VACUUM POLARIZATION IN SPATIALLY-FLAT N-DIMENSIONAL FRIEDMAN–ROBERTSON–WALKER SPACETIMES

Jerzy Matyjasek[†], Paweł Sadurski

Institute of Physics, Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland

We investigate the vacuum polarization, $\langle\phi^2\rangle$, of the quantized massive scalar field with a general curvature coupling parameter in the spatially-flat N-dimensional Friedman–Robertson–Walker spacetime with $4 \leq N \leq 12$. The vacuum polarization is constructed using both adiabatic and Schwinger–DeWitt approaches and the full final results up to $N = 7$ are explicitly demonstrated. The behavior of $\langle\phi^2\rangle$ for $4 \leq N \leq 12$ is examined in the exponentially expanding universe, in the power-law and inflationary power-law models. In the case of exponential expansion, $\langle\phi^2\rangle$ is constant and for a given mass it depends solely on the Hubble constant and the curvature coupling parameter. In the power-law models its behavior is more complicated and, generally, decays in time as t^{-n}, where $n/2$ is the integer part of $N/2$. The $2 + 1$-dimensional case is also briefly analyzed. The relevance of the present results to the stress-energy tensor is examined.

1. Introduction

The physical content of the quantum field theory constructed in a general curved spacetime is given by the most important operator — the renormalized mean value of the stress-energy tensor of the quantized field(s), $\langle T_a^b \rangle$, calculated in physically motivated states. Indeed, it defines the quantum part of the total source term of the semi-classical Einstein field equations. With the right-hand side of the semiclassical Einstein field equations known, one can attempt to determine the evolution of the system, consisting of the classical background and the classical and quantum matter, unless the quantum gravity effects become dominant [1–3]. On the other hand, the field

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[†] Corresponding author: `jurek@kft.umcs.lublin.pl`

fluctuation $\langle\phi^2\rangle$, often regarded as a lesser relative to its more prominent cousin, $\langle T_a^b\rangle$, also gives useful informations about the vacuum polarization effects and the spontaneous symmetry breaking phenomena, being simultaneously easier to calculate. Generally speaking: If some calculation can be performed for the vacuum polarization, it can, in principle, be performed for the stress-energy tensor itself. Of course, the computational complexity of the latter is higher and rapidly grows with dimension, but it usually does not bring any unexpected conceptual complications. Conversely, any sign of unusual behavior should be addressed with care as it inevitably leads to much more complication in the calculations of the stress-energy tensor.

The literature on the quantized fields and their characteristics in various curved backgrounds is abundant and covers both massive and massless fields of various spins. The background geometries range from wormholes [4–7] to black holes (and their singular or regular interiors) and from cosmology to topologically nontrivial spacetimes [8]. In the context of the present paper, the most important are the studies of the quantized fields in cosmology (see *e.g.*, Refs. [9–22] and the references cited therein) and the black hole interiors [23–25]. However, special emphasis — and it is natural — is put on the four dimensional calculations. (For the N-dimensional calculations, see [26–31].) On the other hand, according to many modern theories, the physical world has more than the familiar 4 dimensions and recent renewal of interest in such models stems from the considerable progress made in the string theory and its low-energy limit, Kaluza–Klein-type theories and various extra-large dimensional scenarios, the braneworld scenario included. Additionally, the ADS/CFT correspondence which tries to relate not only conformal field theory with the anti-de Sitter geometry but also such seemingly distant branches as gravitation and condensed matter physics also requires additional dimensions. To be more specific: to unify the fundamental interaction within the framework of the Kaluza–Klein models up to $N = 11$ is required, whereas self-consistency of the string theories favor $N = 10$ or 26.

In this note, we shall calculate $\langle\phi^2\rangle$ of the quantized massive scalar field in the N-dimensional spatially flat Friedman–Robertson–Walker spacetime (FRW) using two distinctive techniques: explicit construction of the adiabatic modes on the one hand [11–16, 20–22] and the calculations carried out within the framework of the Schwinger–DeWitt technique, heavily dependent on the heat kernel coefficients, on the other [21, 22, 29, 32–37]. We shall concentrate on the cosmological models satisfying $4 \leq N \leq 12$ and briefly analyze the case of the lower-dimensional geometry. Since $\langle\phi^2\rangle$ is a product of the field operators which is to be evaluated at the same spacetime point, it necessarily diverges and the finite results can be obtained only after suitable regularization (see *e.g.*, Ref. [38] and the references therein).

Both methods are local and hence do not tell much about the particle creation in the curved background, which is, of course, a nonlocal process. On the other hand, they are useful when the vacuum polarization effects dominate. The criterion for validity of the adiabatic expansion of the mode functions in the FRW spacetime is provided by the chain of inequalities $\dot{a}/a$, $\ddot{a}/a \ldots \ll (m^2 a^2 + k^2)^{1/2}$, which must be satisfied by a scale factor $a(\eta)$, where a dot denotes differentiation with respect to the conformal time and $0 \leq k < \infty$, [2]. Similarly, the Schwinger–DeWitt expansion, which is an expansion in the inverse of mass, is valid as long as the Compton length associated with the field is much smaller than the characteristic radius of curvature of the spacetime, *i.e.*, $\lambda_C/L \ll 1$.

The present paper is, to the authors' knowledge, the first attempt to use the higher-order Hadamard–DeWitt coefficients and the adiabatic mode functions in the calculations of this type. Admittedly, the coefficient $[a_4]$ can be used in the calculations of the next-to-leading term of the approximate stress-energy tensor [39, 40], however, they are limited to $N = 4$ geometries.

Besides a natural curiosity there are a few reasons for our choice of the quantized field, geometry and methods. Due to spatial simplicity of the spacetime, it is possible to construct the mode functions and calculate the vacuum polarization within a framework of the adiabatic approach. It is very important because one can check the validity of the results using various methods. It should be emphasized that equality of the results obtained from both methods considered in this paper must not be taken for granted. Indeed, on general grounds we know that the singularity structure of the Green functions should be the same, however, because of the boundary-dependent terms, this does not mean that the functions are the same. Consequently, the equality of the final results is really impressive. Further, one has a unique opportunity to analyze the dependence of the quantum effects on the dimensions of the background spacetime. Here, we have limited ourselves to $N \leq 12$. It should be noted, however, that to construct the (general) stress-energy tensor in 12-dimensional spacetime, the detailed knowledge of the coincidence limit of $[a_7]$ is required. This is even more important, as the calculations with the coefficients $[a_i]$ for $i > 2$ in $N = 4$-dimensional geometries are extremely hard. Finally, a careful analysis of $\langle \phi^2 \rangle$ (and the steps required in its construction) is certainly helpful in the calculations of the stress-energy tensor.

As the classical background geometry is kept fixed during the calculation, the important problem of the back reaction is not addressed in this paper. This would require a detailed knowledge of the renormalized stress-energy tensor and profound analysis of the semi-classical Einstein field equations, that is beyond the scope of this study. The renormalized stress-energy tensor and the back reaction of the quantized fields upon N-dimensional FRW spacetime will be studied in a subsequent paper.

The paper is organized as follows. The detailed calculations of the vacuum polarization of the quantized massive scalar field in the spatially-flat FRW cosmologies within the framework of the adiabatic approximation are presented in Sec. 2. It is shown that the most important ingredient of the method is construction of the chain of the iterative solutions, $\omega_0, \omega_2, \omega_4, \ldots$, to the dynamical equations and systematic regularization of the bilinear expression. In the second part of this section, the vacuum polarization is constructed using the Schwinger–DeWitt approximation to the Green function. This result is quite general, and, when applied to the spatially flat FRW cosmologies, it reduces to the analogous result obtained with the aid of the adiabatic approximation. In Section 3, the special results for the exponentially expanding scale factor and power-law cosmological models are constructed and examined. The results presented in Sections 2 and 3 comprise the core results of this paper. The last section contains brief discussion and the final remarks. A qualitative discussion of the stress-energy tensor in N-dimensional spacetime as well as the calculations of the vacuum fluctuation in $N = 3$-dimensional model are also presented there. Throughout the paper the natural units are chosen and we follow the Misner, Thorne and Wheeler conventions [41].

2. Approximate expressions for $\langle \phi^2 \rangle_{\mathrm{ren}}$ in N-dimensions

2.1. Adiabatic expansion in spatially-flat Friedman–Robertson–Walker spacetimes

In this section, we will be concerned with the neutral massive scalar field, satisfying the covariant Klein–Gordon equation in N-dimensional spatially flat Friedman–Robertson–Walker spacetime

$$\Box \phi - \left(m^2 + \xi R \right) \phi = 0 \,, \tag{1}$$

where m is the mass of the field, R is the curvature scalar and ξ is the (arbitrary) curvature coupling constant. The two particular values of the parameter ξ are of principal interest: the minimal and conformal coupling, for which $\xi = 0$ and $\xi = (N-2)/(4N-4)$, respectively. The line element can be written in the form

$$ds^2 = a^2(\eta) \left(-d\eta^2 + \delta_{ij} dx^i dx^j \right) \,, \tag{2}$$

where $i, j = 1, \ldots, N-1$. The choice of the conformal time η simplifies calculations.

We shall start our calculations of the vacuum polarization with the substitution

$$\phi(x) = a^{-(N-2)/2} \mu(x) \,, \tag{3}$$

where $\mu(x)$ can be decomposed

$$\mu(x) = (2\pi)^{-(N-1)/2} \int d^{N-1}k \left(\mu_{\boldsymbol{k}}(\eta)e^{ik_ax^a}a_{\boldsymbol{k}} + \mu_{\boldsymbol{k}}^*(\eta)e^{-ik_ax^a}a_{\boldsymbol{k}}^\dagger \right) . \quad (4)$$

The standard commutation relations of the field operator and the conjugate momentum give the relations for the operators $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$

$$
\begin{aligned}
{[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}]} &= \left[a_{\boldsymbol{k}}^\dagger, a_{\boldsymbol{k}'}^\dagger \right] = 0 \,, \\
\left[a_{\boldsymbol{k}}, a_{\boldsymbol{k}'}^\dagger \right] &= \delta\left(\boldsymbol{k} - \boldsymbol{k}' \right) .
\end{aligned}
\quad (5)
$$

The ground state of the field is defined as

$$a_{\boldsymbol{k}}|0\rangle = 0 \,, \quad (6)$$

and the canonical commutators of the field operator and the conjugate momentum lead to commutation relations of $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^\dagger$ provided the functions $\mu_{\boldsymbol{k}}$ satisfy the Wronskian condition

$$\mu_{\boldsymbol{k}}\dot{\mu}_{\boldsymbol{k}}^* - \dot{\mu}_{\boldsymbol{k}}\mu_{\boldsymbol{k}}^* = i \,. \quad (7)$$

Now, substituting (3) into (1), one has

$$\ddot{\mu}_k + \left(k^2 + m^2a^2 \right)\mu_k + (\xi - \xi_c)\left(2(N-1)\frac{\ddot{a}}{a} + (N-4)(N-1)\frac{\dot{a}^2}{a^2} \right)\mu_k = 0 \,, \quad (8)$$

where $k = \sqrt{k_1^2 + \ldots k_{N-1}^2}$ and $\xi_c = (N-2)/(4N-4)$. The formal (*i.e.,* divergent) expression describing the vacuum polarization, $\langle\phi^2\rangle_{\text{formal}}$, is therefore given by

$$\langle\phi^2\rangle_{\text{formal}} = \frac{\pi^{(1-N)/2}(2a)^{2-N}}{\Gamma\left(\frac{N-1}{2}\right)} \int\limits_0^\infty dk\, k^{N-2}|\mu_k|^2 \,. \quad (9)$$

The problem with Eq. (8) is that it is rather complicated and cannot be solved exactly. It is inevitable, therefore, that one should look for reasonable approximations or treat the problem numerically. Here, we shall employ the WKB method and construct the appropriate solution iteratively. To this end, let us introduce the functions $\Omega_k(\eta)$ defined as

$$\mu_k = \frac{1}{\sqrt{2\Omega}}e^{-i\int\Omega d\eta} \quad (10)$$

and substitute it into (8). The form of (10) guarantees that the Wronskian condition (7) is automatically satisfied and the resulting equation assumes the more transparent form

$$
\begin{aligned}
\Omega^2 = {} & k^2 + m^2 a^2 - \frac{\ddot{\Omega}}{2\Omega} + \frac{3\dot{\Omega}^2}{4\Omega^2} \\
& + \left(\xi - \frac{N-2}{4(N-1)} \right) \left(2(N-1)\frac{\ddot{a}}{a} + (N-4)(N-1)\frac{\dot{a}^2}{a^2} \right) .
\end{aligned}
\tag{11}
$$

The equation can be solved iteratively, assuming expansion

$$
\Omega_k = \omega_0 + \omega_2 + \omega_4 + \dots
\tag{12}
$$

with $\omega_0 = \sqrt{k^2 + m^2 a^2}$. The role of the small parameter is played by number of differentiations with respect to the conformal time. Alternatively, one can introduce the (dimensionless) parameter ε by means of the formulas

$$
\frac{d}{d\eta} \to \varepsilon \frac{d}{d\eta} \qquad \text{and} \qquad \Omega_k = \sum_{i=0} \varepsilon^{2i} \omega_{2i} ,
\tag{13}
$$

and collect the terms of the expansion with the like powers of ε. The parameter ε should be set to 1 at the final stage of calculations. The calculation, limited only by its computational complexity, can be carried out term by term up to the required order.

Since

$$
|\mu_k|^2 = \frac{1}{2\Omega_k} ,
\tag{14}
$$

it is clear that the problem at hand reduces to expanding the function Ω_k^{-1} in terms of the small parameter to some definite order and integration of the thus constructed result over k with the appropriate measure. Now, putting

$$
\frac{1}{\Omega_k} = \tilde{\omega}_0 + \varepsilon^2 \tilde{\omega}_2 + \varepsilon^4 \tilde{\omega}_4 + \dots
\tag{15}
$$

and calculating $\tilde{\omega}_i$, it can be shown that for a given dimension N, the integrals of the first $2\lfloor N/2 \rfloor$ terms diverge, where $\lfloor x \rfloor$ denotes the floor function. This statement needs clarification. Not all terms of a given order lead to the divergent integrals. The regularization prescription, however, requires to subtract all the terms of a given order if at least one of them is divergent.

The first finite term, which also provides the first-order approximation of the vacuum polarization is given by

$$
\langle \phi^2 \rangle_{\text{reg}} = \frac{\pi^{(1-N)/2}(2a)^{2-N}}{2\Gamma\left(\frac{N-1}{2}\right)} \int\limits_0^\infty dk\, k^{N-2} \tilde{\omega}_n ,
\tag{16}
$$

where

$$\tilde{\omega}_n = \frac{(-1)^{n/2}}{\omega_0^{n/2+1}} \, \det \begin{pmatrix} \omega_2 & \omega_0 & 0 & 0 & \cdots & 0 \\ \omega_4 & \omega_2 & \omega_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-2} & \omega_{n-4} & \omega_{n-6} & \omega_{n-8} & \cdots & \omega_0 \\ \omega_n & \omega_{n-2} & \omega_{n-4} & \omega_{n-6} & \cdots & \omega_2 \end{pmatrix} \qquad (17)$$

and $n = 2\lfloor N/2 \rfloor$. We believe that the formulas (16) and (17) describing $\langle \phi^2 \rangle_{\text{reg}}$ are new.

The k-dependent parts of the integrands in (16) are of the form

$$\frac{k^p}{(k^2 + m^2 a^2)^{q/2}} \qquad (18)$$

and lead to finite results provided $q - p - 1 > 0$. Indeed, simple consequences of the definition of the beta function, B, and its basic features, give

$$\int\limits_0^\infty dk \frac{k^p}{(k^2 + m^2 a^2)^{q/2}} = \frac{1}{2}(am)^{1+p-q} B\left(\frac{1+p}{2}, \frac{q-p-1}{2}\right)$$

$$= \frac{1}{2}(am)^{1+p-q} \frac{\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{q-p-1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)} . \qquad (19)$$

It is seen from the foregoing discussion that in order to construct the next terms of the WKB approximation in the N-dimensional spacetime, it is necessary to calculate $\tilde{\omega}_{n+2}, \tilde{\omega}_{n+4}, \ldots$, and this can be done with a slight abuse of Eq. (17). In what follows, we shall restrict ourselves only to the leading terms and investigate the vacuum polarization in the (spatially-flat) FRW spacetimes up to $N = 12$. The calculation of the higher order term presents no problems and is limited only by time and the complexity of the intermediate formulas.

2.2. Schwinger–DeWitt approximation

In the Schwinger–DeWitt approach in N dimensions, the approximate Green function that satisfies the covariant scalar field equation

$$\left(\Box - m^2 - \xi R\right) G^{\text{F}}\left(x, x'\right) = -\delta\left(x, x'\right) \equiv -\frac{\delta\left(x - x'\right)}{|g|^{1/2}}, \qquad (20)$$

can be written in the form

$$G^{\text{F}}\left(x, x'\right) = \frac{i \Delta^{1/2}}{(4\pi)^{N/2}} \int\limits_0^\infty i ds \frac{1}{(is)^{N/2}} \exp\left[-im^2 s + \frac{i\sigma\left(x, x'\right)}{2s}\right] A\left(x, x'; is\right)$$

$$(21)$$

with

$$A\left(x, x'; is\right) = \sum_{k=0}^{\infty} (is)^k a_k\left(x, x'\right), \qquad (22)$$

where $\sigma(x, x')$ is half the square of the geodetic distance between x and x', $\Delta(x, x')$ is the van Vleck–Morette determinant and s is a (fictitious) time parameter. The biscalars $a_k(x, x')$ are the celebrated Hadamard–DeWitt coefficients of the heat kernel equation.

Having in mind further applications, let us introduce

$$A_{\text{reg}}^{(N)}\left(x, x'; is\right) = A\left(x, x'; is\right) - \sum_{k=0}^{\lfloor \frac{N}{2} \rfloor - 1} a_k\left(x, x'\right)(is)^k, \qquad (23)$$

and substitute $A(x, x'; is)$ in Eq. (21) by $A_{\text{reg}}^{(N)}(x, x'; is)$. We shall denote the thus obtained "regularized" Green function by $G_{\text{reg}}^{(N)}$. The reason for introducing the regularized sum (23) is simply that the first $\lfloor \frac{N}{2} \rfloor - 1$ terms lead to divergences. The field fluctuation that characterizes the vacuum polarization effects in N-dimensional spacetime can be written as

$$\left\langle \phi^2 \right\rangle_{\text{reg}} = -i \lim_{x' \to x} G_{\text{reg}}^{(N)}. \qquad (24)$$

It follows then that to construct $\langle \phi^2 \rangle$ it is necessary to know the coincidence limit of the Hadamard–DeWitt coefficients, *i.e.*,

$$[a_k] = \lim_{x' \to x} a_k\left(x, x'\right). \qquad (25)$$

The coincidence limit of a_k is constructed from the Riemann tensor, its covariant derivatives and contractions, and the type of the field is encoded in the numerical coefficients depending on ξ. Moreover, when properly calculated, the Hadamard–DeWitt coefficients do not depend on the dimension. Although there is enormous literature devoted to the calculation of the coefficients $[a_k(x, x')]$, their general exact form for the scalar field is known only for $k \leq 5$. It is simply because the complexity of the coefficients rapidly grows with the order. It should be emphasized that even if known, the Hadamard–DeWitt coefficients can be very hard to evaluate in practice, except for the metrics with a high degree of symmetry.

To make the series (21) "steerable" we shall truncate it. Let n' denote the number of the last retained term in $A_{\text{reg}}^{(N)}(x, x'; is)$. Assuming that m^2 has small (negative) imaginary part, *i.e.*, putting $m^2 \to m^2 - i\varepsilon$ ($\varepsilon > 0$) and integrating term by term with the aid of the well-known formula

$$\int_0^{\infty} x^{\nu-1} e^{-\mu x} dx = \mu^{-\nu} \Gamma(\nu), \qquad (26)$$

valid if $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\nu) > 0$, one obtains

$$\langle \phi^2 \rangle_{\mathrm{reg}} = \frac{1}{(4\pi)^{N/2}} \sum_{k=\lfloor N/2 \rfloor}^{n'} \frac{[a_k]}{(m^2)^{k+1-N/2}} \Gamma\left(k+1-\frac{N}{2}\right). \qquad (27)$$

This result is a generalization to arbitrary dimension of the result derived by Frolov in Ref. [42] and coincides with the formula obtained by Lemos and Thompson [31]. It should be noted, however, that the derivation presented here is simpler. For simplicity, in what follows, we shall omit the subscript reg, assuming that all mean values have already been regularized.

Retaining the first term in the above sum and calculating, for a given N, the coincidence limit of the coefficient $a_{\lfloor N/2 \rfloor}$ in the spatially-flat universe gives precisely the same result as (16). Although this equality is expected, we have calculated the vacuum polarization of the quantized field in the large mass limit in the *general* FRW spacetimes of various dimensions up to $N = 9$, and explicitly demonstrated that for the spatially flat geometries the thus obtained results coincide. This may be thought of as a useful check of calculations. It should be noted, however, that (27) is more general and can virtually be applied in any background, provided the Compton length associated with the field is much smaller that the characteristic radius of curvature. On the other hand, even in the spatially-flat FRW geometry, the calculation of the Hadamard–DeWitt coefficients is much more complicated and time-consuming than the construction of the iterative solution of the adiabatic modes.

3. Results

3.1. The general scale factor

In this section, we make use of the general formulas (16) and (27) and calculate $\langle \phi^2 \rangle^{(N)}$ in the spatially-flat FRW cosmologies. Although the calculations of the vacuum polarization that were carried out within the framework of the adiabatic regularization employed (η, x_i) coordinates, one can easily transform the final result into a more familiar (t, x_i) coordinates. (Within the framework of the Schwinger–DeWitt approximation, one can start with the proper time t from the very beginning.) Making use of the relation $dt = a(\eta)d\eta$, after some algebra, one obtains the chain of relations of ascending complexity with the dimension. For $N = 4$ and $N = 5$, one has respectively

$$\pi^2 m^2 \left\langle \phi^2 \right\rangle^{(4)} = -\frac{a^{(4)}}{80a} + \frac{3a''^2}{160a^2} + \frac{a'^4}{32a^4} - \frac{3a^{(3)}a'}{80a^2} + \frac{29a'^2a''}{240a^3}$$
$$+\xi^2 \left(\frac{9a''^2}{8a^2} + \frac{9a'^4}{8a^4} + \frac{9a'^2a''}{4a^3} \right)$$
$$+\xi \left(\frac{a^{(4)}}{16a} - \frac{5a''^2}{16a^2} - \frac{3a'^4}{8a^4} + \frac{3a^{(3)}a'}{16a^2} - \frac{17a'^2a''}{16a^3} \right) \quad (28)$$

and

$$\pi^2 m \left\langle \phi^2 \right\rangle^{(5)} = -\frac{a^{(4)}}{120a} + \frac{a''^2}{96a^2} + \frac{41a'^4}{480a^4} - \frac{a^{(3)}a'}{24a^2} + \frac{29a'^2a''}{240a^3}$$
$$+\xi^2 \left(\frac{a''^2}{a^2} + \frac{9a'^4}{4a^4} + \frac{3a'^2a''}{a^3} \right)$$
$$+\xi \left(\frac{a^{(4)}}{24a} - \frac{a''^2}{4a^2} - \frac{7a'^4}{8a^4} + \frac{5a^{(3)}a'}{24a^2} - \frac{29a'^2a''}{24a^3} \right) . \quad (29)$$

Identical result can be obtained from the coincidence limit of the coefficient $a_2(x, x')$. Similarly, for $N = 6$ and $N = 7$, one can calculate the vacuum polarization either from Eq. (16) or from the coincidence limit of the coefficient $a_3(x, x')$. Slightly more involved calculations give

$$\left\langle \phi^2 \right\rangle^{(6)} = \frac{1}{\pi^3 m^2} \left(\phi_0^{(6)} + \phi_1^{(6)}\xi + \phi_2^{(6)}\xi^2 + \phi_3^{(6)}\xi^3 \right) , \quad (30)$$

where

$$\phi_0^{(6)} = \frac{a^{(6)}}{1792a} + \frac{31a^{(3)^2}}{16128a^2} - \frac{73a''^3}{2688a^3} + \frac{5a^{(5)}a'}{896a^2} - \frac{1717a^{(3)}a'a''}{16128a^3} + \frac{a'^6}{6a^6}$$
$$+\frac{a^{(4)}a''}{16128a^2} - \frac{263a^{(4)}a'^2}{16128a^3} - \frac{1889a^{(3)}a'^3}{16128a^4} + \frac{6593a'^4a''}{16128a^5} - \frac{61a'^2a''^2}{1792a^4} ,$$

$$\phi_1^{(6)} = -\frac{a^{(6)}}{384a} + \frac{7a^{(3)^2}}{384a^2} + \frac{13a''^3}{96a^3} - \frac{5a'^6}{2a^6} - \frac{5a^{(5)}a'}{192a^2} + \frac{7a^{(4)}a''}{128a^2}$$
$$+\frac{217a^{(4)}a'^2}{1152a^3} + \frac{1271a^{(3)}a'^3}{1152a^4} - \frac{6071a'^4a''}{1152a^5} - \frac{49a'^2a''^2}{128a^4} + \frac{1217a^{(3)}a'a''}{1152a^3} ,$$

$$\phi_2^{(6)} = -\frac{25a^{(3)^2}}{192a^2} + \frac{25a''^3}{48a^3} + \frac{25a'^6}{2a^6} - \frac{25a^{(4)}a''}{96a^2} - \frac{25a^{(4)}a'^2}{48a^3}$$
$$-\frac{125a^{(3)}a'^3}{48a^4} + \frac{1075a'^4a''}{48a^5} + \frac{375a'^2a''^2}{64a^4} - \frac{125a^{(3)}a'a''}{48a^3} ,$$

$$\phi_3^{(6)} = -\frac{125a''^3}{48a^3} - \frac{125a'^6}{6a^6} - \frac{125a'^4a''}{4a^5} - \frac{125a'^2a''^2}{8a^4} ,$$

and

$$\langle \phi^2 \rangle^{(7)} = \frac{1}{\pi^3 m} \left(\phi_0^{(7)} + \phi_1^{(7)} \xi + \phi_2^{(7)} \xi^2 + \phi_3^{(7)} \xi^3 \right) , \qquad (31)$$

where

$$
\begin{aligned}
\phi_0^{(7)} = {} & \frac{3a^{(6)}}{8960a} + \frac{103a^{(3)2}}{53760a^2} - \frac{467a''^3}{20160a^3} + \frac{167a'^6}{504a^6} + \frac{39a^{(5)}a'}{8960a^2} + \frac{23a^{(4)}a''}{26880a^2} \\
& - \frac{55a^{(4)}a'^2}{5376a^3} - \frac{93a^{(3)}a'^3}{640a^4} + \frac{5603a'^4a''}{13440a^5} - \frac{263a'^2a''^2}{2560a^4} - \frac{2227a^{(3)}a'a''}{26880a^3} ,
\end{aligned}
$$

$$
\begin{aligned}
\phi_1^{(7)} = {} & \frac{23a^{(4)}a''}{640a^2} - \frac{a^{(6)}}{640a} + \frac{7a^{(3)2}}{640a^2} + \frac{21a''^3}{160a^3} - \frac{301a'^6}{64a^6} - \frac{13a^{(5)}a'}{640a^2} \\
& + \frac{19a^{(4)}a'^2}{128a^3} + \frac{221a^{(3)}a'^3}{160a^4} - \frac{1867a'^4a''}{320a^5} + \frac{67a'^2a''^2}{320a^4} + \frac{291a^{(3)}a'a''}{320a^3} ,
\end{aligned}
$$

$$
\begin{aligned}
\phi_2^{(7)} = {} & \frac{3a''^3}{8a^3} + \frac{1425a'^6}{64a^6} - \frac{3a^{(4)}a''}{16a^2} - \frac{15a^{(4)}a'^2}{32a^3} - \frac{105a^{(3)}a'^3}{32a^4} \\
& + \frac{435a'^4a''}{16a^5} + \frac{39a'^2a''^2}{8a^4} - \frac{39a^{(3)}a'a''}{16a^3} - \frac{3a^{(3)2}}{32a^2} ,
\end{aligned}
$$

$$
\phi_3^{(7)} = -\frac{9a''^3}{4a^3} - \frac{1125a'^6}{32a^6} - \frac{675a'^4a''}{16a^5} - \frac{135a'^2a''^2}{8a^4} .
$$

This procedure can be extended to higher dimensions. Indeed, we have calculated $\langle \phi^2 \rangle^{(N)}$ up to $N = 12$ using the adiabatic modes and up to $N = 9$ using the coincidence limit of $a_2(x, x'), a_3(x, x')$ and $a_4(x, x')$. A general lesson that follows from our calculations is that for $N \geq 8$ the adiabatic method is more efficient and less time-consuming (at least with our implementation of the algorithms). It can be explained by counting the number of operations required to construct the final result. It should be noted that the first-order approximation is proportional to m^{-2} for the even dimensional cosmologies and m^{-1} for the odd ones. Similarly, the approximation to the vacuum polarization can be written in terms of $m^{-(2s-1)}$ for N odd and m^{-2s} for N even $(s = 1, 2 \ldots)$. This however requires, for any N, a slight modification of (17) in the adiabatic method and detailed knowledge of the coincidence limit of the coefficients $a_s(x, x')$, where $s = n/2 + k$, $(k = 1, 2 \ldots)$.

The results for $N \geq 8$ are much more complicated and they are not presented here for obvious reasons. The formulas describing the vacuum fluctuation for $4 \leq N \leq 12$ are available on request from the first author or from our computer code repository[1].

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[1] Computer code repository at `http://kft.umcs.lublin.pl/jurek/computer.html`

3.2. The inflationary universe

The thus obtained results are valid for any scale-factor, $a(t)$, provided the appropriate conditions of applicability are satisfied. As they stand, however, they are not very informative and to gain a better understanding of the nature of the field fluctuation let us restrict ourselves to a few exemplary cases. First, consider the scale factor $a(t)$ satisfying the condition [43, 44]

$$a'(t) = H_0 a(t) \,, \tag{32}$$

where H_0 is a positive constant. Simple manipulations give the general expression for the field fluctuation as a function of the curvature coupling

$$\langle \phi^2 \rangle^{(N)} = \frac{H_0^n}{\pi^{n/2} m^{2/(N-n+1)}} \sum_{i=0}^{\frac{n}{2}} \alpha_i^{(N)} \xi^i \,, \tag{33}$$

where $n = 2\lfloor N/2 \rfloor$. The vacuum polarization is constant, as expected, and for a given mass it depends solely on the Hubble constant (H_0) and the parameter ξ. The numerical coefficients $\alpha_i^{(N)}$ for $N \leq 9$ are tabulated in Table I, whereas the coefficients for $N \geq 10$ can be calculated using the general formulas. By construction, the rate of complexity of the formulas describing the vacuum polarization grows with n and for a given n, the polynomials are of the same order in ξ. The overall behavior of the rescaled field fluctuation, $\sum_{i=0}^{\frac{n}{2}} \alpha_i^{(N)} \xi^i$, for small and positive ξ is shown in Fig. 1. On the other hand, Table II shows the sign of the field fluctuation for the physical values of the coupling parameter. For the minimally coupled massive scalar fields $\langle \phi^2 \rangle^{(N)}$ is always positive, whereas for the conformally coupled field the sign oscillates.

TABLE I

The numerical coefficients $\alpha_i^{(N)}$.

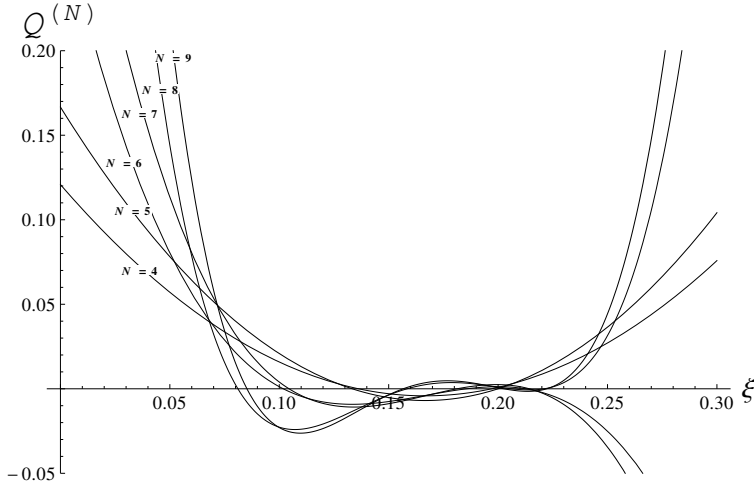|	$\alpha_0^{(N)}$	$\alpha_1^{(N)}$	$\alpha_2^{(N)}$	$\alpha_3^{(N)}$	$\alpha_4^{(N)}$
$N = 4$	$\frac{29}{240}$	$-\frac{3}{2}$	$\frac{9}{2}$	0	0
$N = 5$	$\frac{1}{6}$	$-\frac{25}{12}$	$\frac{25}{4}$	0	0
$N = 6$	$\frac{1139}{4032}$	$-\frac{45}{8}$	$\frac{1125}{32}$	$-\frac{1125}{16}$	0
$N = 7$	$\frac{501}{1280}$	$-\frac{4949}{640}$	$\frac{3087}{64}$	$-\frac{3087}{32}$	0
$N = 8$	$\frac{32377}{34560}$	$-\frac{105}{4}$	$\frac{1029}{4}$	$-\frac{9604}{9}$	$\frac{4802}{3}$
$N = 9$	$\frac{46}{35}$	$-\frac{507}{14}$	$\frac{7047}{20}$	-1458	2187

Fig. 1. $Q^{(N)} = \sum_{i=0}^{\frac{n}{2}} \alpha_i^{(N)} \xi^i$ as the function of the coupling parameter ξ in the spatially flat FRW spacetimes of various dimensions. Top to bottom (in the left part of the figure) the curves are drawn for $N = 9, 8, \ldots, 4$-dimensional, spatially-flat FRW geometries.

TABLE II

The sign of the vacuum polarization for two physical choices of the coupling parameter $\xi = 0$ (minimal coupling) and $\xi_c = (N-2)/(4N-4)$ (conformal coupling).

$N =$	4	5	6	7	8	9	10	11	12
$\xi = 0$	+	+	+	+	+	+	+	+	+
$\xi = \xi_c$	−	−	+	+	−	−	+	+	−

We have not attempted to present an exhaustive study of the quantized massive fields with more exotic values of ξ as they are of somewhat lesser importance. Nevertheless, even this example strongly suggests that the behavior of the stress-energy tensor, which is certainly much more complex, may lead to very interesting dimension-dependent back-reaction phenomena.

3.3. The power-law cosmologies

In our next example, we assume that the scale factor describes the power-law cosmological models

$$a(t) = \left(\frac{t}{t_0}\right)^p, \tag{34}$$

where t_0 is a positive constant. The power-law cosmological models are the solutions of the Einstein field equations with the stress-energy tensor characterized by the energy density, ρ, and pressure, $\tilde{p}$, given respectively by

$$\rho = \frac{(N-1)(N-2)p^2}{16\pi t^2} \tag{35}$$

and

$$\tilde{p} = -\frac{(N-2)p\left[p(N-1)-2\right]}{16\pi t^2} . \tag{36}$$

The corresponding equation of state is given by $\tilde{p} = \alpha\rho$ with

$$\alpha = -\frac{p(N-1)-2}{p(N-1)} . \tag{37}$$

Such cosmological models can also be realized by introducing the scalar field with the exponential potential to the source term of the Einstein field equations.

The power-law models lead to slightly more complicated results, generally described by

$$\langle \phi^2 \rangle = \frac{1}{\pi^{n/2} m^{2/(N-n+1)} t^n} \sum_{i=0}^{\frac{n}{2}} \tilde{Q}_i^{(N)}(p)\xi^i , \tag{38}$$

where the functions $\tilde{Q}_i^{(N)}(p)$ are polynomials in the power-law exponent p. It should be noted that now the vacuum polarization decays in time as t^{-n}. Because of the length of the expressions describing

$$W_N = \sum_{i=0}^{\frac{n}{2}} \tilde{Q}_i^{(N)}(p)\xi^i \tag{39}$$

for $N \geq 6$, we will restrict here only to W_4 and W_5. Nonetheless, some features of the vacuum polarization for $4 \leq N \leq 12$ will be presented graphically. After some algebra, one has

$$\begin{aligned} W_4(p) &= \frac{29p^4}{240} + \frac{7p^3}{240} - \frac{31p^2}{160} + \frac{3p}{40} + \xi^2\left(\frac{9p^4}{2} - \frac{9p^3}{2} + \frac{9p^2}{8}\right) \\ &+ \xi\left(-\frac{3p^4}{2} + \frac{3p^3}{4} + \frac{3p^2}{4} - \frac{3p}{8}\right) \end{aligned} \tag{40}$$

and

$$\begin{aligned} W_5(p) &= \frac{p^4}{6} + \frac{p^3}{30} - \frac{79p^2}{480} + \frac{p}{20} + \xi^2\left(\frac{25p^4}{4} - 5p^3 + p^2\right) \\ &+ \xi\left(-\frac{25p^4}{12} + \frac{5p^3}{6} + \frac{5p^2}{8} - \frac{p}{4}\right) . \end{aligned} \tag{41}$$

The results for $N \geq 6$ are more complicated functions of the exponent p and can be constructed from the general formulas stored in the repository.

Since the most important qualitative information that can be extracted form the derived formulas is the sign of the vacuum polarization, in Figs. 2 and 3 we have plotted the regions in ξ–p plane where $\langle \phi^2 \rangle^{(N)}$ is strictly negative. Inspection of the figures shows that for $-0.3 \leq \xi \leq 0.3$, which seems to be a reasonable choice of the curvature coupling constant, and the power-law exponent satisfying $0 \leq p \leq 3$, the vacuum polarization exhibits a complicated pattern. A closer examination of the figures (and appropriate formulas) shows that there is qualitative complementarity between the shaded regions for pairs of $(i, i+2)$-dimensional spacetimes for $i = 5, 6, 9, 10$.
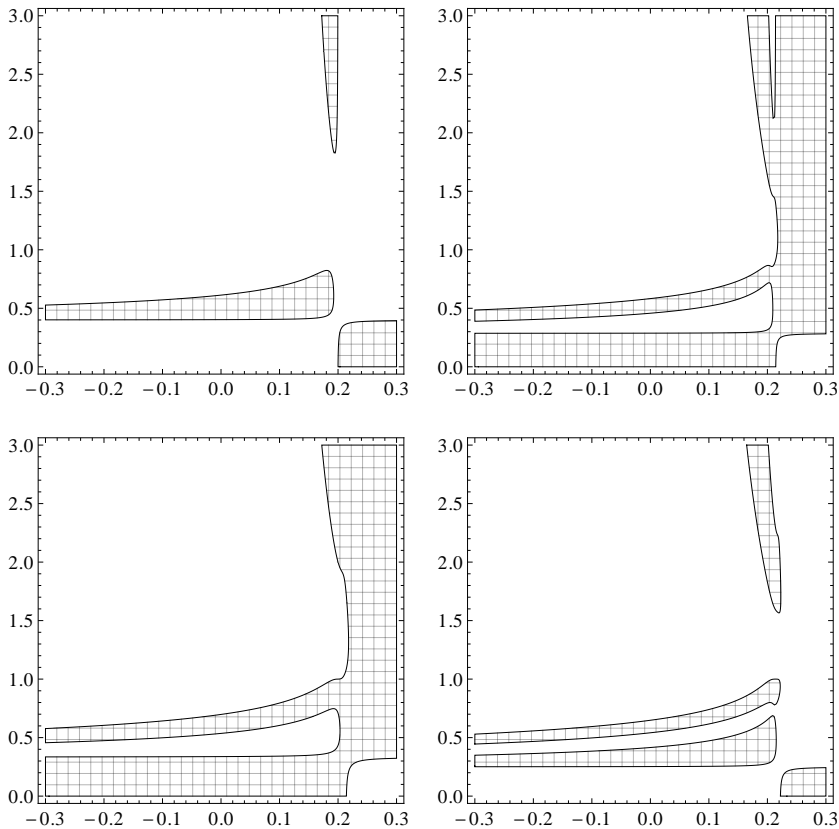

Fig. 2. Points within the shaded region represent values of ξ and p for which the vacuum polarization is negative. The figures (top to bottom) are drawn for $N = 5, 6$ (left panel) and $N = 7, 8$ (right panel). All figures are plotted for $-0.3 \leq \xi \leq 0.3$ (horizontal axis) and $0 \leq p \leq 3$ (vertical axis).

Indeed, as can be easily verified, the shaded regions in the diagram plotted for i-dimensional spacetime fit approximately into the ones drawn for $i + 2$ dimensions.
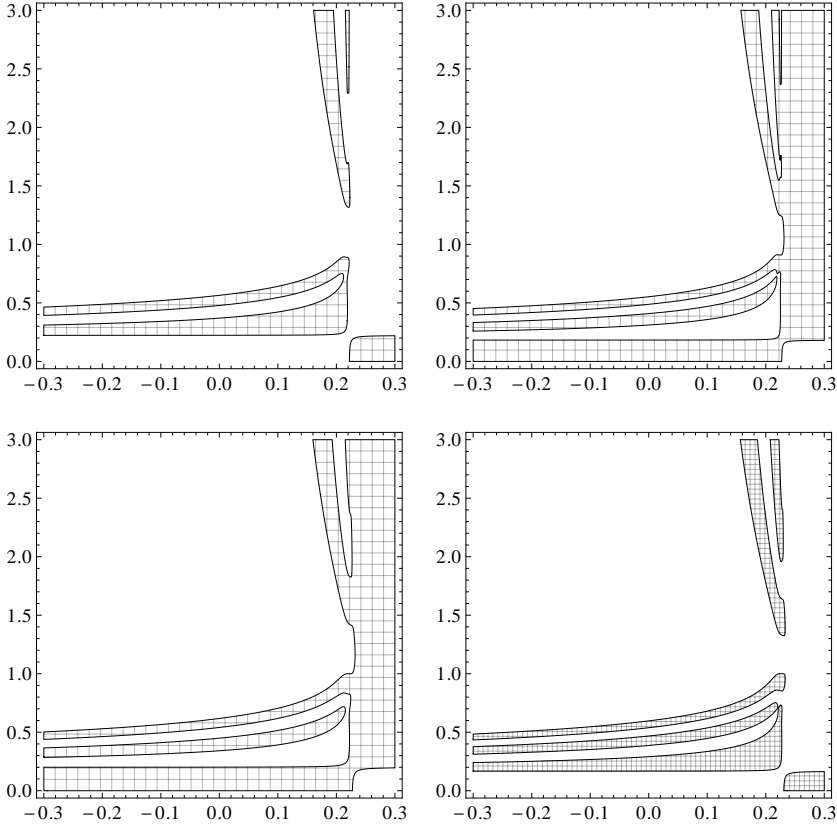

Fig. 3. Points within the shaded region represent values of ξ and p for which the vacuum polarization is negative. The figures (top to bottom) are drawn for $N = 9, 10$ (left panel) and $N = 11, 12$ (right panel). All figures are plotted for $-0.3 \leq \xi \leq 0.3$ (horizontal axis) and $0 \leq p \leq 3$ (vertical axis).

As a final example, let us consider the power-law cosmologies with a large p. The power-law expansion of the FRW universe with the scale factor satisfying (34) with $p \gg 1$ has been proposed as a generalized model of inflation. Here, we examine $\langle \phi^2 \rangle$ concentrating solely on the physical values of the parameter ξ. Inspection of the functions W_N shows that in the range $4 \leq N \leq 12$, the vacuum polarization is always positive for minimal coupling for $p \gtrsim 0.7$. On the other hand, the conformally coupled scalar field exhibits more complex behavior, which, for $p \gg 1$, is described by Table II, as expected.

4. Final remarks

In the above, we have presented a rather detailed derivation and analysis of the vacuum polarization of the massive scalar fields in a large mass limit. Thus far, however, the main emphasis has been on $N = 4$ and the higher-dimensional models. Here, we shall briefly discuss the $(2 + 1)$-dimensional case. As is well known, the $(2+1)$-dimensional spacetime has no local degrees of freedom, *i.e.*, the curvature and the matter distribution are related in a simple way [45, 46]. Because of the triviality of the vacuum polarization as given by Eq. (27) (one needs only the curvature scalar), there is no need for adiabatic calculations. Indeed, since

$$[a_1] = \left(\tfrac{1}{6} - \xi\right) R,\tag{42}$$

the vacuum polarization of the massive scalar field is given by

$$\left\langle \phi^2 \right\rangle^{(3)} = \frac{R}{8\pi m} \left(\frac{1}{6} - \xi\right)\tag{43}$$

and in the spatially flat $N = 3$ FRW spacetime, one obtains

$$\left\langle \phi^2 \right\rangle^{(3)} = \frac{1}{4\pi m} \left(\frac{a'^2}{a^2} + 2\frac{a''}{a}\right) \left(\frac{1}{6} - \xi\right).\tag{44}$$

The coefficient ξ is arbitrary, but the physically interesting choices are limited to $\xi = 0$ (minimal coupling) and $\xi = 1/8$ (conformal coupling). It can be easily checked that summing the adiabatic modes yields precisely the same result.

When the scale factor satisfies (32), one has

$$\left\langle \phi^2 \right\rangle^{(3)} = \frac{3H_0^2}{4\pi m} \left(\frac{1}{6} - \xi\right)\tag{45}$$

which is nonnegative for $\xi < \tfrac{1}{6}$. On the other hand, for a power-law cosmologies

$$\left\langle \phi^2 \right\rangle^{(3)} = \frac{3p^2 - 2p}{4\pi m t^2} \left(\frac{1}{6} - \xi\right).\tag{46}$$

The vacuum polarization is negative in the regions defined by the inequalities $p < \tfrac{2}{3}$, $\xi < \tfrac{1}{6}$ and $p > \tfrac{1}{6}$, $\xi > \tfrac{1}{6}$, respectively, and positive elsewhere. That means, for example, that $\left\langle \phi^2 \right\rangle^{(3)}$ is negative for physical choices of the coupling constant in the first region and positive in the second.

Finally, let us discuss briefly the relevance of our results for the stress-energy tensor. Our calculations and dimensional analysis suggest that the energy density, $\rho^{(N)}$, in the FRW universe with the scale factor (32) is given by

$$\rho^{(N)} = -\left\langle T_t^t \right\rangle^{(N)} \propto \frac{H_0^{n+2}}{\pi^{n/2} m^{2/(N-n+1)}} \,, \tag{47}$$

where $N \geq 4$ and proportionality constant depends on the curvature coupling parameter. On the other hand, for the power-law cosmologies, one has

$$\rho^{(N)} \propto \frac{1}{\pi^{n/2} m^{2/(N-n+1)} t^{n+2}} \,. \tag{48}$$

A similar behavior is expected for the "pressure component" $\tilde{p} = \langle T_x^x \rangle^{(N)}$, where x is the spatial coordinate. The general techniques presented in this paper, with some additional effort, can be extended to construction of the regularized stress-energy tensor. It is interesting to examine how the stress-energy tensor varies with dimension of the background spacetime and to study the semiclassical Einstein field equations. This and related problems are under active investigations and we intend to report on this calculations in future publications.

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