

ON THE QUANTIZATION OF NONLOCAL THEORY

KATARZYNA BOLONEK-LASOŃ

Faculty of Economics and Sociology, Department of Statistical Methods
University of Lodz, Rewolucji 1905 41/43, 90-214 Łódź, Poland
kbolonek1@wp.pl

PIOTR KOSIŃSKI

Faculty of Physics and Applied Informatics, Department of Informatics
University of Lodz, Pomorska 149/153, 90-236 Łódź, Poland
pkosinsk@uni.lodz.pl

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A simple nonlocal theory is put into Hamiltonian form and quantized by using the modern version of Ostrogradski approach.

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1. Introduction

Some attention has been attracted in the last decade by the problem of quantizing the models defined by the Lagrangians which are nonlocal in time. This is mainly due to the fact that such nonlocalities do appear in field theories on noncommutative space-time [1]. However, the very problem dates back to the seminal paper of Pais and Uhlenbeck [2]. They have shown how the quadratic nonlocal Lagrangians can be converted into (alternating) sum of independent oscillators (possibly complex). The corresponding transformation is defined slightly formally in terms of infinite order differential operators which single out particular frequencies of classical motion. Once the Lagrangian is represented as a sum of independent oscillators, the Hamiltonian form of dynamics can be readily constructed and quantization performed. However, the method seems to be applicable only to some particular models.

The Hamiltonian formalism for general higher-derivative theories has been constructed long time ago by Ostrogradski [3]. Its more sophisticated form, in principle applicable to arbitrary nonlocal theories, has been given by a number of authors [4–7]. Its applicability is restricted by the fact that the original dynamical equations appear here as a constraint which has to be dealt with by use of Dirac formalism. This is very difficult because the explicit solutions to the equations of motion are hardly known. However, for particular linear theories, one is able to find solutions which allows to describe explicitly the dynamics in Hamiltonian form.

It has been noted quite recently that the Pais–Uhlenbeck model exhibits a remarkable symmetry provided its eigenfrequencies are proportional to the consecutive odd integers. More precisely, the Noether symmetries of the Pais–Uhlenbeck oscillator of the order of $2l + 1$ (l being half integer) form the so-called l -conformal Galilei group [8–14] which gained recently much attention. The nonlocal model introduced by Pais and Uhlenbeck provides a natural infinite order generalization of their finite order oscillators with special eigenfrequencies. Therefore, it deserves a more careful study. In particular, it is interesting to quantize it using the generalized Ostrogradski formalism which would also provide proper framework for studying the relevant symmetries.

Let us conclude this short introduction with the following remarks. The Hamiltonian formalism proposed by Ostrogradski has a significant disadvantage: the energy is not bounded from below. The resulting dynamics is, therefore, unstable under external perturbations. The instability stems from the fact that some of the momenta enter the Hamiltonian linearly; consequently, it is related to the unbounded regions in phase space (contrary, for example, to the case of hydrogen atom where it follows from the behaviour of potential energy in the neighbourhood of the origin of coordinate space). Such kind of instability cannot be cured on the quantum level by uncertainty principle. In fact, Pais and Uhlenbeck have shown that the resulting quantum models continue to be unstable. In their quantization procedure, the metric in the space of states is positively definite. On the other hand, if one admits indefinite metric, the consistent quantization preserving positivity of energy is possible [15, 16]. In particular, the method proposed in Ref. [16] can be applied to the case under consideration. However, below we restrict ourselves to the standard quantization procedure preserving the positivity of the space of states metric.

In the present note, we use the method developed in Refs. [4–7] to put into Hamiltonian form and quantize the nonlocal model Pais and Uhlenbeck. The final results coincide, as expected, with those of Pais and Uhlenbeck but the method used is quite different. The model is so simple that all details of the generalized Ostrogradski method can be revealed.

2. Hamiltonian formalism and quantization

Our starting point is the following nonlocal Lagrangian

$$L = -\frac{m}{\alpha^2} q(t) q(t + \alpha) , \quad (1)$$

where m and α are some constants of dimensions of mass and time, respectively. L provides a natural generalization of standard harmonic oscillator. Indeed, expanding $q(t + \alpha)$ to the second order in α , one obtains

$$L = \frac{m\dot{q}^2}{2} - \frac{m\omega^2 q^2}{2} + \frac{d}{dt} \left(\frac{m}{2\alpha} q^2 - \frac{m}{2} q\dot{q} \right) + O(\alpha) . \quad (2)$$

Skipping total derivative, one obtains harmonic oscillator with the frequency $\omega^2 \equiv \frac{2}{\alpha^2}$.

The equation of motion

$$\delta S \equiv \int_{-\infty}^{\infty} dt' \frac{\delta L(t')}{\delta q(t)} = 0 \quad (3)$$

following from Eq. (1) reads

$$q(t - \alpha) + q(t + \alpha) = 0 . \quad (4)$$

In order to quantize our theory, one has to put it into Hamiltonian form. To this end, we use the formalism proposed in Refs. [4–6] which provides a far-reaching generalization of the Ostrogradski approach [3]. According to the prescription of Refs. [4–6], one introduces a continuous index λ and makes the following replacements

$$\begin{aligned} q(t) &\rightarrow Q(t, \lambda) , & q(t + \alpha) &\rightarrow Q(t, \lambda + \alpha) , \\ \dot{q}(t) &\rightarrow Q'(t, \lambda) \equiv \frac{\partial Q(t, \lambda)}{\partial \lambda} , & \dot{q}(t + \alpha) &\rightarrow Q'(t, \lambda + \alpha) . \end{aligned} \quad (5)$$

The Lagrangian is defined as

$$\tilde{L}(t) \equiv \int_{-\infty}^{\infty} d\lambda \delta(\lambda) L(Q(t, \lambda), Q'(t, \lambda), \dots) , \quad (6)$$

where $L(Q(t, \lambda), Q'(t, \lambda), \dots)$ is obtained from the original Lagrangian by making the replacements (5).

In our case,

$$L(Q(t, \lambda), Q'(t, \lambda), \dots) = -\frac{m}{\alpha^2} Q(t, \lambda) Q(t, \lambda + \alpha) \quad (7)$$

and

$$\tilde{L}(t) = -\frac{m}{\alpha^2} Q(t, 0) Q(t, \alpha) . \quad (8)$$

The Hamiltonian and the Poisson bracket read

$$\begin{aligned} H(t) &\equiv \int_{-\infty}^{\infty} d\lambda P(t, \lambda) Q'(t, \lambda) - \tilde{L}(t) \\ &= \int_{-\infty}^{\infty} d\lambda P(t, \lambda) Q'(t, \lambda) + \frac{m}{\alpha^2} Q(t, 0) Q(t, \alpha) , \end{aligned} \quad (9)$$

$$\{Q(t, \lambda), P(t, \lambda')\} = \delta(\lambda - \lambda') . \quad (10)$$

Equations (9) and (10) generate the following dynamics

$$\dot{Q}(t, \lambda) = Q'(t, \lambda) , \quad (11)$$

$$\dot{P}(t, \lambda) = P'(t, \lambda) - \frac{m}{\alpha^2} (\delta(\lambda) Q(t, \alpha) + \delta(\lambda - \alpha) Q(t, 0)) . \quad (12)$$

In order to recover the original dynamics, one has to impose new constraints. First, we define the primary momentum constraints

$$P(t, \lambda) - \frac{1}{2} \int_{-\infty}^{\infty} d\sigma (\operatorname{sgn}(\lambda) - \operatorname{sgn}(\sigma)) \varepsilon(t, \sigma, \lambda) \approx 0 , \quad (13)$$

where

$$\varepsilon(t, \sigma, \lambda) \equiv \frac{\delta L(Q(t, \sigma), Q'(t, \sigma) \dots)}{\delta Q(t, \lambda)} . \quad (14)$$

In our case, Eqs. (13) and (14) imply

$$P(t, \lambda) - \frac{m}{2\alpha^2} (\operatorname{sgn}(\lambda - \alpha) - \operatorname{sgn}(\lambda)) Q(t, \lambda - \alpha) \approx 0 . \quad (15)$$

By differentiating a number of times with respect to time, one obtains the secondary constraints

$$\int_{-\infty}^{\infty} d\sigma \varepsilon(t, \sigma, \lambda) = 0 \quad (16)$$

giving here

$$Q(t, \lambda - \alpha) + Q(t, \lambda + \alpha) = 0 . \quad (17)$$

Summarizing, we have the following set of constraints

$$\varphi_1(t, \lambda) \equiv P(t, \lambda) - \frac{m}{2\alpha^2} (\operatorname{sgn}(\lambda - \alpha) - \operatorname{sgn}(\lambda)) Q(t, \lambda - \alpha) \approx 0, \quad (18)$$

$$\varphi_2(t, \lambda) \equiv Q(t, \lambda - \alpha) + Q(t, \lambda + \alpha) \approx 0. \quad (19)$$

In order to convert the constraints, which are here of second kind, into strong inequalities, one can define the Dirac bracket

$$\{A, B\}_D \equiv \{A, B\} - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda d\lambda' \{A, \varphi_i(\lambda)\} C_{ij}^{-1}(\lambda, \lambda') \{\varphi_j(\lambda'), B\}, \quad (20)$$

where

$$C(\lambda, \lambda') = \begin{bmatrix} \{\varphi_1(\lambda), \varphi_1(\lambda')\} & \{\varphi_1(\lambda), \varphi_2(\lambda')\} \\ \{\varphi_2(\lambda), \varphi_1(\lambda')\} & \{\varphi_2(\lambda), \varphi_2(\lambda')\} \end{bmatrix} \quad (21)$$

and

$$\int_{-\infty}^{\infty} d\lambda'' C_{ij}(\lambda, \lambda'') C_{jk}^{-1}(\lambda'', \lambda') = \delta_{ik} \delta(\lambda - \lambda'). \quad (22)$$

However, instead of proceeding via direct computation as sketched above, one can do better. First, φ_1 can be used to solve explicitly for $P(t, \lambda)$. Then, the reduced phase space is spanned by $Q(t, \lambda)$ subject to the constraint φ_2 . The Hamiltonian (9), expressed in terms of basic coordinates, reads

$$H(t) = \int_{-\infty}^{\infty} d\lambda \Delta(\lambda, \alpha) Q(t, \lambda - \alpha) Q'(t, \lambda) + \frac{m}{\alpha^2} Q(t, 0) Q(t, \alpha), \quad (23)$$

where

$$\Delta(\lambda, \alpha) \equiv \frac{m}{2\alpha^2} (\operatorname{sgn}(\lambda - \alpha) - \operatorname{sgn}(\lambda)). \quad (24)$$

Due to the constraints φ_2 , the original dynamics is recovered provided the Hamiltonian equations implied by (20) and (23) coincide with Eq. (11). This is the first condition imposed on Dirac bracket

$$\{Q(t, \lambda), Q(t, \lambda')\} = F(\lambda, \lambda'). \quad (25)$$

The remaining ones are:

$$F(\lambda, \lambda') = -F(\lambda', \lambda), \quad (26)$$

$$F(\lambda - \alpha, \lambda') + F(\lambda + \alpha, \lambda') = 0, \quad (27)$$

the latter being the consequence of the strong equality $\varphi_2 = 0$.

The unique solution to Eqs. (26), (27) which produces Eq. (11) reads

$$F(\lambda, \lambda') = \frac{\alpha^2}{m} \sum_{k=-\infty}^{\infty} (-1)^k \delta(\lambda - \lambda' + (2k+1)\alpha) . \quad (28)$$

The remaining brackets are easily recovered by demanding that they respect constraints

$$\{Q(t, \lambda), P(t, \lambda')\}_D = \Delta(\lambda', \alpha) F(\lambda, \lambda' - \alpha) , \quad (29)$$

$$\{P(t, \lambda), P(t, \lambda')\}_D = \Delta(\lambda, \alpha) \Delta(\lambda', \alpha) F(\lambda, \lambda') . \quad (30)$$

We shall now solve explicitly the second constraint. To this end, we define new variables

$$\tilde{Q}(t, \lambda) \equiv Q(t, \lambda) e^{-\frac{i\pi}{2\alpha}\lambda} . \quad (31)$$

Then, by virtue of Eq. (19), $\tilde{Q}(t, \lambda)$ is periodic in λ , the period being 2α . It can be expanded in Fourier series. Taking this into account, we conclude that $Q(t, \lambda)$ can be expanded as follows

$$Q(t, \lambda) = \sum_{n=-\infty}^{\infty} a_n(t) \Psi_n(\lambda) , \quad (32)$$

where

$$\Psi_n(\lambda) \equiv \frac{1}{\sqrt{2\alpha}} e^{\frac{i\pi}{2\alpha}(2n+1)\lambda} . \quad (33)$$

$\{\Psi_n(\lambda)\}_{n=-\infty}^{\infty}$ form an orthonormal

$$\int_{-\alpha}^{\alpha} d\lambda \bar{\Psi}_n(\lambda) \Psi_m(\lambda) = \delta_{nm} \quad (34)$$

and complete set. Additionally,

$$\bar{\Psi}_n(\lambda) = \Psi_{-(n+1)}(\lambda) . \quad (35)$$

By virtue of Eq. (35), the reality condition for $Q(t, \lambda)$ reads

$$\bar{a}_n = a_{-(n+1)} , \quad (36)$$

a'_n s providing new dynamical variables. Equations (32) and (34) imply

$$a_n(t) = \int_{-\alpha}^{\alpha} d\lambda \bar{\Psi}_m(\lambda) Q(t, \lambda) . \quad (37)$$

It is now straightforward to compute the Dirac bracket for new variables

$$\{a_m, a_n\} = \frac{i\alpha^2}{m} (-1)^m \delta_{m+n+1,0}. \quad (38)$$

Also the Hamiltonian is easily expressible in terms of new variables

$$H = \frac{m}{2\alpha^2} \sum_{k=-\infty}^{\infty} (-1)^k \frac{\pi}{2\alpha} (2k+1) a_k a_{-(k+1)}. \quad (39)$$

By defining

$$a_n = \begin{cases} \frac{\alpha}{\sqrt{m}} c_n, & n - \text{odd positive} \\ \frac{\alpha}{\sqrt{m}} c_{-(n+1)}, & n - \text{odd negative} \\ \frac{\alpha}{\sqrt{m}} \bar{c}_n, & n - \text{even nonnegative} \\ \frac{\alpha}{\sqrt{m}} \bar{c}_{-(n+1)}, & n - \text{even negative} \end{cases}, \quad (40)$$

one obtains

$$\{c_m, \bar{c}_n\} = -i\delta_{mn} \quad (41)$$

and

$$H = \sum_{k=0}^{\infty} (-1)^k \frac{\pi}{2\alpha} (2k+1) \bar{c}_k c_k. \quad (42)$$

Equations (41), (42) tell us that our theory is an alternating sum of independent harmonic oscillators of frequencies $\omega_k = \frac{\pi}{2\alpha} (2k+1)$.

Equations (41), (42) can be readily quantized. Upon rescaling $c_m \rightarrow \sqrt{\hbar} \hat{c}_m$ and keeping the order of factors as in Eq. (42) (which implies the subtraction of zero-energy value), one arrives at

$$[c_m, c_n^+] = \delta_{mn}, \quad (43)$$

$$H = \hbar \sum_{k=0}^{\infty} (-1)^k \frac{\pi}{2\alpha} (2k+1) c_k^+ c_k. \quad (44)$$

Our results coincide with those obtained by the Pais–Uhlenbeck method.

3. α -expansion

The model defined by Eq. (1) can be approximated by finite-order oscillators. To this end, let us expand the right-hand side of Eq. (1) in Taylor series in α

$$L = -\frac{m^2}{\alpha^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} q(t) q^{(n)}(t). \quad (45)$$

Due to the identity

$$qq^{(2k+1)} = \frac{d}{dt} \left(\sum_{i=0}^{k-1} q^{(i)} q^{(2k-i)} + \frac{(-1)^k}{2} q^{(k)^2} \right), \quad (46)$$

L can be rewritten as

$$L = -\frac{m^2}{\alpha^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} q(t) q^{(2n)}(t), \quad (47)$$

or

$$L = -\frac{m^2}{\alpha^2} q(t) \cos \left(i\alpha \frac{d}{dt} \right) q(t). \quad (48)$$

Consider the Lagrangian L_k obtained by truncating the expansion (47) on k^{th} term. Let us denote

$$F_k(x) \equiv \sum_{n=0}^k \frac{(-1)^n x^{2n}}{(2n)!}. \quad (49)$$

Using $\frac{d^2 F_k}{dx^2} = -F_{k-1}$, we find easily that F_k has $2k$ real roots $\pm \nu_n^{(k)}$, $n = 1, \dots, k$. Consequently,

$$L_k = \frac{(-1)^{k-1} m^2 \alpha^{2(k-1)} q(t)}{(2k)!} \prod_{n=1}^k \left(\frac{d^2}{dt^2} + \frac{\nu_n^{(k)^2}}{\alpha^2} \right) q(t). \quad (50)$$

According to Pais and Uhlenbeck, the Hamiltonian corresponding to the above Lagrangian can be written as an alternating sum of harmonic oscillators with frequencies $\frac{\nu_n^{(k)}}{\alpha}$. By virtue of Eq. (48)

$$\lim_{k \rightarrow \infty} \nu_n^{(k)} = (2n+1) \frac{\pi}{2}. \quad (51)$$

We conclude that taking the limit $k \rightarrow \infty$ in the theories described by the Lagrangians L_k , we arrive at the original nonlocal model (42).

4. Final remarks

We quantized simple nonlocal model of Pais and Uhlenbeck using the generalized Ostrogradski method described in Refs. [4–7]. The main point of this method is that the proper equations of motion are imposed as a constraint (Eqs. (15) and (17) in our case). The reason for the necessity of imposing such a constraint is the following. In the standard Ostrogradski

approach, applicable to the theories of arbitrary but finite order, all canonical equations but one, related to the highest time derivative, serve to define the consecutive time derivatives of the initial coordinate variable. The proper equation of motion results from the last Hamiltonian equation related to the highest time derivative. But for systems of infinite order, this equation is lacking. This is why one has to introduce the constraint.

Let us also note that, in the case under consideration, we could modify the quantization procedure in such a way that the energy remains positive definite. In fact, once the dynamics is formulated in completely decoupled form (*cf.* Eqs. (43), (44)) one can replace the Hamiltonian (44) by the one in which all terms enter with positive sign. However, in general, we cannot expect additional integrals of motion beyond the Ostrogradski Hamiltonian which results from time translation invariance and we are left with the alternative: unbounded energy or indefinite metric.

The formalism described above is well suited for the description of symmetries of nonlocal generalization of the set of harmonic oscillators. This will be the subject of the forthcoming paper.

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