LANDAU ENERGY SPECTRUM AND QUANTUM OSCILLATOR MODEL FOR TWISTED N-ENLARGED NEWTON–HOOKE SPACE-TIME

Marcin Daszkiewicz

Institute of Theoretical Physics, University of Wroclaw
pl. Maxa Borna 9, 50-206 Wrocław, Poland
marcin@ift.uni.wroc.pl

(Received October 24, 2014)

We derive the energy levels for oscillator model defined on the twisted N-enlarged Newton–Hooke space-time, *i.e.*, we find time-dependent eigenvalues and corresponding time-dependent eigenstates. We also demonstrate that for a particular choice of deformation parameters of phase space, the above spectrum can be identified with the time-dependent Landau one.

DOI:10.5506/APhysPolB.45.2047
PACS numbers: 02.20.Uw, 02.40.Gh, 03.65.–w

1. Introduction

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on Quantum Gravity [2, 3] and String Theory models [4, 5], indicating that space-time at Planck scale should be noncommutative, *i.e.*, it should have a quantum nature. Consequently, there are a number of papers dealing with noncommutative classical and quantum mechanics (see *e.g.* [6–8]) as well as with field theoretical models (see *e.g.* [9–11]) in which the quantum space-time is employed.

It is well-known that a proper modification of the Poincare and Galilei Hopf algebras can be realized in the framework of Quantum Groups [12, 13]. Hence, in accordance with the Hopf-algebraic classification of all deformations of relativistic and nonrelativistic symmetries (see [14, 15]), one can distinguish three types of quantum spaces [14, 15] (for details, see also [16]):

canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [17–19]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}\,, \tag{1}$$

Lie-algebraic modification of classical space-time [19–22]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho} x_\rho \,, \tag{2}$$

and quadratic deformation of Minkowski and Galilei spaces [19, 22–24]

$$[x_\mu, x_\nu] = i\theta_{\mu\nu}^{\rho\tau} x_\rho x_\tau \,, \tag{3}$$

with coefficients $\theta_{\mu\nu}$, $\theta_{\mu\nu}^{\rho}$ and $\theta_{\mu\nu}^{\rho\tau}$ being constants.

Moreover, it has been demonstrated in [16], that in the case of the so-called N-enlarged Newton–Hooke Hopf algebras $\mathcal{U}_0^{(N)}(\mathrm{NH}_\pm)$ the twist deformation provides the new space-time noncommutativity of the form[1,2]

$$[t, x_i] = 0 \,, \qquad [x_i, x_j] = i f_\pm\left(\frac{t}{\tau}\right)\theta_{ij}(x) \,, \tag{4}$$

with time-dependent functions

$$\begin{aligned} f_+\left(\frac{t}{\tau}\right) &= f\left(\sinh\left(\frac{t}{\tau}\right), \cosh\left(\frac{t}{\tau}\right)\right) \,, \\ f_-\left(\frac{t}{\tau}\right) &= f\left(\sin\left(\frac{t}{\tau}\right), \cos\left(\frac{t}{\tau}\right)\right) \,, \end{aligned}$$

$\theta_{ij}(x) \sim \theta_{ij} = \mathrm{const}$ or $\theta_{ij}(x) \sim \theta_{ij}^{k} x_k$, and τ denoting the time scale parameter — the cosmological constant. Besides, it should be noted that the mentioned above quantum spaces (1), (2) and (3) can be obtained by the proper contraction limit of the commutation relations (4)[3].

As mentioned above, recently, there has been discussed the impact of different kinds of quantum spaces on the dynamical structure of physical systems (see *e.g.* [6–10] and [25–36]). Particularly, it has been demonstrated, that in the case of a classical oscillator model [28] as well as in the case of a nonrelativistic particle moving in constant external field force $\vec{F}$ [29], there are generated by space-time noncommutativity additional force terms. Such a type of investigation has been performed for quantum oscillator model as well [28], *i.e.*, it was demonstrated that the quantum space in nontrivial way affects the spectrum of the energy operator. Besides, in the paper [30] there has been considered a model of a particle moving on the κ-Galilei space-time in the presence of gravitational field force. It has been demonstrated,

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[1] $x_0 = ct$.

[2] The discussed space-times have been defined as the quantum representation spaces, the so-called Hopf modules (see e*e.g.* [17, 18]), for the quantum N-enlarged Newton–Hooke Hopf algebras.

[3] Such a result indicates that the twisted N-enlarged Newton–Hooke Hopf algebra plays a role of the most general type of quantum group deformation at nonrelativistic level.

that in such a case there is produced a force term, which can be identified with the so-called Pioneer anomaly [32], and the value of the deformation parameter κ can be fixed by a comparison of obtained result with observational data. Moreover, especially interesting results have been obtained in the series of papers [33–36] concerning the Hall effect for canonically deformed space-time (1). Particularly, there has been found the θ-dependent (Landau) energy spectrum of an electron moving in uniform magnetic as well as in uniform electric field. Such results have been generalized to the case of the twisted N-enlarged Newton–Hooke Hopf algebra [16], *i.e.*, there has been derived the time-dependent Landau levels for the particle moving in the corresponding quantum space in the presence of external fields [37].

In this article, we find the time-dependent energy levels for oscillator model defined on noncommutative space-time (11). In particular, we demonstrate that for a special choice of deformation parameters of corresponding phase space (see formula (35)) the above spectrum can be identified with time-dependent Landau one.

The paper is organized as follows. In Sect. 2 we recall basic facts concerning the twisted N-enlarged Newton–Hooke space-times provided in the article [16]. The third section is devoted to the calculation of energy spectrum (as well as Landau levels) for the twist-deformed oscillator model. The final remarks are presented in the last section.

2. Twisted N-enlarged Newton–Hooke space-times

In this section, we recall the basic facts associated with the twisted N-enlarged Newton–Hooke Hopf algebra $\mathcal{U}_\alpha^{(N)}(\mathrm{NH}_\pm)$ and with the corresponding quantum space-times [16]. Firstly, it should be noted, that in accordance with the Drinfeld twist procedure, the algebraic sector of twisted Hopf structure $\mathcal{U}_\alpha^{(N)}(\mathrm{NH}_\pm)$ remains undeformed, *i.e.*, it takes the form

$$[M_{ij}, M_{kl}] = i\left(\delta_{il}\, M_{jk} - \delta_{jl}\, M_{ik} + \delta_{jk} M_{il} - \delta_{ik} M_{jl}\right), \qquad [H, M_{ij}\,] = 0\,, \tag{5}$$

$$\left[M_{ij}, G_k^{(n)}\right] = i\left(\delta_{jk}\, G_i^{(n)} - \delta_{ik}\, G_j^{(n)}\right), \qquad \left[G_i^{(n)}, G_j^{(m)}\right] = 0\,, \tag{6}$$

$$\left[G_i^{(k)}, H\right] = -ikG_i^{(k-1)}\,, \qquad \left[H, G_i^{(0)}\right] = \pm\frac{i}{\tau}G_i^{(1)}\,; \qquad k > 1\,, \tag{7}$$

where τ, M_{ij}, H, $G_i^{(0)}(= P_i)$, $G_i^{(1)}(= K_i)$ and $G_i^{(n)}(n > 1)$ can be identified with cosmological time parameter, rotation, time translation, momentum, boost and accelerations operators respectively. Besides, the coproducts and antipodes of considered algebra are given by[4]

$$\Delta_\alpha(a) = \mathcal{F}_\alpha \circ \Delta_0(a) \circ \mathcal{F}_\alpha^{-1}\,, \qquad S_\alpha(a) = u_\alpha S_0(a)\, u_\alpha^{-1} \tag{8}$$

[4] $\Delta_0(a) = a \otimes 1 + 1 \otimes a$, $S_0(a) = -a$.

with $u_\alpha = \sum f_{(1)} S_0(f_{(2)})$ (we use the Sweedler's notation $\mathcal{F}_\alpha = \sum f_{(1)} \otimes f_{(2)}$) and with the twist factor $\mathcal{F}_\alpha \in \mathcal{U}_\alpha^{(N)}(\mathrm{NH}_\pm) \otimes \mathcal{U}_\alpha^{(N)}(\mathrm{NH}_\pm)$ satisfying the classical cocycle condition

$$\mathcal{F}_{\alpha 12} \cdot (\Delta_0 \otimes 1)\, \mathcal{F}_\alpha = \mathcal{F}_{\alpha 23} \cdot (1 \otimes \Delta_0)\, \mathcal{F}_\alpha\,, \tag{9}$$

and the normalization condition

$$(\epsilon \otimes 1)\ \mathcal{F}_\alpha = (1 \otimes \epsilon)\ \mathcal{F}_\alpha = 1 \tag{10}$$

such that $\mathcal{F}_{\alpha 12} = \mathcal{F}_\alpha \otimes 1$ and $\mathcal{F}_{\alpha 23} = 1 \otimes \mathcal{F}_\alpha$.

The corresponding quantum space-times are defined as the representation spaces (Hopf modules) for the N-enlarged Newton–Hooke Hopf algebra $\mathcal{U}_\alpha^{(N)}(\mathrm{NH}_\pm)$. Generally, they are equipped with two the spatial directions commuting to classical time, *i.e.* they take the form

$$[t, \hat{x}_i] = [\hat{x}_1, \hat{x}_3] = [\hat{x}_2, \hat{x}_3] = 0\,, \qquad [\hat{x}_1, \hat{x}_2] = i f(t)\,, \qquad i = 1,2,3\,. \tag{11}$$

However, it should be noted that this type of noncommutativity has been constructed explicitly only in the case of the 6-enlarged Newton–Hooke Hopf algebra, with [16][5]

$$\begin{aligned}
f(t) &= f_{\kappa_1}(t) = f_{\pm,\kappa_1}\left(\frac{t}{\tau}\right) = \kappa_1\, C_\pm^2\left(\frac{t}{\tau}\right)\,,\\
f(t) &= f_{\kappa_2}(t) = f_{\pm,\kappa_2}\left(\frac{t}{\tau}\right) = \kappa_2 \tau\, C_\pm\left(\frac{t}{\tau}\right) S_\pm\left(\frac{t}{\tau}\right)\,,\\
&\;\;\vdots\\
f(t) &= f_{\kappa_{35}}\left(\frac{t}{\tau}\right) = 86400\kappa_{35}\,\tau^{11}\left(\pm C_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right)\\
&\quad\times\left(S_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{6}\left(\frac{t}{\tau}\right)^3 - \frac{t}{\tau}\right)\,,\\
f(t) &= f_{\kappa_{36}}\left(\frac{t}{\tau}\right) = 518400\kappa_{36}\,\tau^{12}\left(\pm C_\pm\left(\frac{t}{\tau}\right) \mp \frac{1}{24}\left(\frac{t}{\tau}\right)^4 - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mp 1\right)^2\,,
\end{aligned} \tag{12}$$

[5] $\kappa_a = \alpha$ ($a = 1, \ldots, 36$) denote the deformation parameters.

and

$$C_{+/-}\left(\frac{t}{\tau}\right) = \cosh / \cos \left(\frac{t}{\tau}\right) \quad \text{and} \quad S_{+/-}\left(\frac{t}{\tau}\right) = \sinh / \sin \left(\frac{t}{\tau}\right) .$$

Moreover, one can easily check that when τ is approaching the infinity limit, the above quantum spaces reproduce the canonical (1), Lie-algebraic (2) and quadratic (3) type of space-time noncommutativity, *i.e.*, for $\tau \to \infty$, we get

$$\begin{aligned} f_{\kappa_1}(t) &= \kappa_1 \,, \\ f_{\kappa_2}(t) &= \kappa_2 \, t \,, \\ &\cdot \\ &\cdot \\ &\cdot \\ f_{\kappa_{35}}(t) &= \kappa_{35} \, t^{11} \,, \\ f_{\kappa_{36}}(t) &= \kappa_{36} \, t^{12} \,. \end{aligned} \tag{13}$$

Of course, for all deformation parameters κ_a going to zero, the above deformations disappear.

3. Quantum oscillator model for twisted N-enlarged Newton–Hooke space-time

Let us now return to the main aim of our investigations, *i.e.*, to the oscillator model defined on quantum space-times (11)–(13). In the first step of our construction, we extend the described in previous section spaces to the whole algebra of momentum and position operators as follows

$$[\hat{x}_1, \hat{x}_2] = 2if_{\kappa_a}(t) \,, \qquad [\hat{p}_1, \hat{p}_2] = 2ig_{\kappa_a}(t) \,, \tag{14}$$

$$[\hat{x}_i, \hat{x}_j] = i\delta_{ij} \left[1 + f_{\kappa_a}(t) g_{\kappa_a}(t)\right] \tag{15}$$

with the arbitrary function $g_{\kappa_a}(t)$. One can check that relations (14), (15) satisfy the Jacobi identity and for deformation parameters κ_a approaching zero become classical.

Next, by analogy to the commutative case, we define the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_1^2 + \hat{p}_2^2\right) + \frac{1}{2} m\omega^2 \left(\hat{x}_1^2 + \hat{x}_2^2\right) \tag{16}$$

with m and ω denoting the mass and frequency of a particle, respectively.

In order to analyse the above system, we represent the noncommutative operators $(\hat{x}_i, \hat{p}_i)$ by the classical ones (x_i, p_i) as (see *e.g.* [26–28])

$$\hat{x}_1 = x_1 - f_{\kappa_a}(t) p_2 \,, \tag{17}$$
$$\hat{x}_2 = x_2 + f_{\kappa_a}(t) p_1 \,, \tag{18}$$
$$\hat{p}_1 = p_1 + g_{\kappa_a}(t) x_2 \,, \tag{19}$$
$$\hat{p}_2 = p_2 - g_{\kappa_a}(t) x_1 \,, \tag{20}$$

where

$$[x_i, x_j] = 0 = [p_i, p] \,, \qquad [x_i, p_j] = i\hbar\delta_{ij} \,. \tag{21}$$

Then, the Hamiltonian (16) takes the form

$$\hat{H} = \hat{H}(t) = \frac{1}{2M(t)}\left(p_1^2 + p_2^2\right) + \frac{1}{2}M(t)\Omega^2(t)\left(x_1^2 + x_2^2\right) - S(t)L \tag{22}$$

with

$$L = x_1 p_2 - x_2 p_1 \,, \tag{23}$$
$$1/M(t) = 1/m + m\omega^2 f_{\kappa_a}^2(t) \,, \tag{24}$$
$$\Omega(t) = \sqrt{\left(1/m + m\omega^2 f_{\kappa_a}^2(t)\right)\left(m\omega^2 + g_{\kappa_a}^2(t)/m\right)} \,, \tag{25}$$

and

$$S(t) = m\omega^2 f_{\kappa_a}(t) + g_{\kappa_a}(t)/m \,. \tag{26}$$

In accordance with the scheme proposed in [28], we introduce a set of time-dependent creation $(a_A^\dagger(t))$ and annihilation $(a_A(t))$ operators

$$\hat{a}_\pm(t) = \frac{1}{2}\left[\frac{(p_2 \pm ip_1)}{\sqrt{M(t)\Omega(t)}} - i\sqrt{M(t)\Omega(t)}(x_2 \pm ix_1)\right] \tag{27}$$

satisfying the standard commutation relations

$$[\hat{a}_A, \hat{a}_B] = 0 \,, \qquad \left[\hat{a}_A^\dagger, \hat{a}_B^\dagger\right] = 0 \,, \qquad \left[\hat{a}_A, \hat{a}_B^\dagger\right] = \delta_{AB} \,; \qquad A, B = \pm \,. \tag{28}$$

Then, it is easy to see that in terms of the objects (27), the Hamiltonian function (22) can be written as follows

$$\hat{H}(t) = \Omega_+(t)\left(\hat{N}_+(t) + \tfrac{1}{2}\right) + \Omega_-(t)\left(\hat{N}_-(t) + \tfrac{1}{2}\right) \tag{29}$$

with the coefficient $\Omega_\pm(t)$ and the particle number operators $\hat{N}_\pm(t)$ given by

$$\Omega_\pm(t) = \Omega(t) \mp S(t) \,, \tag{30}$$
$$\hat{N}_\pm(t) = \hat{a}_\pm^\dagger(t)\hat{a}_\pm(t) \,. \tag{31}$$

Besides, one can observe that the eigenvectors of Hamiltonian (29) can be written as

$$|n_+, n_-, t\rangle = \frac{1}{\sqrt{n_+!}}\frac{1}{\sqrt{n_-!}}\left(\hat{a}_+^\dagger(t)\right)^{n_+}\left(\hat{a}_-^\dagger(t)\right)^{n_-}|0\rangle\,, \tag{32}$$

while the corresponding eigenvalues take the form

$$E_{n_+,n_-}(t) = \Omega_+(t)\left(n_+ + \tfrac{1}{2}\right) + \Omega_-(t)\left(n_- + \tfrac{1}{2}\right)\,. \tag{33}$$

Let us now consider an interesting situation such that

$$\Omega(t) = S(t) = m\omega^2 f_{\kappa_a}(t) + \frac{1}{f_{\kappa_a}(t)m}\,. \tag{34}$$

One can check that it appears when functions $f_{\kappa_a}(t)$ and $g_{\kappa_a}(t)$ satisfy the following condition

$$f_{\kappa_a}(t)\cdot g_{\kappa_a}(t) = 1\,. \tag{35}$$

Then, we have

$$\Omega_-(t) = 2\Omega(t)\,, \qquad \Omega_+(t) = 0\,, \tag{36}$$

and, consequently, the spectrum (33) provides the Landau energy levels with the time-dependent frequency $\Omega(t)$ [6]

$$E_n(t) = \Omega(t)\,(2n+1)\;; \qquad n = 0,1,2,3,\ldots \tag{37}$$

It should be mentioned, however, that the above result can be obtained in simpler (but less general) way as well. Firstly, one can observe that under the condition (35) the Hamiltonian (22) takes a particular form

$$\hat{H} = \hat{H}(t) = \frac{1}{2M(t)}\left(p_1^2 + p_2^2\right) + \frac{1}{2}M(t)\Omega^2(t)\left(x_1^2 + x_2^2\right) - \Omega(t)L\,. \tag{38}$$

Next, if one defines the following creation and annihilation operators[7]

$$\hat{A}(t) = \frac{1}{\sqrt{2}}\left(\pi_1 + i\pi_2\right)\,, \qquad \hat{A}(t) = \frac{1}{\sqrt{2}}\left(\pi_1 - i\pi_2\right) \tag{39}$$

[6] For canonical deformation ($f_{\kappa_1}(t) = \kappa_1$), we get constant frequency $\Omega(\kappa_1)$.

[7] $\left[\hat{A}(t), \hat{A}^\dagger(t)\right] = 1$.

with

$$\pi_1(t) = \frac{1}{\sqrt{2M(t)\Omega(t)}}\left(p_1 + M(t)\Omega(t)x_2\right) , \tag{40}$$

$$\pi_2(t) = \frac{1}{\sqrt{2M(t)\Omega(t)}}\left(p_2 - M(t)\Omega(t)x_1\right) , \tag{41}$$

then, the function (38) can be written as

$$\hat{H}(t) = \Omega(t)\left[\pi_1^2 + \pi_2^2\right] = 2\Omega(t)\left(\hat{A}^\dagger(t)\hat{A}(t) + \tfrac{1}{2}\right) . \tag{42}$$

Consequently, it is easy to notice that (in fact) the corresponding eigenvectors and the corresponding eigenvalues are given by

$$|n,t\rangle = \frac{1}{\sqrt{n!}}\left(\hat{A}^\dagger(t)\right)^n |0\rangle , \tag{43}$$

$$E_n(t) = \Omega(t)\,(2n+1) \; ; \qquad n = 0,1,2,3,\ldots , \tag{44}$$

respectively.

4. Final remarks

In this paper, we find the time-dependent energy levels for an oscillator model defined on the twisted N-enlarged Newton–Hooke space-time (11). Moreover, we demonstrate that for a special choice of the deformation parameters of the corresponding phase space (see formula (35)) the above spectrum can be identified with the time-dependent Landau one.

As it was already mentioned in Introduction the time-dependent Landau energies for quantum space (11) (with function $g_{\kappa_a}(t)$ equal zero) has been already found in [37]. Precisely, there has been provided the energy levels for a nonrelativistic particle moving in uniform magnetic (B) as well as in uniform electric (external) field; they read as follows

$$\hat{E}_n(t) = \hat{\Omega}(t)\,(2n+1) \; ; \qquad n = 0,1,2,3,\ldots \tag{45}$$

with frequency $\hat{\Omega}(t)$ given by

$$\hat{\Omega}(t) = \left(1 - \frac{f_{\kappa_a}(t)B}{4}\right)\frac{B}{2m} , \tag{46}$$

where m denotes the mass of particle.

Finally, it should be mentioned that the presented investigation has been performed for most general (constructed explicitly) type of space-time noncommutativity at nonrelativistic level.

The author would like to thank J. Lukierski for valuable discussions. This paper has been financially supported by the Polish National Science Centre grant No 2011/01/B/ST2/03354.

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