetry

In order to investigate the impact of the nuclear point-group symmetries, it is important to be able to construct mean-field Hamiltonians symmetric under any given group proposed by physicist. This problem has been solved in Ref. [5] and we summarise the result shortly.

Consider given symmetry point group \mathcal{G} composed of elements $\hat{g} \in \mathcal{G}$; they can always be parametrized using Wigner functions $D_{\mu'\mu}^\lambda(\Omega_g)$ in terms of triplets of Euler angles $\Omega_g \equiv \{\alpha, \beta, \gamma\}_g$. We wish to construct a surface, say Σ , invariant under the action of all the symmetry elements of this group

$$\forall \hat{g} \in \mathcal{G} : \Sigma \xrightarrow{\hat{g}} \Sigma' = \Sigma \Rightarrow \sum_{\lambda\mu} \alpha_{\lambda\mu}^* \hat{g} Y_{\lambda\mu}(\vartheta, \varphi) = \sum_{\lambda\mu} \alpha_{\lambda\mu}^* Y_{\lambda\mu}(\vartheta, \varphi). \quad (1)$$

This is a system of as many relations as the number of group elements which, after transforming, gives a system of equations for the deformation parameters $\alpha_{\lambda\mu}$, separately at each order λ . These coupled equations can be, somewhat schematically, represented in the form (for details *cf.* Ref. [5])

$$\forall \hat{g} \in \mathcal{G} : \sum_{\mu=-\lambda}^{\lambda} \left[D_{\mu'\mu}^\lambda(\Omega_g) - \delta_{\mu'\mu} \right] \alpha_{\lambda\mu}^* = 0. \quad (2)$$

Equations being uniform we may take one of the deformation parameters, *e.g.* $\alpha_{\lambda\bar{\mu}}$, as an independent variable and determine the remaining ones as functions of the latter. For tetrahedral symmetry the lowest-order solution of the problem is possible for $\lambda = 3$ with deformations $\alpha_{3\pm 2} \neq 0$. There are no solutions for $\lambda = 4, 5$ and 6 , the next possible being the solution with $\lambda = 7$. It takes the form: $\alpha_{7,\pm 2}$ -arbitrary and $\alpha_{7,\pm 6} = -\sqrt{11/13} \alpha_{7,\pm 2}$. The higher order solutions correspond to odd- λ only, where $\alpha_{\lambda-\text{odd},\pm 2}$ can be taken as arbitrary and all other non-zero deformation parameters have $\mu = \pm(2+4k) \leq \lambda$, with an integer k ; they are known functions of $\alpha_{\lambda-\text{odd},\pm 2}$.

With this information we can build an arbitrary surface invariant under the tetrahedral symmetry point-group with $\lambda = 3, 7, 9, \dots$. Using the latter, the phenomenological, *e.g.* deformed Woods–Saxon mean-field Hamiltonian, invariant with respect to the tetrahedral group can be constructed. Similarly one can construct an ensemble of constraint multipole moments using

exactly the same selection of λ and μ . Thanks to this selection, the phenomenological or constrained Hartree–Fock mean field Hamiltonians can be constructed and the implied symmetries studied.

In principle, the presence of a rotation (cranking) axis breaks down the exact tetrahedral symmetry — and yet, some exotic discrete symmetries remain such as doublex and triplex, conditioned by the presence of the tetrahedral one — *cf.* Section 2.8.

2.3. Relating Hamiltonian-symmetry groups and nuclear stability

A systematic mechanism according to which point-group symmetries may stabilise nuclear systems by possibly producing big gaps in the single particle spectra has been described for the first time in Ref. [1]. Formulated qualitatively, the argument can be presented as follows: (a) Symmetry groups having more irreducible representations than the others will likely lead to stronger deformed shell-gaps in the single-nucleon spectra. (b) This is because each irreducible representation generates one family of levels which, within their family, repel each other thus filling-in the potential well of $U_o \approx -60$ MeV depth. (c) Energy levels within different families are independent of each other in the sense that they cross in function of the deformation (no repulsion). (d) For instance, with the total number of 6 irreducible representations (characteristic for octahedral symmetry) we will generate six families of single-particle levels with, on the average, 6 times higher level spacing as compared to the group with one irreducible representation.

Indeed, with the total number of bound levels denoted N_o , such an average spacing per family is roughly given by $U_o/(N_o/6) = 6(U_o/N_o)$ (*cf.* Fig. 2 in Ref. [4] for illustration). Realistic calculations show that the discussed mechanism often leads to considerable single-particle gaps and thus to an increase in nuclear stability for certain chains of ‘magic numbers’. According to such an approach each symmetry group may *generate its characteristic magic numbers*. For instance, proton and neutron tetrahedral magic numbers have been calculated in Refs. [1, 2, 6] to be $Z_t, N_t = 32, 40, 56, 64, 70, 90$ and 136. Such results generalise for other groups and chains of magic numbers thus offering a guideline so as to where to look for the most stable nuclear mean-field configurations, *cf. e.g.* Ref. [7].

2.4. Constructing quantum rotor of predefined symmetry

Suppose that a static minimum on the total energy surface corresponds to the tetrahedral (or another point group) symmetry configuration. Since the corresponding nuclear shapes are not spherical, the nuclear orientation in space can be defined and it follows that the corresponding nuclei may

rotate collectively. Point group symmetries lead not only to the characteristic grouping of single-nucleon levels but also influence the spectra generated by the corresponding quantum rotors with a given predefined symmetry. This problem has been for the first time discussed in the nuclear context in Ref. [8] where the so-called *generalised quantum rotor* Hamiltonian has been constructed allowing to model the rotational spectra of quantum objects with a point group symmetry predefined *a priori*. This has been followed up in Refs. [9–11], where the formalism related to transition probabilities and, in particular, the reduced transition-probability formulae have been derived.

The well known, so-called triaxial rotor Hamiltonian is invariant under relatively simple point-group, D_{2h} ; it has the form

$$\hat{H}_{D_{2h}} = \frac{\hat{I}_x^2}{2\mathcal{J}_x} + \frac{\hat{I}_y^2}{2\mathcal{J}_y} + \frac{\hat{I}_z^2}{2\mathcal{J}_z}. \tag{3}$$

The corresponding solutions can be conveniently expressed in terms of the Wigner functions, $D_{\mu'\mu}^\lambda(\Omega)$, depending on the Euler angles, $\Omega \equiv \{\alpha, \beta, \gamma\}$, the latter expressing the orientation of the body-fixed frame with respect to laboratory and the former probability density of the orientation of the nucleus.

The generalisation proceeds through introducing first the tensor-operator basis built out of the angular-momentum vector $\hat{I} = \{\hat{I}_{+1}, \hat{I}_0, \hat{I}_{-1}\}$

$$\hat{T}_\mu^\lambda \equiv \left(\left(\dots \left((\hat{I} \otimes \hat{I})^{\lambda_2=2} \otimes \hat{I} \right)^{\lambda_3=3} \otimes \dots \otimes \hat{I}^{\lambda_{n-1}=n-1} \right) \otimes \hat{I} \right)_\mu^{\lambda_n=\lambda=n}. \tag{4}$$

Expressing the above with the help of the Clebsch–Gordan coupling, we may construct a *generic* form of a rotor Hamiltonian as a scalar, which, after adjusting appropriately the coupling constants $h_{\lambda\mu}$ takes the form

$$\hat{\mathcal{H}} = h_{00}\hat{T}_0^0 + \sum_{\lambda=1}^{\infty} \left[h_{\lambda 0}\hat{T}_0^\lambda + \sum_{\mu=1}^{\lambda} \left(h_{\lambda\mu}\hat{T}_\mu^\lambda + (-1)^\mu h_{\lambda\mu}^* \hat{T}_{-\mu}^\lambda \right) \right], \tag{5}$$

with $\hat{T}_0^0 \equiv (\hat{I} \otimes \hat{I})_0^0$. It can be used to represent any symmetry we may be interested in. For instance, using the results of Sect. 2.2, for the tetrahedral symmetric rotor Hamiltonian, in the lowest order, ($\lambda = 3$), we obtain

$$T_d\text{-symmetry} : \hat{\mathcal{H}} = h_{00}\hat{T}_0^0 + h_{32} \left(\hat{T}_{+2}^3 - \hat{T}_{-2}^3 \right). \tag{6}$$

With the coupling constant h_{32} chosen imaginary, we obtain a Hermitian model Hamiltonian, invariant under the T_d -symmetry thus allowing to model the implied T_d -spectra, study the corresponding spectral degeneracies, the effective inertia behaviour *etc.*, *cf.* Refs. [6, 8, 9] for illustrations.

2.5. Multi-dimensional deformation spaces: isotropy groups and orbits

In order to obtain the information about the privileged, *i.e.* the lowest-energy, nuclear configurations (shapes), one may chose to perform the total energy calculations in a certain sub-space of the full deformation space which is composed of, say, $D \equiv \{\alpha_{\lambda\mu} : \lambda \in [\lambda_{\min}, \lambda_{\max}]\}$ for $\alpha_{\lambda\mu}^{\min} \leq \alpha_{\lambda\mu} \leq \alpha_{\lambda\mu}^{\max}$. One shows easily that in such a space there may exist, in general, several sectors which correspond to equivalent shapes. For instance, in the case of the triaxial ellipsoid geometry, by applying the discrete rotations about the angle of π through \mathcal{O}_x -, \mathcal{O}_y - or \mathcal{O}_z -axes, we obtain exactly the same shapes (therefore the energies). By performing the calculations in the range of deformations D specified above, we repeat the calculations of the same energies 8 times. Whereas the discussion for an obvious type of geometry, like the triaxial ellipsoid above, the position and the solution of the problem may be considered obvious — it is much less so for an arbitrary class of shapes.

The general solution of this problem has been proposed in Ref. [12] using the group-theory approach which involves the notions of isotropy groups and associated orbits; below, this solution is schematised briefly. Let us consider a deformed mean-field Hamiltonian, $\hat{H}(\bar{\alpha})$, whose spatial geometry is defined with the help of a fixed deformation set, say $\bar{\alpha} \in D$. Let G_S with elements g_s be a group of symmetry of $\hat{H}(\bar{\alpha})$ such that acting with g_s on $\bar{\alpha}$ leads to a new deformation set $g_s \circ \bar{\alpha}$ such that the nuclear surfaces, $\Sigma_{\bar{\alpha}}$ and $\Sigma_{g_s \circ \bar{\alpha}}$ coincide and thus the corresponding Hamiltonian produces the same energy. The ensemble

$$\{\bar{\alpha}\}_{G_S} \equiv \{g_s \circ \bar{\alpha} : g_s \in G_S\} \quad (7)$$

is called ‘orbit of element $\bar{\alpha}$ ’ and from the context it becomes clear that it will be sufficient to consider one element in the orbit as representative of the full ensemble. By constructing such orbits (for details *cf.* [12]), we may directly identify the sub-sectors in D which repeat the information; the actual calculations are performed for just one representative of the orbit. This solution allows to save in terms of computing time and storage up to an order of magnitude which gives a non-neglegible gain in the case of the large scale calculations.

Let us emphasise that although the computing aspects are non-negligible — yet *not the most important* in the case of the large scale calculations. As it turns out the energy minimisation in multi-dimensional deformation spaces is a mathematically a non-trivial problem especially when only the unique solutions have physical significance. In those cases the discussed group theory approach offers a mathematically elegant — yet simple solution. [For an example of the total energy surfaces of doubly-magic zirconium nuclei obtained according to the discussed scheme *cf.* Fig. 3 of Ref. [4].]

2.6. *Between the laboratory and rotating frames: symmetrization group*

Most of theoretical methods describing collective motion of many-body systems use the body-fixed reference frame. This allows to transform time-dependent formulations into formulations in terms of stationary problems what is both physically plausible and offers considerable simplifications, particularly fruitful when the adiabaticity condition can be applied — what is often the case. However, experimental results are obtained in laboratory frames and the theory results need to be transformed appropriately to be able to compare observables such as, for instance, electromagnetic transition probabilities. The mathematical background of the problem has been formulated and discussed in Ref. [13] and the solutions presented in Ref. [14].

Using nuclear deformation parameters which are already spherical tensors, such as $\alpha_{\lambda\mu}$ introduced earlier, offers important advantages from the point of view of transformation properties between various reference frames. In the laboratory frame, the equation of the moving nuclear surface can be expressed with the help of time-dependent deformation parameters $\alpha_{\lambda\mu}^{(\text{lab.})}(t)$

$$\Sigma^{(\text{lab.})} : R(\vartheta, \varphi) \sim R_0 \left[1 + \sum_{\lambda\mu} \alpha_{\lambda\mu}^{(\text{lab.})}(t) Y_{\lambda\mu}(\vartheta, \varphi) \right], \quad (8)$$

whereas the same geometrical form can be considered static in the intrinsic (rotating) coordinate frame and expressed using time-independent $\alpha_{\lambda\mu}^{(\text{intr.})}$ as

$$\Sigma^{(\text{intr.})} : \alpha_{\lambda\mu}^{(\text{intr.})} = \sum_{\mu'} D_{\mu'\mu}^{\lambda}[\Omega(t)] \alpha_{\lambda\mu'}^{(\text{lab.})}(t). \quad (9)$$

Let the number of independent deformation parameters in Eq. (8) be n , and let us consider from now on the instantaneous transformations at given t . Since the relative orientation of the two frames, $\Sigma^{(\text{lab.})}$ and $\Sigma^{(\text{intr.})}$, requires three degrees of freedom [in Eq. (9) taken as three Euler angles Ω] it becomes clear that the static deformation parameters in the intrinsic frame may involve only $(n - 3)$ independent coefficients $\{\alpha_{\lambda\mu}^{(\text{intr.})}\}$, the remaining three parameters describing the orientation of the rigid surface.

Introduce a compact notation: for the ensemble of $(n - 3)$ independent body-fixed deformation parameters, $\{\alpha\}$, and for the ensemble of n laboratory parameters $\{\alpha^{(\text{lab.})}\}$. Let symmetry group of the family of surfaces considered [*e.g.* for quadrupole-triaxial nuclei] be G . We are interested in expressing the (unique) laboratory wave function of the nucleus in $\Sigma^{(\text{lab.})}$, whereas the relative orientation of $\Sigma^{(\text{lab.})}$ and $\Sigma^{(\text{intr.})}$ is characterised by Ω . We consider the solutions in the body-fixed frame, $\psi^{(\text{intr.})}(\alpha, \Omega)$, as known.

The problem associated with the symmetry G is that

$$\forall \hat{g} \in G : \quad \hat{g} \psi^{(\text{intr.})}(\alpha, \Omega) = \varphi^{(\text{intr.})}(\alpha', \Omega') \neq \psi^{(\text{intr.})}(\alpha, \Omega) \quad (10)$$

whereas the energies of the solutions of the G -transformed Hamiltonian, $\psi^{(\text{intr.})}(\alpha, \Omega)$ and $\varphi^{(\text{intr.})}(\alpha', \Omega')$ are the same, the two wave functions themselves generally differ. It is well known from group-representation theory, that the invariance of the Hamiltonian, here $\hat{H}^{(\text{intr.})}$, implies that

$$\hat{H}^{(\text{intr.})} \psi^{(\text{intr.})} = E \psi^{(\text{intr.})} \rightarrow \left[\hat{g} \overbrace{\hat{H}^{(\text{intr.})}}^{\hat{H}^{(\text{intr.})}} \hat{g}^{-1} \right] \left[\overbrace{\hat{g} \psi^{(\text{intr.})}}^{\varphi^{(\text{intr.})}} \right] = E \left[\overbrace{\hat{g} \psi^{(\text{intr.})}}^{\varphi^{(\text{intr.})}} \right]. \quad (11)$$

For n_G elements in G , there will be, in general, n_G different functions of the type $\varphi^{(\text{intr.})}$ which are physically equivalent, because corresponding to the n_G physically-indistinguishable positions of the considered surface. They are physically indistinguishable because the corresponding Hamiltonian remains invariant — yet distinguishable mathematically as coinciding images of the same surface through the transformations $\hat{g} \in G$. It is not acceptable to select one of the functions $\varphi^{(\text{intr.})}$ to obtain the laboratory result (through the usual transformation involving Wigner functions) because the corresponding result will be dependent on a particular choice of a position of the nuclear surface among n_G equivalent positions. Therefore, in order to make the physical laboratory solutions independent of arbitrary choices, they must be constructed with help of the appropriately symmetrized forms as discussed in Ref. [14].

A complication, apparently not recognised earlier in the literature, consists in the fact that the choice leading from n degrees of freedom in the laboratory frame, the latter including all the deformations *sensu stricto* and the orientation with respect to the rotating frame — *vs.* the specification of the $(n - 3)$ degrees of freedom defining the shapes alone — is not unique. It then follows that the group needed for the symmetrization, G_s , does not need to be identical with the physical symmetry-group G of Hamiltonian $\hat{H}^{\text{intr.}}$ — and the issue needs to be studied on the case-by-case basis. We refer to such groups as symmetrization groups (*cf.* Ref. [14] for more details).

2.7. Microscopic theories involving angular-momentum projection

Total angular momentum of an isolated nucleus is conserved in the laboratory (inertial) reference frame. The corresponding wave functions, Ψ_{IM} , are characterised by the laboratory angular momentum quantum numbers, I and M , and transform as spherical tensors of the group of rotations. On the other hand, the deformed nuclei and the underlying mean-field Hamiltonians are generally *not* invariant under the group of rotations in the intrinsic

reference frame. This inconsistency caused by the strategy of using the mean-field theory as an excellent (although approximate) tool implies the necessity of restoring the broken symmetries by another approximation: The angular momentum and (if necessary) parity projection techniques.

Projection techniques are rather standard these days, but applied to the realistic Hamiltonians become complex and computer-time consuming. Recently, new achievements related to the projection techniques have been reported in Ref. [15] and results focussed on the tetrahedral symmetry in nuclei illustrated in Ref. [16]. These calculations use the anti-symmetrised determinant-type wave functions Φ constructed out of the single-nucleon solutions of (*e.g.* tetrahedral) deformed mean-field Hamiltonian on which the standard angular-momentum, \hat{P}_{MK}^I , and parity \hat{P}^\pm projection operators are applied leading to the double-spherical-tensor auxiliary wave-functions $\Phi_{MK}^{I\pm} \equiv \hat{P}_{MK}^I \hat{P}^\pm \Phi$ which are further used to construct the final physical solutions.

A possibility explored in the above references is to model the K -quantum number mixing caused by both the presence of the non-axial deformations and of the Coriolis effects induced by nuclear rotation and looking for the final solutions in the form

$$\Psi_{M,q}^{I\pm} = \sum_K g_{K,q}^{I\pm} \Phi_{MK}^{I\pm}, \quad (12)$$

where q denotes some extra quantum numbers enumerating the solutions. The mixing coefficients can be sought in the framework of the Hill–Wheller formalism

$$\sum_{K'} \mathcal{H}_{KK'}^{I(\pm)} g_{K',\alpha}^{I(\pm)} = E_\alpha^{I(\pm)} \sum_{K'} \mathcal{N}_{KK'}^{I(\pm)} g_{K',\alpha}^{I(\pm)}, \quad (13)$$

with the Hamiltonian and norm kernel matrices being defined as usual as

$$\begin{pmatrix} \mathcal{H}_{KK'}^{I(\pm)} \\ \mathcal{N}_{KK'}^{I(\pm)} \end{pmatrix} = \langle \Phi | \begin{pmatrix} \hat{\mathcal{H}} \\ 1 \end{pmatrix} \hat{P}_{KK'}^I \hat{P}_\pm | \Phi \rangle. \quad (14)$$

The choice of the Hamiltonian $\hat{\mathcal{H}}$ is determined by the physical mechanisms and interactions considered of importance in a given physics context. In the case of examining point-group symmetries the scalar structure of $\hat{\mathcal{H}}$ is important. The choice in Refs. [15, 16] focuses on the two-body interaction Hamiltonian involving multipole–multipole interactions both in the particle-hole and the particle–particle channel. Calculations performed for a few doubly-magic tetrahedral nuclei demonstrate that the resulting excitation spectra reproduce the excitation patterns expected from the group-representation theory (*cf. e.g.* Ref.[17] and [16] for details) and provides a good basis for

the microscopic model calculations of the electromagnetic transition probabilities — a decisive element in searching for the experimental confirmation of the presence of tetrahedral symmetry in nuclei.

2.8. Point groups and new quantum numbers: doublex and triplex

In studies of the structure of fast spinning axially-symmetric nuclei, the rotational bands have been traditionally classified, after Bohr and Mottelson, in terms of parity and signature quantum numbers. Assuming that \mathcal{O}_z is the nuclear symmetry axis and that the nucleus is rotating about perpendicular to it \mathcal{O}_y -axis, the mean-field cranking Hamiltonian takes the form

$$\hat{H}^{\omega_y} \equiv \hat{H} - \omega_y \hat{j}_y \leftrightarrow \left[\hat{H}^{\omega_y}, \hat{\mathcal{R}}_y(\pi) \right] = 0 \quad (15)$$

i.e. it is invariant under rotation through the angle of π about \mathcal{O}_y -axis. The associated good quantum number is called signature and it can be shown that the strong $E2$ -transitions connect the states of the same parity and signature.

In the case of odd- λ deformations, the intrinsic parity is broken, but the nuclear cranking mean-field Hamiltonian is invariant under the rotary-reflection transformation $\hat{\mathcal{S}}_y \equiv \hat{\mathcal{R}}_y(\pi) \cdot \hat{\mathcal{P}}$ composed of the rotation transformation just introduced and inversion (parity) $\hat{\mathcal{P}}$, as illustrated in Fig. 1. The states with common simplex form bands.

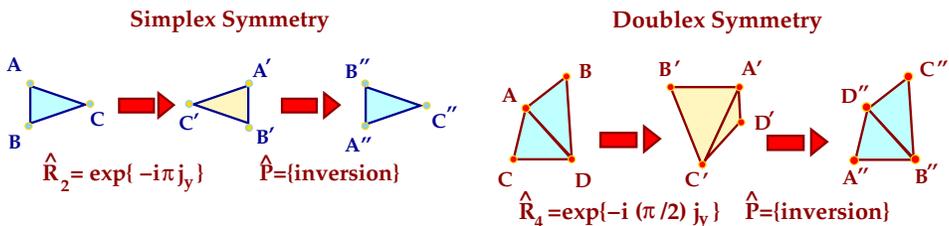


Fig. 1. Discrete symmetries of the cranking mean-field Hamiltonian in the presence of odd- λ multiplicities in the nuclear surface. Left: The so-called *simplex* symmetry, $\hat{\mathcal{S}}_y = \hat{\mathcal{R}}_y(\pi) \cdot \hat{\mathcal{P}}$, applying *e.g.* to the pear-shape surfaces like the ones represented by Y_{30} spherical harmonic, here schematised with a triangle. Right: What we refer to as *doublex* symmetry, defined by the product $\hat{\mathcal{D}}_y = \hat{\mathcal{R}}_y(\pi/2) \cdot \hat{\mathcal{P}}$.

In the case of tetrahedral symmetry, the cranking Hamiltonian is invariant with respect to *doublex* symmetry illustrated in the right-hand side of figure 1. Let us present shortly the consequences of the doublex symmetry on the single-nucleonic level associated with the \mathcal{O}_μ -axis. Since $\hat{\mathcal{D}}_\mu =$

$\hat{\mathcal{R}}_\mu(\pi/2) \cdot \hat{\mathcal{P}}$ we find that doublex and signature are related, *viz.*

$$\hat{D}_\mu^2 = \hat{\mathcal{P}}^2 \left[\hat{\mathcal{R}}_\mu(\pi/2) \right]^2 = \hat{\mathcal{R}}_\mu(\pi) \rightarrow \hat{D}_\mu^2 = \hat{\mathcal{R}}_\mu \rightarrow \left[\hat{D}_\mu, \hat{\mathcal{R}}_\mu \right] = 0 \quad (16)$$

and since signature and doublex operations commute, the single-nucleonic states ψ_n can be classified simultaneously using signature and doublex quantum numbers

$$\hat{\mathcal{R}}_\mu^2 = -1 \rightarrow r_\mu = \pm i \quad \text{and} \quad \hat{D}_\mu^4 = -1 \rightarrow d_\mu = \exp\{i\frac{\pi}{4}\delta\} \quad (17)$$

with $\delta = 0, \frac{1}{2}, 1, \frac{3}{2}$. These properties have been discussed in Refs. [18] but so far remained unexplored from the experimental point of view.

In analogy, triplex quantum number can be associated with the cranking Hamiltonian in which the direction of the axis of rotation passes through one of the tips and the triangular base — the symmetry is associated with the discrete rotation angle of $2\pi/3$.

3. Search of tetrahedral symmetry — experimental perspective

One of the questions of high actuality is how to distinguish between the signs of the tetrahedral deformation α_{32} and the ‘traditional’ axial octupole deformation, α_{30} , the latter discussed in the literature over the years. Since at spin $I = 0$ the actual shapes of the even–even nuclei are indistinguishable, the answer must come from the non-zero low spin states; at higher spins the Coriolis interactions and the presence of the distinct spin axis are expected to break the tetrahedral symmetry. To enable a direct comparison requires accurate branching ratios and lifetime measurements and may need new experimental technical developments that are envisaged within the TetraNuc Collaboration both for the Rare-Earth and the Actinides regions, Ref. [19].

The early discussions of possible signs of tetrahedral symmetry, Ref. [24], were based on the assumption of the static tetrahedral deformation, the assumption which implies that the associated quadrupole moments vanish. As a result, one would expect the ‘usual’ $E2$ transitions to be weak or very weak and the dipole transitions to the lower-lying ground-state band possibly winning. In fact, in the early experiments, no $E2$ transitions were observed in the tetrahedral-suspect band in ^{156}Gd below spin 9^- , the corresponding band-states decaying via $E1$ transitions to the g.s. band. The historically first measurements performed on this nucleus in the framework of the TetraNuc Collaboration aimed at the lowest-lying odd-spin negative-parity band with apparently missing $E2$ transitions. In the meantime the theoretical analysis has been extended to include the large amplitude collective motion by solving the corresponding collective Schrödinger equation rather

than *assuming* the static tetrahedral deformation. Since the predicted tetrahedral and the ground-state minima lie close and the separating potential barriers are not very high, the new class of solutions, originally unexpected — has been discovered, Ref. [21]. The solutions manifest the simultaneous combination of the probability maxima at both tetrahedral and ground-state minima complicating the interpretation. According to this scenario, tetrahedral states may live in a perfect coexistence with the quadrupole deformed ones.

A fascinating possibility corresponds to the scenario according to which the nucleus may spent most of its life-time in a tetrahedral configuration from which the $E2$ transitions are very improbable — whereas during a certain small portion of the life-time they can acquire the ground-state quadrupole deformation and emit the $E2$ gamma rays with the $B(E2)$ equal to that of the quadrupole-deformed ground-state.

Our first experiment on the lowest negative parity band in the nucleus ^{156}Gd , Ref. [20], and more recently performed GAMS measurements at ILL, Grenoble, employed the unique Bragg-spectrometry techniques to measure the lifetimes and branching ratios at the very bottom of the band. The results suggest that this originally proposed tetrahedral-symmetry band carries a relatively large quadrupole moment, Ref. [22], that is comparable to the one of the ground-state band. On the one hand, this result disfavors the static tetrahedral symmetry interpretation for this particular band but on the other hand, it is in perfect agreement with the scenario of Ref. [21] just described.

Taking into account recent evolution in theoretical considerations, we re-examined the interpretation of the bands in question in ^{156}Gd in terms of octupole vibrations as proposed so far in the literature. In fact, the odd-spin negative-parity band, which according to the early discussions was a possible tetrahedral-symmetry candidate, is accompanied by an even-spin negative-parity band, traditionally interpreted as the signature/simplex partner of the other. In such a case, the two bands should have the same intrinsic structure and thus identical dipole and quadrupole moments. On the other hand, if the dipole and quadrupole moments of the two bands so far interpreted as signature-partners are significantly different, the understanding of the discussed structures would need to be seriously revised, possibly ruling out the octupole-vibrational view and inviting a more modern interpretation.

We have performed a series of experiments at the ILL, Grenoble — preliminary results, Ref. [25], showing that the $B(E1)$ transition probabilities of these so-called partner-bands are different by almost an order of magnitude and that the $B(E2)/B(E1)$ branching ratios of the two bands have a totally different behavior (see Fig. 2). These preliminary results do question the

interpretation of the two bands in terms of vibrational octupole partners and even if it is too soon to draw a definite conclusion, they strongly encourage the new measurements at the ILL planned for 2013.

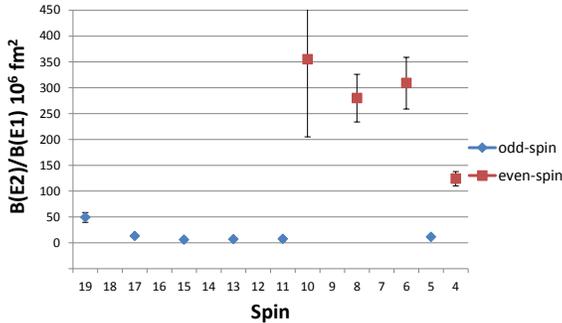


Fig. 2. Experimental information about the $B(E2)/B(E1)$ ratios in the lowest negative parity bands in ^{156}Gd for odd- and even-spins as indicated.

4. Summary

We have reviewed what, in our opinion, represents a considerable progress achieved during the years in both conceptual and the group-theoretical developments of advanced quantum-mechanical methods needed to describe nuclear geometrical symmetries. At present, the methods are operational which allow to generate the mean-field Hamiltonians of *a priori* predefined symmetry — the same applies to the quantum rotor Hamiltonians. Advanced projection techniques allow to generate the wave-functions which, although generated from the mean-field formalism, in the laboratory frame transform as spherical tensors and are perfectly suited for calculating the electromagnetic transition probabilities. These latter calculations have not been done so far but all the necessary quantum-mechanical tools are almost there and the work is in progress.

On the conceptual level, the use of the intrinsic groups, and the idea and systematic development of the formalism of the symmetrization group allow for large scale calculations whose results can be meaningfully transformed to the laboratory reference frame and used on the way to the final identification of the point-group symmetries in general and tetrahedral symmetry in particular — through the branching-ratios of electromagnetic transitions.

On the experimental level, encouraging results have been obtained which indicate that the so far used interpretation of the one-and-only pear-shape octupole degrees of freedom may turn out to be insufficient to interpret the variety of already known data on the negative-parity rotational bands in nuclei and encourages further advanced studies in the direction of the new

ideas — as the ones suggested by the group-theory predictions combined with the realistic microscopic nuclear structure theories — such as the mean-field theory.

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