THE ROLE OF THE POTENTIAL OF A NEW BISTABLE SYSTEM ON STOCHASTIC RESONANCE*

Guitian He†, Chao Min, Yan Tian

School of Sciences, Southwest Petroleum University, Chengdu 610500, P.R. China

(Received October 4, 2013; revised version received October 14, 2013)

The phenomenon of stochastic resonance (SR) in a new bistable system is investigated in the presence of periodic force and additive noise. The exact expression of signal-to-noise ratio (SNR) of the new bistable system is obtained. Theoretical analysis and numerical simulation results show that the output SNR is a non-monotonic function of the noise intensity. The influence of parameters of system on SR is studied, moreover, the relationship between the barrier height of potential and the performance of SR is discussed. The target of this work is to explore the influence of the shape of potential function on SR and give a tool when dealing with particle diffusion, SR control and information processing.

DOI:10.5506/APhysPolB.45.29
PACS numbers: 05.40.–a 02.50.Ey

1. Introduction

Noise and their effects on physical, chemical and biological systems have been extensively studied over the last decade. One important effect is stochastic resonance (SR), in which a weak external periodic signal can be amplified by additive noise as it passes through a non-linear system [1–11]. The concept of SR was originally introduced by Benzi *et al.* [1] and the two-state theory was investigated to describe SR [3, 6–10]. Today, SR is a well-known behavior of non-linear stochastic dynamics [5]. The SR has been widely discussed both theoretically and experimentally in many fields, such as periodically recurrent of ice ages, laser physics, information processing, chemistry, and other scientific fields [12–17].

Recently, there is a growing interest in exploring SR in complex systems such as the chaotic system, neuronal network, bistable system with time delay [2–4, 6–29]. Especially, a typical bistable potential $U(x) =$

* This work was supported by the Science and Technology Project of the Education Department of the Sichuan Province (Grant No. 13ZA0191).

† heguitian100@163.com

$-\alpha x^2/2 + \beta x^4/4$ in which a Brownian particle is present was extensively investigated. The system is driven by periodic perturbation $s(t) = A\cos(\Omega t)$ and random noise $\xi(t)$. Due to perturbation, the particle randomly jumps between the two wells with the Kramers escape rate r_{K}. When the average escape time ($\tau_{\mathrm{K}} = 1/r_{\mathrm{K}}$) of the particle over the barrier approximately equals the half-time period τ_{Ω} of periodic perturbation $s(t)$, the noise induced inter-well jumps. This phenomenon is the stochastic resonance (SR). However, only a few works discussed the effects of different potential functions on the output of SR. Note that various potential functions can be generated and easily applied to real mechanical systems and information processing. Consequently, it is necessary to investigate the SR mechanism in different potential functions.

When studying the problem of a Brownian particle in a symmetric double well periodically tilted in time, the corresponding potential $U(x)$ is usually assumed to diverge like $U(x) \sim x^4$ at large x. However, the divergence of the potential for $|x| \to \infty$ strongly affects the response of the system to an external time-periodic forcing. Motivated by the above ideas, the goal of the present paper is to investigate the effects of various potential functions on SR. For simplicity, we assume that $U(x) \sim |x|^q$ for $x \to \pm\infty$.

There are several technical methods to describe the SR phenomenon of particle diffusion, for instance, adiabatic approximation(or two-state) theory, Fokker–Plank equation. There are also several quantities to quantify the SR, such as signal-to-noise ratio (SNR), spectral power amplification (SPA), the position amplitude of the particle, the first moment of the position of the particle. In this paper, some of the previous methods and quantities to study the SR phenomenon in a new bistable system are used.

This paper is organized as follows. Section 2 presents the new bistable system driven by sinusoidal signal and addictive Gaussian white noise. Moreover, the dynamic of potential function of new bistable system under various parameters is studied. Section 3 gets the exact expression of SNR of the system. With the output SNR, the conventional SR takes place with the intensity. Section 4 draws some conclusions.

2. Potential function

In Refs. [12, 20], the researchers studied the following bistable potential

$$U(x) = a\,\exp\left(-\frac{x^2}{b^2}\right) + k\frac{|x|^q}{q}\,. \tag{1}$$

Furthermore, they obtained material results about SR. In our work, the potential function which we will investigate is given by

$$U(x) = a\,\exp\left(-\frac{x^2}{b^2}\right) - \frac{c}{2}x^2 + k\frac{|x|^q}{q}\,, \tag{2}$$

where $a \geq 0,\ b,\ c,\ k > 0$. If $a = 0$ and $q = 4$, then the potential function $U(x)$ is deduced to the typical bistable potential function $-cx^2/2 + kx^4/4$. We consider a Brownian particle described by the overdamped Langevin equation

$$\dot{x} = \frac{2a}{b^2}x\,\exp\left(-\frac{x^2}{b^2}\right) + cx - k\frac{|x|^q}{x} + s(t) + \xi(t)\,, \tag{3}$$

where $s(t)$ is a weak sinusoidal signal $A\cos(\Omega t)$ and $\xi(t)$ is an addictive Gaussian white noise with zero-mean and auto-correlation function

$$\xi(t)\xi\left(t'\right) = 2D\delta\left(t - t'\right)\,, \tag{4}$$

in which D is the noise intensity.

First, the dynamic of potential function Eq. (2) under various parameters will be studied. For numerical purpose, it is convenient to choose a, b, c and k as 1. Then we investigate the relation between the barrier height and parameter q. Figure 1 demonstrates that the barrier height, ΔU, and the potential minima, $\pm x_{\mathrm{m}}$, weakly depend on q. As q is increased, the slope of the potential wall increases as shown in Fig. 1. Figure 1 depicts also that

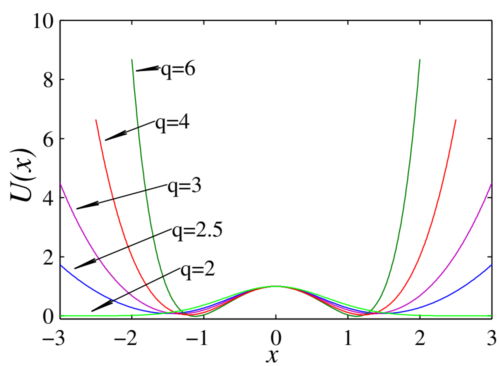

Fig. 1. Potential function (2) for $a = b = c = k = 1$.

the potential $U(x)$ (about $x > 2$) is horizontal for $q = 2$. If parameters $a = b = c = 1$, $q = 3$ are fixed, the relation between the barrier height and parameter k is depicted in Fig. 2. Figure 2 shows that both the barrier height and potential minima depend on k. Figure 2 demonstrates that the barrier height and potential minima shrink with k increasing which means that the particle is more confined between the two minima. As k is increased,

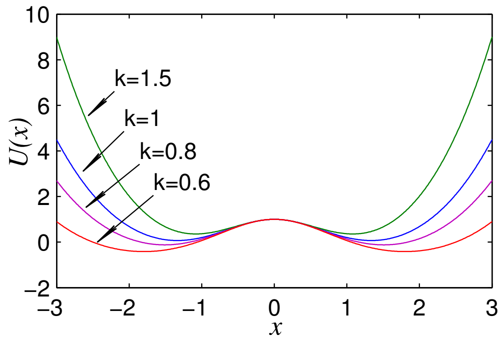

Fig. 2. Potential function (2) for $a = b = c = 1$, $q = 3$.

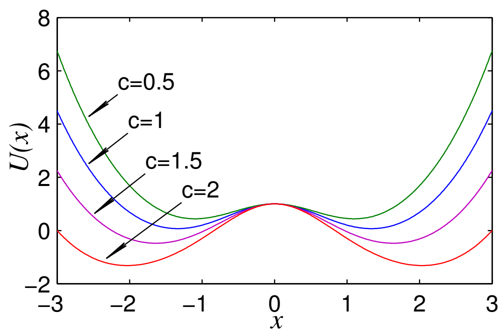

Fig. 3. Potential function (2) for $a = b = k = 1$, $q = 3$.

the slope of the potential wall increases as shown in Fig. 2. The relation between barrier height and parameter c is depicted in Fig. 3. The barrier height and potential minima expend with c increasing which means that the particle goes more farther from one well to the other. Figure 4 depicts that the barrier height depends on b. However, the potential minima weakly

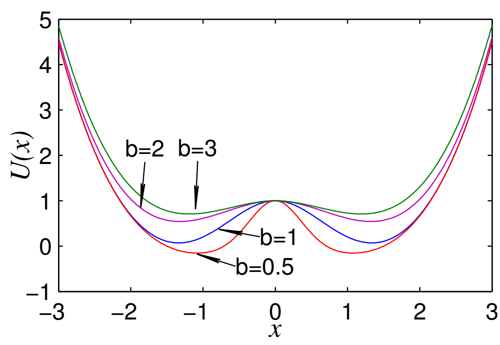

Fig. 4. Potential function (2) for $a = c = k = 1$, $q = 3$.

depend on b. The valley is shallow with b increasing which implies that the particle jumps over the barrier more easily with b increased. The parameters $b = c = k = 1,\ q = 3$ are fixed and the relation between barrier height and parameter a is demonstrated in Fig. 5. The barrier height depends on a, however, the potential minima weakly depend on a. The peak is lifted with a increasing which means that the particle jumps over the barrier much harder.

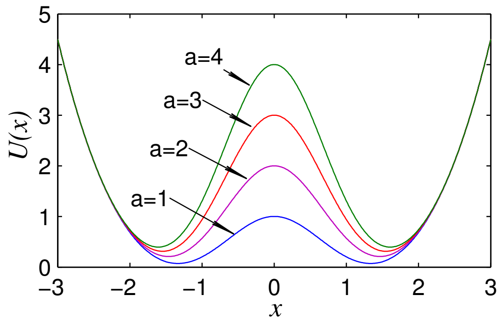

Fig. 5. Potential function (2) for $b = c = k = 1$, $q = 3$.

3. Stochastic resonance in a new bistable system

For small amplitudes, the response of the system to the periodic input signal can be written as [4]

$$\langle x(t) \rangle = \bar{x} \cos\left(\Omega t - \bar{\phi}\right) \tag{5}$$

with amplitude $\bar{x}$ and a phase lag $\bar{\phi}$. Approximate expressions for the amplitude and phase shift read

$$\bar{x}(D) = \frac{A \left\langle x^2 \right\rangle_0}{D} \frac{2r_{\mathrm{K}}}{4r_{\mathrm{K}}^2 + \Omega} \tag{6}$$

and

$$\bar{\phi} = \arctan\left(\frac{\Omega}{2r_{\mathrm{K}}}\right). \tag{7}$$

Here, the angular brackets $\langle \cdot \rangle$ denote ensemble averaging over a large number of phase space trajectories. $\langle x^2 \rangle_0$ is the variance of the stationary unperturbed process $x(t)$ $(A = 0)$.

According to Eq. (6), the behavior of the amplitude reads (for details refer to [12, 20])

$$\lim_{D \to \infty} \bar{x}(D) \sim D^{\frac{2}{q}-1}. \tag{8}$$

From Eq. (8), we can obviously see that $\bar{x}(D)$ goes to zero for $D \to 0$, $D \to \infty$ and $q > 2$ [12, 20]. This implies that there exists a maximum of $x(D)$ for some D. However, the $\bar{x}(D)$ goes monotonically with D increased. The case $q = 2$ is the marginal case. Figure 1 also presents the case. Therefore, we will investigate SR of Eq. (3) under the condition $q > 2$.

The signal-to-noise ratio (SNR), spectral power amplification (SPA), and other quantities have been proposed as indices to quantify the efficiency of SR. We will apply the signal-to-noise ratio (SNR) to quantify SR of the new bistable system. The corresponding Fokker–Plank equation (FPE) of Eq. (3) reads

$$\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(U'(x) - A\cos(\Omega t)\right)\rho(x,t) + D\frac{\partial^2}{\partial x^2}\rho(x,t)\,, \tag{9}$$

where $U'(x)$ denotes $dU(x)/dx$. Therefore, the steady state distribution function $\rho_{\mathrm{s}}(x,t)$ can be written as

$$\rho_{\mathrm{s}}(x,t) = N\exp\left(-\frac{\Phi(x,t)}{D}\right)\,, \tag{10}$$

where N is the normalization constant, and $\Phi(x,t)$ is given by

$$\Phi(x,t) = U(x) - Ax\cos(\Omega t)\,. \tag{11}$$

In the absence of modulation ($A = 0$), within the framework of the theory of SR [1–4, 6–10], one can obtain the Kramers rate

$$W_0 = \frac{\sqrt{|U''(x_{\mathrm{m}})||U''(0)|}}{2\pi}\exp\left(-\frac{\Delta U}{D}\right)\,, \tag{12}$$

where

$$\begin{aligned} U''(x) &= -\frac{2a}{b^2}\exp\left(-\frac{x^2}{b^2}\right) + \frac{4ax^2}{b^4}\exp\left(-\frac{x^2}{b^2}\right) + k(q-1)|x|^{q-2} - c\,,\\ \Delta U &= |U(x_{\mathrm{m}}) - U(0)|\,, \end{aligned} \tag{13}$$

and in the presence of modulation ($A \neq 0$), the modified Kramers rate is given by

$$\begin{aligned} W_{\pm} &= \frac{\sqrt{|U''(x_{\mathrm{m}})||U''(0)|}}{2\pi}\exp\left(-\frac{\Phi(0,t) - \Phi(\pm x_{\mathrm{m}},t)}{D}\right)\\ &= \frac{\sqrt{|U''(x_{\mathrm{m}})||U''(0)|}}{2\pi}\exp\left(-\frac{\Delta U \pm x_{\mathrm{m}} A\cos(\Omega t)}{D}\right)\,. \end{aligned} \tag{14}$$

According to the theory of Refs. [1–4, 6, 7], the bistable case is reduced to a two-state system, characterized by the occupation probabilities $n_+(t) = \int_0^{+\infty} P(x,t)dx$ and $n_-(t) = 1 - n_+$, respectively. The master equation for these occupation probabilities is

$$\begin{aligned}\frac{dn_+(t)}{dt} &= -\frac{dn_-(t)}{dt} = W_-(t)n_-(t) - W_+(t)n_+(t)\\ &= W_-(t) - [W_-(t) + W_+(t)]n_+(t)\,.\end{aligned} \tag{15}$$

The general solution of Eq. (15) is

$$n_+(t) = g^-(t)\left[n_+(t_0)g(t_0) + \int\limits_{t_0}^{t} W_+(t')g(t')dt'\right] \tag{16}$$

with $g(t) = \exp\int_{t_0}^{t}[W_-(t') + W_+(t')]dt'$.

We assume that the transition rate $W_\pm(t)$ is of the form

$$W_\pm(t) = f(\varDelta \pm s)\,. \tag{17}$$

Since we assume that A is small compared with the barrier height, then it is possible to make a Taylor expansion of the function $W_\pm(t)$

$$W_\pm(t) = \tfrac{1}{2}\left[W_0 \mp W_s A + o\left(A^2\right)\right]\,, \tag{18}$$

where W_0 and W_s are given by

$$\frac{W_0}{2} = f(\varDelta)\,, \qquad \frac{W_s}{2} = -\frac{df(\varDelta)}{ds}|_{A=0}\,. \tag{19}$$

The power spectrum is given by

$$S(\omega) = \frac{\pi W_s^2 A^2}{2\left(W_0^2 + \varOmega^2\right)}\delta(\omega - \varOmega) + \frac{2W_0}{W_0^2 + \omega^2}\left(1 - \frac{W_s^2 A^2}{2\left(W_0^2 + \varOmega^2\right)}\right)\,. \tag{20}$$

The SNR reads

$$\mathrm{SNR} = \frac{\pi W_s^2 A^2}{4W_0}\left(1 - \frac{W_s^2 A^2}{2\left(W_0^2 + \varOmega^2\right)}\right)^{-1}\,, \tag{21}$$

where

$$\begin{aligned}W_0 &= \frac{\sqrt{|U''(x_{\mathrm{m}})||U''(0)|}}{\pi}\exp\left(-\frac{\Delta U}{D}\right)\,,\\ W_s &= \frac{x_{\mathrm{m}} W_0}{D}\,.\end{aligned} \tag{22}$$

Since the SNR is always positive, namely, $1 - \frac{W_s^2 A^2}{2(W_0^2+\Omega^2)} > 0$, we can obtain the valid range of the noise intensity

$$D > \sqrt{\frac{x_{\mathrm{m}}^2 W_0^2 A^2}{2(W_0^2 + \Omega^2)}} \simeq \frac{x_{\mathrm{m}} A}{\sqrt{2}}\,. \tag{23}$$

SNR shows a rich structure as a function of D, a, b, c, k, A and Ω. The influence of noise intensity on the SNR is shown from Fig. 6 to Fig. 12. In order to compare with Fig. 1, the parameters a, b, c and k of Fig. 6 are the same as in Fig. 1. Figure 6 shows the SNR as a function of noise intensity D with various values q. The peak of the curve is sharp and there is an optimal noise intensity at which SNR takes its maximum that identifies a characteristic of the SR phenomenon. When D is small, the system is dominated by the effect of the standard SR model. The peak of the curve decreases slowly with the parameter q increasing. Comparing Fig. 1 and Fig. 6, it can be seen that the parameter q has a little influence on the barrier height and the SR phenomenon.

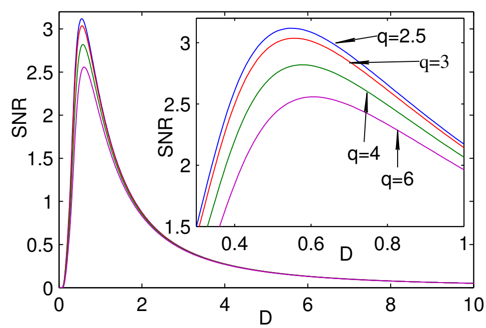

Fig. 6. SNR *versus* the noise intensity D with varied q for $a = b = c = k = 1$, $A = 0.4$, $\Omega = 0.1$.

The parameters of Fig. 7 are specified in figure caption. In Fig. 7, SNR is plotted as a function of the noise intensity D with varied k. There exists a maximum of SNR at some D which induces SR and the curve of SNR turns flat as D increases. It can be seen that SNR goes to zero as $D \to 0$ or $D \to \infty$. Comparing Fig. 2 and Fig. 7, we can see that the barrier height shrinks and the peak of SNR is lifted with k increased. The larger k, the lower barrier and the more easily SR is induced.

The parameters of Fig. 8 are specified in figure caption. In Fig. 8, the SNR as a function of noise intensity D with various value c is depicted. There is a maximum of SNR at some D. Figure 8 demonstrates that the resonance peaks decrease with c increasing. For large c, the curve is rather flat. Comparing Fig. 3 and Fig. 8, the larger c, the higher barrier, and the harder SR is induced.

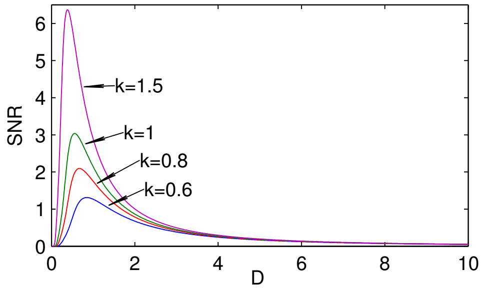

Fig. 7. SNR *versus* the noise intensity D with varied k for $a = b = c = 1$, $q = 3$, $A = 0.4$, $\Omega = 0.1$.

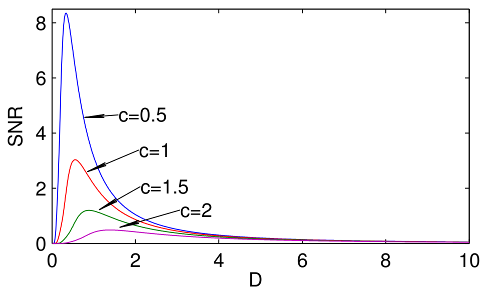

Fig. 8. SNR *versus* the noise intensity D with varied c for $a = b = k = 1$, $q = 3$, $A = 0.4$, $\Omega = 0.1$.

In Fig. 9, SNR is plotted as a function of the noise intensity D with varied b. Figure 9 shows that the resonance peak is lifted with b increasing. For large b, the peak is sharp, however, the peak is flat when b is small. Comparing Fig. 4 and Fig. 9, the larger b, the lower barrier, and the more easily SR is induced.

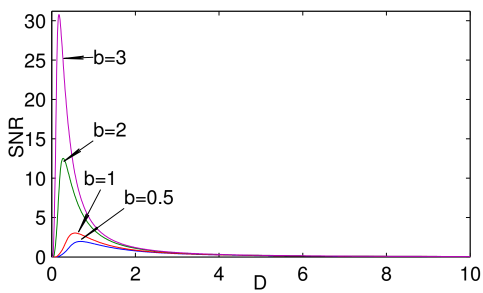

Fig. 9. SNR *versus* the noise intensity D with varied b for $a = c = k = 1$, $q = 3$, $A = 0.4$, $\Omega = 0.1$.

The parameters of Fig. 10 are specified in figure caption. In Fig. 10, the SNR as a function of noise intensity D with various values a is plotted. There is an optimal noise intensity at which SNR takes its maximum that induces SR. The resonance peak decreases with a increasing, as seen in Fig. 10. For small a, the peak is sharp, whereas the curve becomes flat with a increasing. Comparing Fig. 5 and Fig. 10, the larger a, the higher barrier and the harder SR is induced.

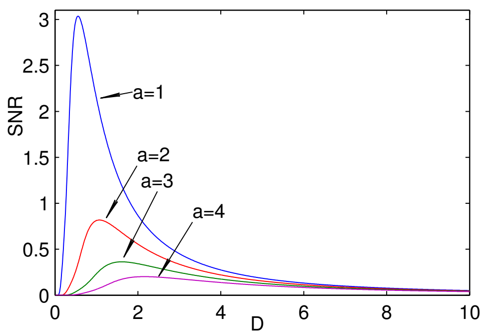

Fig. 10. SNR *versus* the noise intensity D with varied a for $b = c = k = 1$, $q = 3$, $A = 0.4$, $\Omega = 0.1$.

The parameters of Fig. 11 are specified in figure caption. In Fig. 11, SNR is plotted as a function of the noise intensity D with varied A. There is a maximum of SNR at some D which is the characteristic of the SR. Figure 11 depicts that the peak increases with the periodic force A increasing. For small force A, the peak is rather flat, however, the resonance peak becomes sharp with A increasing. The larger A, the more easily SR is induced.

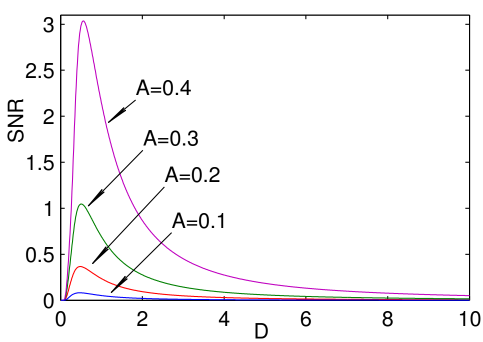

Fig. 11. SNR *versus* the noise intensity D with varied A for $a = b = c = k = 1$, $q = 3$, $\Omega = 0.1$.

Figure 12 plots the SNR as a function of noise intensity D with various value Ω and shows that the resonance peak decrease slowly with the frequency Ω increasing. It can be seen that parameter Ω has a little influence on the SR phenomenon.

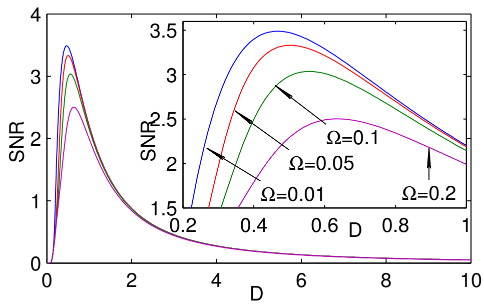

Fig. 12. SNR *versus* the noise intensity D with varied Ω for $a = b = c = k = 1$, $q = 3$, $A = 0.4$.

Comparing Fig. 1–Fig. 5 and Fig.6–Fig.10, the higher the barrier height ΔU, the flatter the resonance peak and the harder SR is induced. On the contrary, the lower the barrier height ΔU, the sharper the resonance peak and the more easy SR is induced. Thus, the parameters present the opposite effects on the barrier height and resonance peak.

In Fig. 13, we have plotted the real time trajectory of the particle. In Fig. 13 (left) (for $q = 3$), because of the potential, the particle travels large distances away from the minima and spends more time in the wings of the potential ($x > x_{\mathrm{m}}$or $x < x_{\mathrm{m}}$) over a duration of many cycles of the applied force without passing over the barrier. On the other hand, in Fig. 13 (right) (for $q = 6$), the potential helps the particle to pass the barrier, and the particle travels short distances away from the minima. This is clear from the figure.

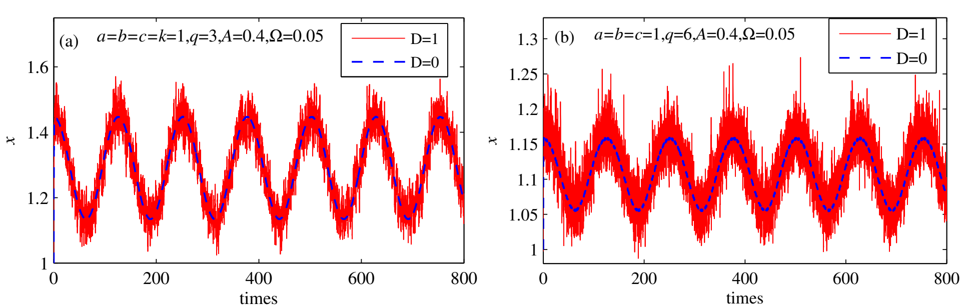

Fig. 13. Positions of particle as a function of time for given trajectories are shown. (Left) for $q = 3$ potential; (Right) for $q = 6$ potential.

4. Conclusion

In this paper, we have investigated the SR phenomenon in a new bistable system with potential function $U(x) \sim |x|^q$ for $q > 2$. A detailed discussion of the effects of the potential function Eq. (2) is presented. For periodic input signal, the SNR is employed to quantify SR of new bistable system. We investigate the influence of parameters of the new bistable system on the SR phenomenon. Moreover, our further investigation reveals the relationship between the barrier height and the mechanism of SR in the new bistable system by simulations. The simulations present the opposite influence on the barrier height of potential and SR phenomenon, which is sensitive to the nature of the confining potential. The results also give the theoretical and experimental tool when dealing with SR control and information processing.

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