GAUSSIAN MOTION COMPETING WITH LÉVY FLIGHTS

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We present a stochastic model where random walker can change its position according to two competing motions: Gaussian motion or Lévy flights and at each step, the type of motion is chosen randomly. We assume that contribution from all processes to the entire process is determined by the probabilities of appearances to each of them. For large times and spatial distance, we derive fractional differential equation describing the evolution of the probability density. This asymptotic form is determined by parameters describing stochastic motion: probabilities of occurrence of the Gaussian motion and Lévy flights, and two diffusion constants. We also show that for the initial density in the form of the Dirac delta function, this model has the analytic solution given in the integral form. For other forms of initial densities, we present results of numerical solutions for various model parameters.

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1. Introduction

In the description of natural phenomena by classical statistical physics, we usually need to decide which theoretical model is sufficient to explain observations. It occurs that many phenomena are well modeled by diffusion-like processes, *i.e.*, normal diffusion, subdiffusion or superdiffusion [1–3]. In the present paper, we investigate a model which takes into account two competing processes: normal diffusion and superdiffusion. Our motivation to consider such a model is related to the specific behaviour of chaotic trajectories in a deterministic Hamiltonian system in the presence of acceleration modes in the phase space. An example of such a system is the well known standard map [4–6], however similar effects may be observed for the whole family of mappings. In Fig. 1, where we present an evolution of an angular momentum for some initial conditions, one can see that the system performs short jumps corresponding to the Gaussian motion and from time to time is captured by an accelerating island what results in the long jump, *i.e.*, a long lasting movement with an almost linear momentum increase.

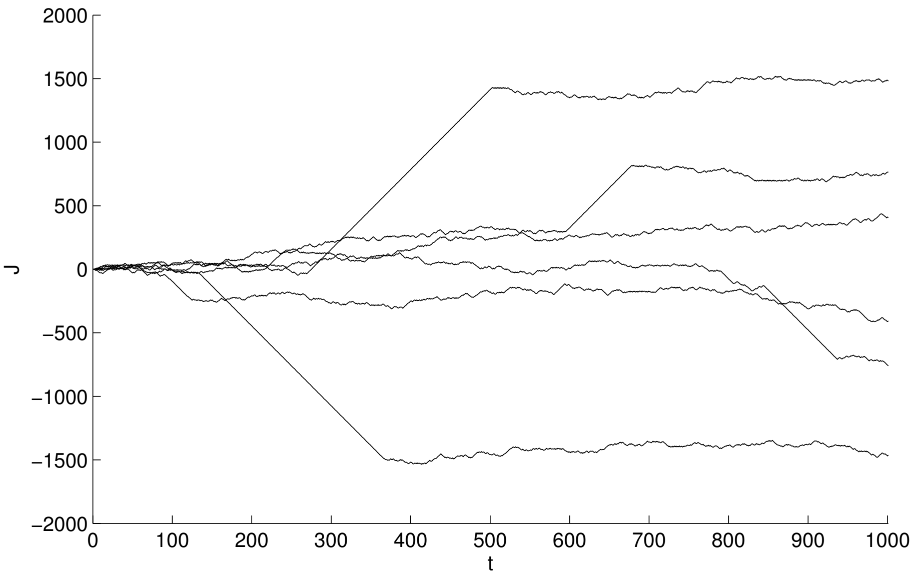

Fig. 1. Selected trajectories in phase space of standard map with accelerating modes.

Performing numerical statistics for such models, one can notice that the momentum density for sufficiently long time has the inverse power tails (compare Fig. 2). Based on this observation and the interplay of short and long jumps for individual trajectories, we propose a model being a mixture of two competing motions: Gaussian and Lévy.

Recently, competition between Leŕy flights and Gaussian motion was considered in context of biological systems [7–9], however their models and results cannot be directly applied to the problem treated in the present paper.

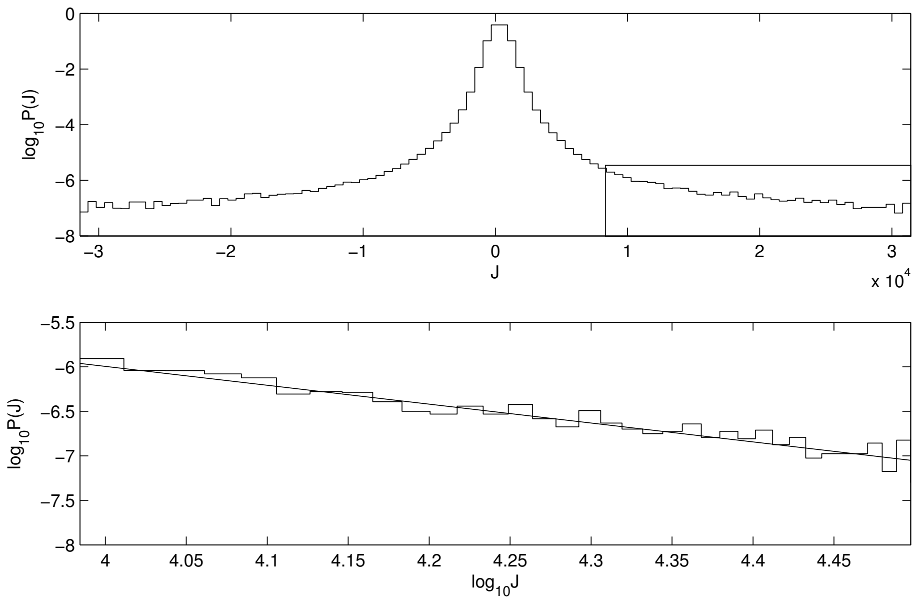

Fig. 2. Logarithm of the angular momentum distribution function for the standard map in the presence of accelerating modes.

2. Assumptions of the model

We assume that the particle goes from point x' to point x in N steps, $N = 1, 2, \ldots$ In each step, the particle jumps the distance x, but this distance is taken from one of two different distributions: $b(x)$ — normal distribution (Gaussian motion) and $l(x)$ — symmetric Lévy distribution (Lévy motion). In each step, the probability that the particle will take a Gaussian jump is equal a, and is independent of t and x; consequently, $1 - a$ denotes probability of the other (Lévy) choice. We further assume that particle changes its position with infinite velocity and between jumps it waits. Waiting time probability density is described by $\psi(t)$, and each waiting time is independent of previous waiting times and of a position x. The probability that the particle does not change its position up to time t is given by the following formula

$$\Omega(t) = 1 - \int_0^t \psi(\tau) d\tau \, . \tag{1}$$

We introduce function $P(N,t)$ which describes the probability that the particle has done N steps until time t

$$P(N,t) = \int_0^t \psi^{*N}(\tau)\Omega(t-\tau)d\tau\,, \tag{2}$$

and another function $P(N,x)$, describing the probability that the particle has done N steps from point x_0 and is at position x. This function may be expressed in terms of probability a and distributions $b(x), l(x)$ as

$$P(N,x) = \sum_{M=0}^{M\leq N} \tbinom{N}{M} \left[ab(x)\right]^{*(N-M)} * \left[(1-a)l(x)\right]^{*M}\,, \tag{3}$$

where symbol $*N$ denotes N^{th} convolution. The probability density that the particle is at time t at position x is obtained as a sum over number of steps

$$W(x,t) = \sum_N P(N,t)P(N,x)\,. \tag{4}$$

It is convenient to change the formula (4) to the following form

$$W(x,t) = \int_0^t \Omega(t-\tau)\sum_N P(x,\tau,N)d\tau\,, \tag{5}$$

where

$$P(x,t,N) = \psi^{*N}(t)P(N,x)$$

is the probability density of just having arrived at position x at time t after N steps.

3. Integral transforms

Applying standard techniques of Laplace and Fourier transforms to the function $P(x,t,N)$, we obtain the following result

$$P(k,s,N) = \psi^N(s)\sum_{M=0}^{M\leq N} \tbinom{N}{M} \left[ab(k)\right]^{N-M}\left[(1-a)l(k)\right]^{M}\,. \tag{6}$$

The right-hand side of (6) can be written as

$$\left[\psi(s)\left(ab(k)+(1-a)l(k)\right)\right]^N\,. \tag{7}$$

Applying the same sequence of transforms to the equation (5) and taking into account (7), we obtain

$$W(s,k) = \Omega(s)\eta(k,s)\,, \tag{8}$$

where

$$\Omega(s) = \frac{1-\psi(s)}{s} \tag{9}$$

and

$$\eta(k,s) = \sum_{N=0}^{\infty} \left[\psi(s)\left[ab(k)+(1-a)l(k)\right]\right]^N = \frac{1}{\left[1-\psi(s)\left(ab(k)+(1-a)l(k)\right)\right]}\,. \tag{10}$$

We want to show how our approach relates to the standard framework of Continuous Time Random Walk (CTRW)[2]; to this aim, it is convenient to write the following identity for the function $\eta(k,s)$

$$\eta(k,s) = \eta(k,s)\psi(s)\left[ab(k)+(1-a)l(k)\right]+1\,. \tag{11}$$

If we use inverse Laplace and inverse Fourier transforms, we can obtain relation for function $\eta(x,t)$

$$\eta(x,t) = \int_0^\infty \int_{-\infty}^\infty \eta\left(x-x',t-t'\right)\psi\left(t'\right)\left[ab(x)+(1-a)l(x)\right]dx'dt' + \delta(x)\delta(t)\,. \tag{12}$$

Inverse Laplace and Fourier transforms of (8) give

$$W(x,t) = \int_0^t \Omega\left(t-t'\right)\eta\left(x,t'\right)dt'\,. \tag{13}$$

Function $\eta(x,t)$ is the probability density that the walker has just arrived to point x at time t. We can compare equation (12) for density $\eta(x,t)$ to formula (22) in [17], where one can find a standard CTRW approach. One easily sees that in our model the walker arrives to point x, starting from some point x', according to density of probability $[ab(x)+(1-a)l(x)]$ (12), which reflects competition of these two processes.

4. Differential equation

We assume the specific form of waiting time probability density, namely the Poisson distribution whose Laplace transform has the following form $\psi(s)=1/(\tau s+1)$. We also assume that Fourier transforms of jump distributions have forms: $b(k)=\exp(-\sigma^2k^2)$ and $l(k)=\exp(-\varepsilon^\alpha|k|^\alpha)$ (symmetric

Lévy distribution with $1 < \alpha < 2$). We make typical approximations: $\psi(s) \sim 1 - \tau s$, $b(k) \sim 1 - \sigma^2 k^2$ and $l(k) = e^{-\varepsilon^\alpha |k|^\alpha} \sim 1 - \varepsilon^\alpha |k|^\alpha$. Taking into account this assumptions in formula (8) and neglecting terms of the order higher than two, we obtain the following approximation

$$W(k,s) \approx \frac{1}{s + a\frac{\sigma^2}{\tau}k^2 + (1-a)\frac{\varepsilon^\alpha}{\tau}|k|^\alpha}\,. \tag{14}$$

This approximation allows us to obtain the following differential equation ($1 < \alpha < 2$)

$$\frac{\partial W(x,t)}{\partial t} = aD_1 \frac{\partial^2 W(x,t)}{\partial x^2} + (1-a)D_2 \frac{\partial^\alpha W(x,t)}{\partial |x|^\alpha}\,, \tag{15}$$

where: $D_1 = \frac{\sigma^2}{\tau}$ and $D_2 = \frac{\varepsilon^\alpha}{\tau}$, and symbol $\frac{\partial^\alpha}{\partial |x|^\alpha}$ denotes a Riesz derivative

$$\frac{\partial^\alpha}{\partial |x|^\alpha} f(x) = -\frac{1}{2\cos\frac{\alpha\pi}{2}} \left[{}_{-\infty}D_x^\alpha f(x) + {}_x D_{+\infty}^\alpha f(x) \right]\,, \qquad 1 < \alpha < 2\,. \tag{16}$$

In (16), symbols ${}_{-\infty}D_x^\alpha$ and ${}_xD_{+\infty}^\alpha$ for $1 < \alpha < 2$ are defined by [10, 11]

$${}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int\limits_{-\infty}^{x} \frac{f(x')}{(x-x')^{\alpha-1}} dx' \tag{17}$$

and

$${}_xD_{+\infty}^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dx^2} \int\limits_{x}^{+\infty} \frac{f(x')}{(x'-x)^{\alpha-1}} dx'\,. \tag{18}$$

5. Solution

It is possible to obtain a solution for this equation with the initial density in a form of Dirac delta function. Taking the inverse Laplace transform of $W(k,s)$ given by (14), one gets

$$W(k,t) = e^{-aD_1k^2t} e^{-(1-a)D_2|k|^\alpha t}\,. \tag{19}$$

Solution $W(x,t)$ is obtained by taking the inverse Fourier transform, and this leads to the following convolution formula

$$W(x,t) = \frac{1}{\sqrt{4\pi a D_1 t}} \int\limits_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{aD_1 t}} \frac{1}{\alpha |x'|} H_{2,2}^{1,1} \left[\frac{|x'|}{[(1-a)D_2 t]^{\frac{1}{\alpha}}} \middle|_{(1,1)(1,\frac{1}{2})}^{(1,\frac{1}{\alpha})(1,\frac{1}{2})} \right] dx'\,, \tag{20}$$

where H is the Fox H-function [12, 13].

For given value of t, asymptotic behaviour ($x \to \infty$) of $W(x,t)$ is determined by small values of k ($k \to 0$) in formula (19). This implies the power-law asymptotics

$$W(x,t) \sim |x|^{-(1+\alpha)} \tag{21}$$

and means that Lévy flights prevail.

On the other hand, numerical solutions of differential equation (15) can be obtained for quite general initial densities. To get results of numerical calculation, we use matrix approach [18, 19], where one considers differential equation on a space-time grid.

It is convenient to introduce new function

$$\tilde{W}(x,t) = W(x,t) - W(x,0) \tag{22}$$

for which equation (15) may be written as

$$\begin{aligned} &\frac{\partial \tilde{W}(x,t)}{\partial t} - aD_1 \frac{\partial^2 \tilde{W}(x,t)}{\partial x^2} - (1-a)D_2 \frac{\partial^\alpha \tilde{W}(x,t)}{\partial |x|^\alpha} \\ &= \left[aD_1 \frac{\partial^2 W(x,t)}{\partial x^2} + (1-a)D_2 \frac{\partial^\alpha W(x,t)}{\partial |x|^\alpha} \right]_{t=0} . \end{aligned} \tag{23}$$

The time interval $[0,t]$ is divided into equidistant time points t_j, where $j = 0, 1, 2, \ldots, N$ and $t_N = t$ with step Δt. The spatial variable is bounded to the interval $[-x_{\max}, x_{\max}]$ and is divided into equidistant spatial points x_i, where $i = 1, 2, \ldots, M$ with step Δx, $x_1 = -x_{\max}$ and $x_M = x_{\max}$. The pairs (x_i, t_j) form a space-time grid on which one solves numerically the fractional differential equation, *i.e.*, computes approximation to $\tilde{W}(x_i, t_j)$. From definition (22), it follows that the function $\tilde{W}$ satisfies the following initial condition: $\tilde{W}(x,0) = 0$. Additionally, we introduce boundary conditions in the form: $\tilde{W}(x_1, t_j) = 0$ and $\tilde{W}(x_M, t_j) = 0$.

Following [19], we define vectors

$$\vec{W} = \left[\tilde{W}(x_1,0), \ldots, \tilde{W}(x_M,0), \ldots, \tilde{W}(x_1,t_1), \ldots, \tilde{W}(x_M,t_N) \right]^T , \tag{24}$$

$$\vec{P} = \left[W^{(2)}(x_1,0), \ldots, W^{(2)}(x_M,0), \ldots, W^{(2)}(x_1,0), \ldots, W^{(2)}(x_M,0) \right]^T , \tag{25}$$

and

$$\vec{R} = \left[W^{(\alpha)}(x_1,0), \ldots, W^{(\alpha)}(x_M,0), \ldots, W^{(\alpha)}(x_1,0), \ldots, W^{(\alpha)}(x_M,0) \right]^T , \tag{26}$$

where T is a transposition. Vector $\vec{W}$ consists of values of density $\tilde{W}$ at all nodes of the space-time grid, while vectors $\vec{P}$ and $\vec{R}$ consists of $N+1$ times replicated values of second derivative and Riesz derivative of the order α of density $\tilde{W}(x_i,0)$, respectively. With these definitions, discretization of equation (23) leads to the following system $(N+1)M$ linear equations

$$\begin{aligned}&\left[B_{N+1}\otimes I_M - aD_1 I_{N+1}\otimes S_M - (1-a)D_2 I_{N+1}\otimes R_M\right]\vec{W}\\&= aD_1\vec{P} + (1-a)D_2\vec{R}\,,\end{aligned}\tag{27}$$

where $\otimes$ is a Kronecker product, I_M and I_{N+1} represent identity matrices, B_{N+1}, S_M, R_M correspond to the time derivative, spatial derivative of the second order and spatial Riesz derivative of the order α, respectively. The lower index $(M, N+1)$ denotes the size of each of these matrices and the three matrices corresponding to derivatives are defined by the following formulas:

$$B_{N+1} = \frac{1}{\Delta t}\begin{pmatrix} 1 & 0 & 0 & \dots & 0\\ -1 & 1 & 0 & \dots & 0\\ 0 & -1 & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots & 0\\ 0 & 0 & \dots & -1 & 1\end{pmatrix}, \tag{28}$$

$$S_M = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \dots & 0\\ 1 & -2 & 1 & \dots & 0\\ 0 & 1 & -2 & 1 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 1 & -2\end{pmatrix}. \tag{29}$$

Matrix R_M is given by

$$R_M = -\frac{1}{2\cos\left(\frac{\alpha\pi}{2}\right)}\left[Q_M + Q_M^T\right], \tag{30}$$

where Q_M is defined by [19]

$$Q_M = \frac{1}{|\Delta x|^\alpha}\begin{pmatrix} r_1^\alpha & r_0^\alpha & 0 & 0 & \dots & 0\\ r_2^\alpha & r_1^\alpha & r_0^\alpha & 0 & \dots & 0\\ r_3^\alpha & r_2^\alpha & r_1^\alpha & r_0^\alpha & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ r_{M-1}^\alpha & r_{M-2}^\alpha & r_{M-3}^\alpha & r_{M-4}^\alpha & \dots & r_0^\alpha\\ r_M^\alpha & r_{M-1}^\alpha & r_{M-2}^\alpha & r_{M-3}^\alpha & \dots & r_1^\alpha\end{pmatrix} \tag{31}$$

and

$$r_j^\alpha = (-1)^j\binom{\alpha}{j} = (-1)^j\frac{\Gamma(1+\alpha)}{\Gamma(1+j)\Gamma(1+\alpha-j)}. \tag{32}$$

Matrix R_M, in the limit $\alpha \to 2$, is equal to the matrix S_M (29). The form of the matrix R_M is a direct consequence of the discretization formula of Riesz derivatives, which is described at [14–16]

$$\left[\frac{\partial^\alpha}{\partial|x|^\alpha} f(x)\right]_{x=x_n} \approx -\frac{1}{2\cos\frac{\alpha\pi}{2}} \left[D_+^\alpha + D_-^\alpha\right] f_n\,, \tag{33}$$

where $f_n = f(x_n)$ and

$$D_\pm^\alpha f_n = \frac{1}{|\Delta x|^\alpha} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} f_{n\pm 1\mp j}\,, \qquad 1 < \alpha < 2\,. \tag{34}$$

In Fig. 3, we present density $W(x,t)$ at time $t = 3$ for three distinct values of the fractional order α. Other parameters have the following values $a = 0.5$, $D_1 = 1$, and $D_2 = 1$. For all cases, the same Gaussian was used as an initial density. One can see that for $\alpha = 1.1$ (dotted line) the distribution for $|x| \to \infty$ tends to zero slower than for the remaining densities. On the other hand, density for $\alpha = 1.9$ (solid line) for $|x| \to \infty$ tends to zero most quickly.

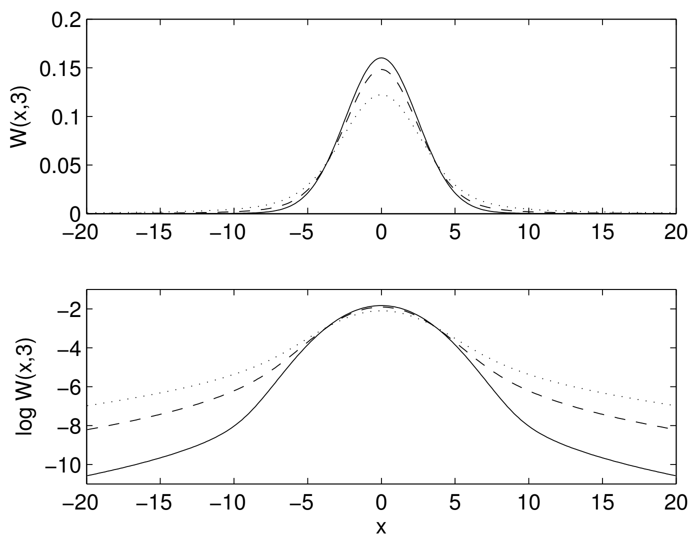

Fig. 3. Densities $W(x,t)$ at time $t = 3$ obtained in numerical integration of fractional differential equation for three values of parameter α: $\alpha = 1.1$ (dotted line), $\alpha = 1.5$ (dashed line) and $\alpha = 1.9$ (solid line). Other parameters have the following values $a = 0.5$, $D_1 = 1$ and $D_2 = 1$. All densities started from the same Gaussian initial density.

In Fig. 4, we present density $W(x,t)$ at time $t=3$ for three distinct values of probability a. Here, we assume that the order of fractional derivative α is equal 1.5, and diffusion constants have the following values: $D_1 = 1$ and $D_2 = 1$. As the initial density, we use the same Gaussian function for all three cases. One can see that, when $|x| \to \infty$, the density for $a = 0.2$ (dotted line) tends to zero slower than for the other two cases, while for $a = 0.8$ it tends to zero faster than the other ones. This property has clear interpretation from microscopic point of view. If a is small, then the particle performs more frequent jumps according to symmetric Lévy distribution and this property is reflected in the macroscopic density of a process.

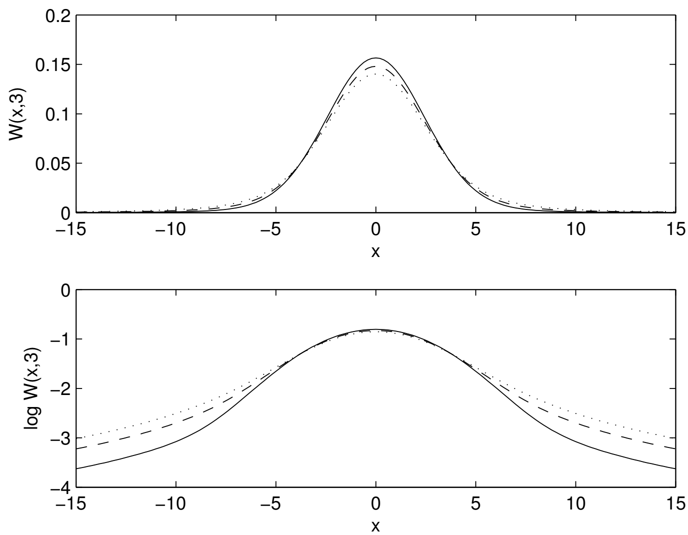

Fig. 4. Densities $W(x,t)$ at time $t=3$ obtained in numerical integration of fractional differential equation with different parameter a: $a = 0.2$ (dotted line), $a = 0.5$ (dashed line) and $a = 0.8$ (solid line). The order of fractional derivative α is equal 1.5. Diffusion constants have following values: $D_1 = 1$ and $D_2 = 1$. All densities started from the same Gaussian initial density.

6. Conclusions

In this paper, we presented the model, which links Gaussian motion with Lévy flights. This connection was achieved as an extension of the well known CTRW model. We formulated this model in two equivalent ways. They lead to the fractional differential equation (15), which is parametrized by four constants: probability a, two diffusion constants D_1, D_2 and an order of fractional derivative α. Using numerical methods and analytical results, we can follow the time evolution of the probability density.

REFERENCES

[1] I.M. Sokolov, J. Klafter, *Chaos* **15**, 026103 (2005).

[2] J. Klafter, I.M. Sokolov, *First Steps in Random Walks, From Tools to Applications*, Oxford University Press Inc., New York 2011.

[3] A. Janicki, A. Weron, *Simulation and Chaotic Behavior of α-Stable Stochastic Processes*, Marcel Dekker, Inc., 1994.

[4] B.V. Chirikov, *Phys. Rep.* **52**, 263 (1979).

[5] C.F.F. Karney, *Physica D* **8**, 360 (1983).

[6] L.E. Reichl, *The Transition to Chaos*, Springer-Verlag, New York 1992.

[7] E. Heinsalu, E. Hernández-García, C. López, *Phys. Rev. Lett.* **110**, 258101 (2013).

[8] E. Heinsalu, E. Hernández-García, C. López, *Phys. Rev.* **E85**, 041105 (2012).

[9] R. Martínez-García, J.M. Calabrese, C. López, *Phys. Rev.* **E89**, 032718 (2014).

[10] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Amsterdam 1993.

[11] R. Gorenflo *et al.*, *Chem. Phys.* **284**, 521 (2002).

[12] A.M. Mathai, R.K. Saxena, *The H-function with Application in Statistics and Other Disciplines*, J. Wiley and Sons, New York–London–Sidney 1978.

[13] H.M. Strivastava, K.C. Gupta, S.P. Goyal, *The H-functions of One and Two Variables*, South Asian Publishers Pvt Ltd, New Delhi 1982.

[14] R. Gorenflo, F. Mainardi, *Fract. Calc. Appl. Anal.* **1**, 167 (1998).

[15] M.M. Meerschaert, Ch. Tadjeran, *J. Comput. Appl. Math.* **172**, 65 (2004).

[16] Q. Yang, F. Liu, I. Turner, *Appl. Math. Model.* **34**, 200 (2010).

[17] R. Metzler, J. Klafter, *Phys. Rep.* **339**, 1 (2000).

[18] I. Podlubny, *Fract. Calc. Appl. Anal.* **3**, 231 (2000).

[19] I. Podlubny *et al.*, *J. Comput. Phys.* **228**, 3137 (2009).