FRACTIONAL FOKKER–PLANCK EQUATION WITH SPACE DEPENDENT DRIFT AND DIFFUSION: THE CASE OF TEMPERED α-STABLE WAITING-TIMES*

Janusz Gajda

Hugo Steinhaus Center, Institute of Mathematics and Computer Science
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

(Received April 11, 2013)

In this paper, we propose a stochastic process with space dependent force and diffusion coefficients. We show that PDF of this process satisfies a generalized tempered fractional Fokker–Planck equation. Thus we obtain a complete description of subdiffusion with tempered α-stable waiting-times and with space dependent force and diffusion. Based on derived stochastic representation, we simulate paths of the underlying process. Moreover, we approximate the solution of the proposed system via Monte Carlo methods.

DOI:10.5506/APhysPolB.44.1149
PACS numbers: 05.10.Gg, 05.40.–a, 05.10.–a, 02.50.Fz

1. Introduction

Usual description of diffusion is based on Fokker–Planck equation, which describes probability density function (PDF) of the moving Brownian particle. However, many experiments confirm, that dynamics of complex systems is not satisfactorily described by this Gaussian process. Let us mention here diffusion on fractals [1], transport of fluid in porous media [2], single molecule spectroscopy [3], and many more (see [4] and references therein). Mentioned here physical systems concern the so-called anomalous diffusion. In contrary to normal, anomalous diffusion is characterized by nonlinear dependence of centered second moment. Thus we can distinguish three cases. When $\left\langle x^2\right\rangle \propto t^\gamma$, where $\gamma > 1$, we consider a superdiffusive process and $\gamma = 1$ characterizes normal diffusion. The case of $\gamma < 1$ is characteristic for

* Presented at the XXV Marian Smoluchowski Symposium on Statistical Physics, "Fluctuation Relations in Nonequilibrium Regime", Kraków, Poland, September 10–13, 2012.

subdiffusive processes, which are a main subject of this paper. Moreover, we assume finite moments waiting times (with finite second moment jump size) scenario to avoid special cases discussed in [5] when the long waiting times can be hidden by long jumps.

In order to properly describe anomalous diffusion phenomenon, several authors introduced fractional Fokker–Planck equations (FFPE), describing PDF of anomalously diffusing particles [4]. For the case of space-dependent force $F(x)$, a FFPE was proposed in [6] (see also [4]). Its stochastic representation was found in [7] and this result revealed that subdiffusion is a combination of two independent mechanisms. First is the standard diffusion represented by a Itô process $X(\tau)$, while the second is the waiting time distribution represented by the so-called inverse α-stable subordinator $S_\alpha(t)$, where $\alpha \in (0,1)$. This heavy-tailed waiting-time distribution is responsible for trapping events (periods when the particle stays motionless).

Recent experiments show, however, another interesting phenomenon. Namely, in many physical systems, we observe a transition from the subdiffusive motion ($\gamma < 1$) in short times to the normal ($\gamma = 1$) motion in long times. This dual behavior was empirically confirmed in a number of systems. One should mention here random motion of bright points at the solar photosphere [8], motion of molecules inside living cells [9–11], and transport of passive tracers in heterogeneous media [12]. To model such situations instead of α-stable, the inverse tempered α-stable subordinator $S_{\alpha,\lambda}(t)$ was proposed [13–15]. This approach resulted in derivation of tempered version of FFPE. Thus the complete description of tempered subdiffusion in the case of space dependent force was given.

In this paper, we propose a stochastic process driven by inverse tempered α-stable subordinator and subordinated Brownian motion whose PDF satisfies tempered version of FFPE with space dependent force and diffusion coefficient. Thus we obtain a new description of subdiffusion in the case of tempered α-stable waiting times. This will extend the results presented recently in [7, 16, 17].

This article is structured as follows. In section 2 we give a short description of tempered α-stable processes and their inverses. We propose a model of tempered subdiffusion with space dependent force and diffusion coefficient. We show that PDF of the introduced model satisfies a tempered version of FFPE with some more general integro-differential operator. In section 3 we present an algorithm of simulating its sample paths. By the application of Monte Carlo methods, we also approximate solutions of the generalized FFPE. Section 4 contains the conclusions.

2. Tempered FFPE and its stochastic representation

In order to capture short time subdiffusive and long time normal character of motion, one needs to modify α-stable heavy-tailed waiting times. The first step in this direction was first proposed in [18]. Their idea was based on cutting of the heavy tails and was further extended to smoothly truncated stable laws in [19]. The complete description of a class of tempered α-stable distributions was given by Rosiński in [20]. His modification is based on the Lévy measure of α-stable process. As a result, the class of tempered distribution has finite moments of all orders, but, at the same time, it resembles stable laws in many aspects (see [20] for more details).

Let us define strictly increasing tempered α-stable Lévy process $T_{\alpha,\lambda}(\tau)$ via its Laplace transform

$$E\left(e^{-uT_{\alpha,\lambda}(\tau)}\right) = e^{-\tau((u+\lambda)^{\alpha}-\lambda^{\alpha})} .$$

Here, the constant $\lambda > 0$ is the tempering parameter and $0 < \alpha < 1$ is the stability parameter. Let us observe that if $\lambda \searrow 0$, we obtain the Laplace transform of one-sided positive stable distribution. The PDF of $T_{\alpha,\lambda}$ has a simple form $ce^{-\lambda x} f_{\alpha}(x)$, where $f_{\alpha}(x)$ is the PDF of one-sided stable distribution and $c > 0$ is the normalizing constant [20]. Based on the above, we define the inverse tempered α-stable subordinator by

$$S_{\alpha,\lambda}(t) = \inf\{\tau > 0 : T_{\alpha,\lambda}(\tau) > t\} , \quad t \geq 0 .$$

The process $S_{\alpha,\lambda}(t)$ serves as a new operational time of a system. In Fig. 1 (top panel) we present a trajectory of the process $S_{\alpha,\lambda}(t)$. One observes the constant intervals distributed according to the tempered α-stable law. This intervals represent the events when the particle stays motionless.

Before we formulate our main result, let us present two important theorems, which will be used later.

Theorem 2.1. *Let $S_{\alpha,\lambda}(t)$ be the inverse tempered α-stable subordinator, and $l(t)$ be an integrable function, then*

$$\mathbb{E}\left[\int_0^t l(Z(S_{\alpha,\lambda}(\tau)))dS_{\alpha,\lambda}(\tau)\right] = \int_0^t M(t-\tau)\mathbb{E}\left[l\left(Z(S_{\alpha,\lambda}(\tau))\right)\right] d\tau , \qquad (1)$$

where the memory kernel $M(t)$ is defined via its Laplace transform

$$\hat{M}(k) = \int_0^{\infty} e^{-kt} M(t)dt = \frac{1}{(k+\lambda)^{\alpha} - \lambda^{\alpha}} \qquad (2)$$

and the process $Z(t)$ is given by

$$dZ(t) = F(Z(t))dt + \sqrt{D\left(Z(t)\right)}dB(t)\,. \tag{3}$$

Proof of this theorem is given in Appendix A.

With the help of Theorem 2.1, we can formulate the following corollary the proof of which is similar as in [21].

Corollary 2.1. *For any continuous function $f(\cdot)$ we have the following equality*

$$\mathbb{E}\left[\int\limits_0^t f(\tau) l\left(Z(S_{\alpha,\lambda}(\tau))\right) dS_{\alpha,\lambda}(\tau)\right] = \int\limits_0^t f(\tau)\Phi_\tau \mathbb{E}\left[l\left(Z(S_{\alpha,\lambda}(\tau))\right)\right] d\tau\,,$$

where Φ_t is the integro-differential operator defined as

$$\Phi_t f(t) = \frac{d}{dt}\int\limits_0^t M(t-y)f(y)dy\,, \tag{4}$$

where the memory kernel $M(t)$ is defined in Eq. (2) and $Z(t)$ is given by Eq. (3).

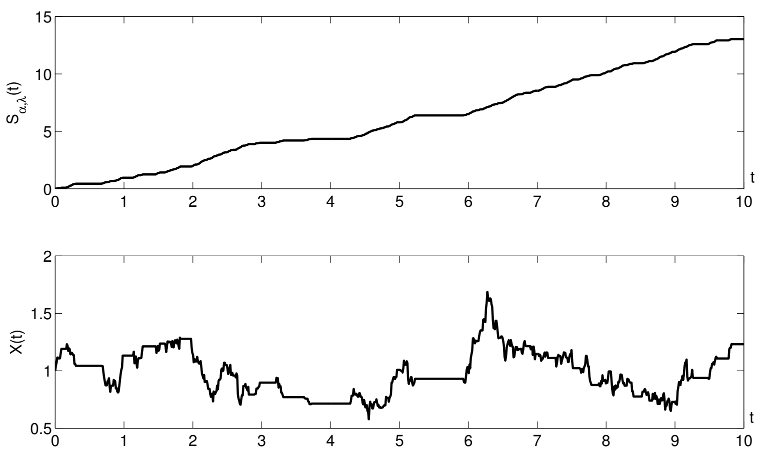

Fig. 1. Sample realizations of the inverse tempered α-stable subordinator $S_{\alpha,\lambda}(t)$ (top panel) and the process $X(t)$, Eq. (8). The parameters are $\alpha = 0.8$ and $\lambda = 0.1$. $F(x) = 0.01x^{1/6}$ and $D(x) = 0.06x^{1/4}$. The constant intervals in the trajectory of the process $X(t)$ represent the periods of stagnation of subdiffusive particle.

Now, let us show that PDF $q(\tau,t)$ of the process $Y(t)=\int_0^t f(t_1)dS_{\alpha,\lambda}(t_1)$ satisfies the following equation [22]

$$\frac{\partial}{\partial t}q(\tau,t)=-f(t)\frac{\partial}{\partial\tau}\Phi_t q(\tau,t)\,. \tag{5}$$

This will extend the results presented in [23]. Equation (5) was introduced in [22] within generalized master equation approach, and it describes the response of a continuous time random walk system to a time-dependent field $f(t)$. Authors showed that the moments

$$m_n(t)=\int\limits_{-\infty}^{\infty}\tau^n q(\tau,t)d\tau\,,$$

of the distribution $q(\tau,t)$ satisfy the following recursive form

$$m_n(t)=n\int\limits_0^t f(t_1)\Phi_{t_1}m_{n-1}(t_1)dt_1\,. \tag{6}$$

The formal assumption is that $m_0=1$.

Thus we can formulate the following theorem.

Theorem 2.2. *Let the process Y be defined as*

$$Y(t)=\int\limits_0^t f(t_1)dS_{\alpha,\lambda}(t_1)\,,\quad t\geq 0 \tag{7}$$

and the function $f(\cdot)$ be integrable. Then the PDF of Y satisfies the tempered fractional Fokker–Planck equation (5), with time dependent force $f(\cdot)$.

Proof of this theorem is given in Appendix B.

In the next theorem, we give our main result which extends previous considerations in [14], to the case when drift coefficient is space dependent. Following [14], we propose a form of a stochastic process $X(t)$ in the case when the force and diffusion coefficient are space dependent. We also prove the corresponding tempered Fokker–Planck equation for PDF $p(x,t)$ of $X(t)$.

Theorem 2.3. *Assume that inverse tempered α-stable subordinator $S_{\alpha,\lambda}(t)$ is independent of the standard Brownian motion $B(\tau)$. Then the PDF of the following process*

$$\begin{aligned} X(0)&=0\,,\\ dX(t)&=F\left(X(t)\right)dS_{\alpha,\lambda}(t)+\sqrt{D\left(X(t)\right)}dB(S_{\alpha,\lambda}(t))\,, \end{aligned} \tag{8}$$

is the stochastic solution of the tempered fractional Fokker–Planck equation

$$\frac{\partial}{\partial t}p(x,t) = \left[-\frac{\partial}{\partial x}F(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}D(x)\right]\Phi_t p(x,t)\,. \tag{9}$$

Here, the initial condition is $p(x,0)=\delta(x)$ *and the functions* $F(x), D(x) \in C^\infty((-\infty,\infty))$.

Let us discuss briefly the proposed model. First, we observe that force and diffusion coefficient are space dependent. When $D=1$ we obtain results presented in [14]. Moreover, we observe that when $\lambda \searrow 0$ the integro-differential operator (9) is proportional to the fractional Riemann–Liouville derivative, thus we recover results derived in [7] in the case when $D=1$. The proof of this theorem is presented in Appendix C.

3. Approximation of sample paths

Here, we will show how to simulate sample paths of the tempered subdiffusion process $X(t)$. In the first step, we need to simulate the trajectory of the inverse subordinator $S_{\alpha,\lambda}(t)$. As was shown in [14], the approximation process can be written as

$$S_{\alpha,\lambda,\Delta t}(t) = (\min\{n\in\mathbb{N} : T_{\alpha,\lambda}(n\Delta t) > t\} - 1)\Delta t\,, \tag{10}$$

where Δt is the step length and $t\in[0,T]$. To obtain accurate approximation we need to take small Δt [23]. From the form of (10), we infer that to simulate the process $S_{\alpha,\lambda,\Delta t}$ we only need to generate the values $T_{\alpha,\lambda}(n\Delta t)$, $n=1,2,\ldots$ Since the process $T_{\alpha,\lambda}(t)$ is a Lévy process this can be accomplished by method of summing up the increments

$$T_{\alpha,\lambda}(0) = 0\,, \tag{11}$$

$$T_{\alpha,\lambda}(n\Delta t) = T_{\alpha,\lambda}((n-1)\Delta t) + T_{\rm i}\,, \tag{12}$$

where $T_{\rm i}$ are independent, tempered α-stable random variables. The algorithm of generating $T_{\rm i}$ was proposed in [27], one can find it also in [14]. For completeness, we include it in the Appendix D. In the second step, we finally approximate the process $X(t)$ by the classical Euler scheme [26]

$$\begin{aligned} X(0) &= 0\,,\\ X(t_k) &= X(t_{k-1}) + F\left(X(t_{k-1})\right)\Delta\tau_k + \sqrt{D\left(X(t_{k-1})\right)}(\Delta\tau_k)^{1/2}\xi_k\,, \end{aligned}$$

where ξ_k are i.i.d. standard normal random variables $\xi_k \sim N(0,1)$ and $\Delta\tau_k = S_{\alpha,\lambda}(t_k) - S_{\alpha,\lambda}(t_{k-1})$.

In Fig. 1, we present sample realizations of the time-change process $S_{\alpha,\lambda}(t)$ and $X(t)$. Observe the characteristic subdiffusive constant periods visible on the plots. In Fig. 2 we obtained the solution of (9) via Monte Carlo methods.

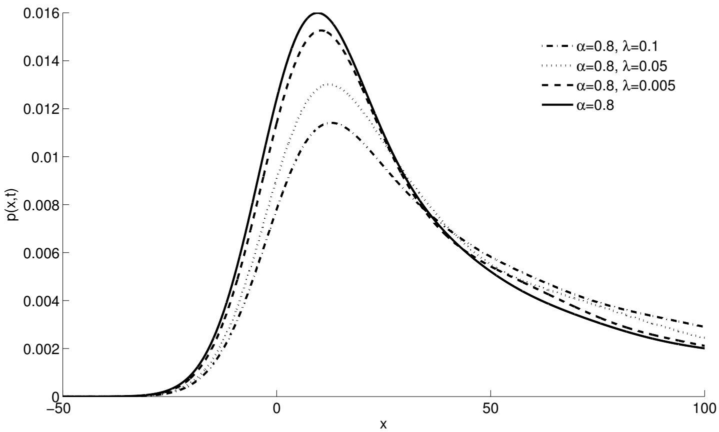

Fig. 2. Approximated solution of Eq. (9). One can observe that when $\lambda \to 0$ we approximate solution of the FFPE with pure α-stable waiting times. The parameter $\alpha = 0.8$, $F(x) = x$, and $D(x) = 0.06x^{1/4}$.

4. Conclusions

It is confirmed by many experiments that tempered processes play an important role in description of such complicated phenomenon as subdiffusion. In this paper, we have extended the model of tempered subdiffusion proposed in [28] to the case of space dependent force and diffusion coefficient. We found that our model is driven by the inverse tempered α-stable subordinator and Brownian motion. We derived the corresponding tempered FFPE equation for the PDF of the considered system. We also presented an algorithm of simulating of the sample paths of the underlying process. We compared our results with the case of purely α-stable waiting times [21]. We believe that presented here methodology will be useful in description of anomalous diffusion phenomenon.

Appendix A

Proof of Theorem 2.1.

Let us denote by $g(\tau, t)$ the PDF of the process $S_{\alpha,\lambda}(t)$ and by $v(z, t)$ the PDF of the process $Z(t)$ given by Eq. (3). Moreover, we assume that both $Z(t)$ and $S_{\alpha,\lambda}(t)$ are independent. From [14] we know that the Laplace transform of g, has the form

$$\hat{g}(\tau, k) = \int_0^\infty e^{-kt} g(\tau, t) dt = \frac{(k+\lambda)^\alpha - \lambda^\alpha}{k} e^{-\tau[(k+\lambda)^\alpha - \lambda^\alpha]} . \tag{A.1}$$

Next, we have

$$\mathbb{E}\left[\int\limits_0^t l(Z(S_{\alpha,\lambda}(\tau)))dS_{\alpha,\lambda}(\tau)\right] = \mathbb{E}\left[\int\limits_0^{S_{\alpha,\lambda}(t)} l(Z(\tau))d\tau\right]$$
$$= \int\limits_0^\infty g(\tau,t)\int\limits_0^\tau\int\limits_{-\infty}^\infty l(z)v(z,t_1)dzdt_1d\tau\,. \tag{A.2}$$

Now, the Laplace transform $\mathcal{L}_{t\to k}$ of the last line of (A.2) yields

$$\mathcal{L}\left\{\int\limits_0^\infty g(\tau,t)\int\limits_0^\tau\int\limits_{-\infty}^\infty l(z)v(z,t_1)dzdt_1d\tau\right\}$$
$$= \frac{1}{k}\int\limits_0^\infty\int\limits_{-\infty}^\infty e^{-\tau[(k+\lambda)^\alpha-\lambda^\alpha]}l(z)v(z,\tau)dzd\tau = \frac{1}{k}\int\limits_{-\infty}^\infty l(z)\hat{v}(z,[(k+\lambda)^\alpha-\lambda^\alpha])dz\,. \tag{A.3}$$

Taking the Laplace transform of the right-hand side (RHS) of Eq. (1).

$$\mathcal{L}\left\{\int\limits_0^t M(t-\tau)\int\limits_0^\infty\int\limits_{-\infty}^\infty l(z)v(z,s)g(s,\tau)dzdsd\tau\right\}$$
$$= \frac{1}{k}\int\limits_0^\infty\int\limits_{-\infty}^\infty e^{-s[(k+\lambda)^\alpha-\lambda^\alpha]}l(z)v(z,s)dzds = \frac{1}{k}\int\limits_{-\infty}^\infty l(z)\hat{v}(z,[(k+\lambda)^\alpha-\lambda^\alpha])dz\,. \tag{A.4}$$

Since in the Laplace domain both sides of Eq. (1) are equal, the result of Eq. (1) follows from uniqueness of the Laplace transform.

Appendix B

Proof of Theorem 2.2.

First, let us show that the moments of the process Y coincide with (6). By the change of variable formula, we can get

$$\left(\int\limits_0^t f(t_1)dS_{\alpha,\lambda}(t_1)\right)^n = n\int\limits_0^t\left(\int\limits_0^{t_1} f(t_2)dS_{\alpha,\lambda}(t_2)\right)^{n-1} f(t_1)dS_{\alpha,\lambda}(t_1)\,.$$

Thus, performing n iterations, we can get that

$$\left(\int_0^t f(t_1)dS_{\alpha,\lambda}(t_1)\right)^n = n!\int_0^t\int_0^{t_1}\ldots\int_0^{t_{n-1}} f(t_1)\ldots f(t_n)dS_{\alpha,\lambda}(t_n)\ldots dS_{\alpha,\lambda}(t_1)\,. \tag{B.1}$$

Now, following the same procedure as in [23], we introduce the following measure on the $[0,\infty)$, by $\Pi((s,t]) = S_{\alpha,\lambda}(t) - S_{\alpha,\lambda}(s)$, where $t > s \geq 0$. Let $\{C(t)\}_{t\geq 0}$ be the Cox process directed by Π. $C(t)$ is the renewal process with renewal function

$$u(t) = \mathbb{E}[C(t)] = \mathbb{E}[S_{\alpha,\lambda}(t)] = \int_0^t M(t_1)dt_1\,.$$

From [23], we have

$$\mathbb{E}[dS_{\alpha,\lambda}(t_1)\ldots dS_{\alpha,\lambda}(t_n)] = \prod_{i=1}^n u'(t_i - t_{i+1})dt_i\,,$$

where $t_1 > t_2 > \ldots > t_n > t_{n+1} = 0$. Combining the above and (B.1), we obtain

$$\begin{aligned} r_n(t) &= \mathbb{E}\left[\left(\int_0^t f(t_1)dS_{\alpha,\lambda}(t_1)\right)^n\right] \\ &= n!\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}}\prod_{i=1}^n f(t_i)M(t_i - t_{i+1})dt_n\ldots dt_1\,. \end{aligned}$$

Thus we have

$$\begin{aligned} r_n(t) &= n\int_0^t f(t_1)\int_0^{t_1} M(t_1-t_2)\frac{d}{dt_2}r_{n-1}(t_2)dt_2dt_1 \\ &= n\int_0^t f(t_1)\frac{d}{dt_1}\int_0^{t_1}\int_0^{t_1-t_2} M(s)\frac{d}{dt_2}r_{n-1}(t_2)dsdt_2dt_1 \\ &= n\int_0^t f(t_1)\Phi_{t_1}r_{n-1}(t_1)dt_1\,. \end{aligned}$$

This shows that $m_n(t)$ coincides with $r_n(t)$. Therefore, to prove that the PDF of the process $Y(t)$ satisfies (5) it is enough to show its characteristic function is holomorphic at a neighborhood of zero. In such case, the moments determine uniquely the distribution. Assume that $t_0 > 0$. We have that

$$\begin{aligned} r_n(t_0) = \mathbb{E}\left[\left(\int_0^{t_0} f(u)dS_{\alpha,\lambda}(u)\right)^n\right] &\le M^n\mathbb{E}\left[S^n_{\alpha,\lambda}(t_0)\right] \\ &= M^n\int_0^\infty x^{n-1}\mathbb{P}\left(S_{\alpha,\lambda}(t_0) > x\right)dx \\ &= M^n\int_0^\infty x^{n-1}\mathbb{P}\left(T_{\alpha,\lambda}(x) < t_0\right)dx \\ &= M^n\int_0^\infty x^{n-1}\mathbb{P}\left(e^{-uT_{\alpha,\lambda}(x)} > e^{-ut_0}\right)dx \\ &\le M^n e^{ut_0}\int_0^\infty x^{n-1}e^{-x[(u+\lambda)^\alpha-\lambda^\alpha]}dx \\ &= M^n\frac{e^{ut_0}\Gamma(n)}{[(u+\lambda)^\alpha-\lambda^\alpha]^n}\,, \end{aligned}$$

where $M = \sup_{0\le s\le t_0}|f(s)|$. In consequence, if we set $|z| < M^{-1}$ and u large enough, we get that the series $\sum_{n=1}^\infty r_n(t_0)z^n/n!$ is convergent. Therefore, indeed, the moments determine the distribution (see [24], chapter VII, Section 3). Thus, the PDF of $Y(t_0)$ is equal to the solution $q(\tau, t_0)$ of (5).

Appendix C

Proof of Theorem 2.3.

The main idea of the proof of Theorem 2.3 comes from the paper [21]. We use the fact that characteristic function determines the distributions [25], thus we need to show that characteristic function of the process $X(t)$ satisfies Fourier transformation of Eq. (9) with respect to the space variable x. We use the assumption that processes $S_{\alpha,\lambda}(t)$ and $Z(t)$ are independent, moreover [23], we have that $X(t) = Z(S_{\alpha,\lambda}(t))$. The process $Z(t)$ is given by (3).

From subdiffusive Itô formula [29] in tempered case, we have

$$\begin{aligned} de^{ikX(t)} &= ike^{ikX(t)}F\left(X(t)\right)dS_{\alpha,\lambda}(t) - \frac{k^2}{2}D\left(X(t)\right)e^{ikX(t)}dS_{\alpha,\lambda}(t) \\ &\quad + ike^{ikX(t)}\sqrt{D\left(X(t)\right)}dB(S_{\alpha,\lambda}(t))\,. \end{aligned} \tag{C.1}$$

Thus, we get

$$\begin{aligned} \mathbb{E}\left[e^{ikX(t)}\right] - \mathbb{E}\left[e^{ikX(0)}\right] &= \mathbb{E}\left[ik\int\limits_0^{S_{\alpha,\lambda}(t)} F\left(Z(\tau)\right)e^{ikZ(\tau)}d\tau\right] \\ &\quad - \mathbb{E}\left[\frac{k^2}{2}\int\limits_0^{S_{\alpha,\lambda}(t)} D\left(Z(\tau)\right)e^{ikZ(\tau)}d\tau\right] \\ &= \mathbb{E}\left[ik\int\limits_0^t F\left(Z(S_{\alpha,\lambda}(t_1))\right)e^{ikZ(S_{\alpha,\lambda}(t_1))}dS_{\alpha,\lambda}(t_1)\right] \\ &\quad - \mathbb{E}\left[\frac{k^2}{2}\int\limits_0^t D\left(Z(S_{\alpha,\lambda}(t_1))\right)e^{ikZ(S_{\alpha,\lambda}(t_1))}dS_{\alpha,\lambda}(t_1)\right]. \end{aligned}$$

Noting that the Fourier transform of function $f(x)$ is defined as $\mathcal{F}\{f(x)\}(k) = \int_{-\infty}^{+\infty} f(x)e^{ikx}$, one can observe that $\mathbb{E}\big(F(X(t))e^{ikX(t)}\big) = \mathcal{F}\{F(x)p(x,t)\}$, where $p(x,t)$ is the PDF of $X(t)$. Similar thing happens with $\mathbb{E}\big(D(X(t))e^{ikX(t)}\big)$. Taking the first derivative w.r.t. t, we can get

$$\begin{aligned} \frac{\partial}{\partial t}\mathbb{E}\left[e^{ikX(t)}\right] &= ik\Phi_t\mathbb{E}\left[F\left(X(t)\right)e^{ikX(t)}\right] \\ &\quad - \frac{k^2}{2}\Phi_t\mathbb{E}\left[D\left(X(t)\right)e^{ikX(t)}\right], \end{aligned} \tag{C.2}$$

which is identical with the Fourier transform of Eq. (9) with initial condition $\mathbb{E}\left[e^{ikX(0)}\right] = 1$. This leads to the conclusion that PDF of $X(t)$ satisfies Eq. (9).

Appendix D

In order to complete the presentation, we show the method of simulating tempered α-stable random variables. This method was proposed in [27]. Thus to simulate the tempered random variable $T > 0$ with the Laplace transform $E(e^{-uT}) = e^{-\Delta t((u+\lambda)^\alpha - \lambda^\alpha)}$, we follow three steps:

(I) Generate exponential random variable E with mean λ^{-1};

(II) Generate totally skewed α-stable random variable S using the formula [30]

$$S = \Delta t^{1/\alpha} \frac{\sin\left(\alpha\left(J+\frac{\pi}{2}\right)\right)}{\cos(J)^{1/\alpha}} \left(\frac{\cos\left(J-\alpha\left(J+\frac{\pi}{2}\right)\right)}{W}\right)^{(1-\alpha)/\alpha}, \quad \text{(D.1)}$$

where, J is uniformly distributed on $[-\pi/2, \pi/2]$, and W exponentially distributed with mean 1;

(III) If $E > S$ put $T = S$, otherwise go to step (I).

REFERENCES

[1] J. Stephenson, *Physica A* **222**, 234 (1995).
[2] H. Spohn, *J. Phys. I* **3**, 69 (1993).
[3] E. Barkai, R. Silbey, *Chem. Phys. Lett.* **310**, 287 (1999).
[4] R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).
[5] B. Dybiec, E. Gudowska-Nowak, *Phys. Rev.* **E80**, 061122 (2009).
[6] R. Metzler, E. Barkai, J. Klafter, *Europhys. Lett.* **46**, 431 (1999).
[7] M. Magdziarz, A. Weron, K. Weron, *Phys. Rev.* **E75**, 016708 (2007).
[8] A.C. Cadavid, J.K. Lawrence, A.A. Ruzmaikin, *Astrophys. J.* **521**, 844 (1999).
[9] M. Platani, I. Goldberg, A.I. Lamond, J.R. Swedlow, *Nat. Cell Biol.* **4**, 502 (2002).
[10] K. Murase *et al.*, *Biophys. J.* **86**, 4075 (2004).
[11] J.-H. Jeon *et al.*, *Phys. Rev. Lett.* **106**, 048103 (2011).
[12] M.M. Meerschaert, Y. Zhang, B. Baeumer, *Geophys. Res. Lett.* **35**, L17403 (2008).
[13] A. Stanislavsky, K. Weron, A. Weron, *Phys. Rev.* **E78**, 051106 (2009).
[14] J. Gajda, M. Magdziarz, *Phys. Rev.* **E82**, 011117 (2010).
[15] J. Gajda, M. Magdziarz, *Phys. Rev.* **E84**, 021137 (2011).
[16] M. Magdziarz, A. Weron, J. Klafter, *Phys. Rev. Lett.* **101**, 210601 (2008).
[17] M. Magdziarz, *J. Stat. Phys.* **135**, 763 (2009).
[18] R.N. Mantegna, H.E. Stanley, *Phys. Rev. Lett.* **73**, 2946 (1994).
[19] I. Koponen, *Phys. Rev.* **E52**, 1197 (1995).
[20] J. Rosiński, *Stoch. Process. Appl.* **117**, 677 (2007).
[21] L. Lv, W. Qiu, F. Ren, *J. Stat. Phys.* **149**, 619 (2012).

[22] I.M. Sokolov, J. Klafter, *Phys. Rev. Lett.* **97**, 140602 (2006).

[23] M. Magdziarz, *Stoch. Process. Appl.* **119**, 3238 (2009).

[24] W. Feller, *An Introduction to Probability Theory and Its Applications*, 2nd edn., vol. 2, Wiley, New York 1971.

[25] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge 1999.

[26] A. Janicki, A. Weron, *Simulation and Chaotic Behaviour of α-Stable Stochastic Processes*, Marcel Dekker, New York 1994.

[27] B. Baeumer, M.M. Meerschaert, *J. Comput. Appl. Math.* **233**, 2438 (2010).

[28] A. Piryatinska, A.I. Saichev, W.A. Woyczynski, *Physica A* **349**, 375 (2005).

[29] A. Weron, S. Orzeł, *Acta Phys. Pol. B* **40**, 1271 (2009).

[30] R. Weron, *Statist. Probab. Lett.* **28**, 165 (1996).