SOLITON SOLUTION OF GENERALIZED CHIRAL NONLINEAR SCHRÖDINGER'S EQUATION WITH TIME-DEPENDENT COEFFICIENTS

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This paper obtains the exact 1-soliton solution to the chiral nonlinear Schrödinger's equation. There are three types of integration architectures that are implemented in this paper. They are the functional variable approach, first integral method as well as the ansatz method. These soliton solutions are obtained. There are constraint conditions that also fall out which must remain valid in order for the solitons and other solutions to exist.

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1. Introduction

The study of chiral solitons arises in the context of nuclear physics with the chiral nonlinear Schrödinger's equation (CNLSE). This equation is the outcome of the Jackiw–Pi Model. CNLSE has been studied in the past few decades. There are several aspects of CNLSE that has been addressed in the past few decades. They are traveling wave solutions, conservation

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laws, perturbation theory, integrability with semi-inverse variational principle, and several others [1–28]. This paper will also address the integrability aspects of the CNLSE and the generalized CNLSE with constant as well as time-dependent coefficients. There are three types of integration tools that will be adopted to integrate and obtain soliton solutions to the model. They are the functional variable method, first integral approach, and the ansatz method. These machineries will reveal solitons, topological solitons and singular soliton solutions to CNLSE.

2. Governing equations

The dimensionless form of generalized CNLSE with time-dependent coefficients is given by

$$i\left(q^m\right)_t + a(t)\left(q^m\right)_{xx} + ib(t)\left(qq_x^* - q^*q_x\right)q^m = 0\,, \tag{1}$$

where the dependent variable q represents the complex wave profile, while x and t are the independent variables that respectively represent the spatial and temporal variables. Moreover, $a(t)$ and $b(t)$ are all functions of t and m is a positive real number that makes the CNLSE general. For $a(t) = a$ and $b(t) = b$, Eq. (1) reduces to the generalized CNLSE [1]

$$i\left(q^m\right)_t + a\left(q^m\right)_{xx} + ib\left(qq_x^* - q^*q_x\right)q^m = 0\,. \tag{2}$$

For $m = 1$, we get the regular CNLSE with time-dependent coefficients [2]

$$iq_t + a(t)q_{xx} + ib(t)\left(qq_x^* - q^*q_x\right)q = 0\,, \tag{3}$$

and when $a(t) = a$ and $b(t) = b$, then Eq. (3) reduces to the regular CNLSE that was first proposed by Jackiw and Pi [3, 4].

The three integration tools that will be adopted in order to address these generalized form of CNLSE are going to be individually studied in the following three sections.

3. Functional variable method

We first describe the functional variable method [5–7], then apply it to construct the nontopological 1-soliton solution of the generalized CNLSE [1] in the following form

$$i\left(q^m\right)_t + a\left(q^m\right)_{xx} + ib\left(qq_x^* - q^*q_x\right)q^m = 0\,. \tag{4}$$

This method by itself is well known. In fact, the method banks on the fact that an ODE of the form $u'' = f(u)$ can be integrated by the introduction of an *integration function* that is alternately known as *functional variable* [29].

Cevikel *et al.* [7] used the functional variable method to obtain exact solutions of Zakharov–Kuznetsov modified equal width (ZK-MEW), the modified Benjamin–Bona–Mahony (mBBM) and the modified KdV-Kadomtsev–Petviashvili (KdV-KP) equations. This method definitely can be applied to nonlinear PDEs which can be converted to a second order ODE through the traveling wave transformation.

This method definitely can be applied to nonlinear PDEs which can be converted to a second order ordinary differential equations (ODE) through the traveling wave transformation.

3.1. Detailed description of the method

Let us consider a NLEE with independent variables x, t and dependent variable u as

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \dots) = 0\,, \tag{5}$$

where P is a polynomial in u and its partial derivatives.

To find the traveling wave solution of Eq. (5), we introduce the wave variable $\xi = x - ct$ so that

$$u(x,t) = U(\xi)\,. \tag{6}$$

The nonlinear partial differential equation can be converted to an ODE as

$$Q\left(U, U', U'', \dots\right) = 0\,, \tag{7}$$

where Q is a polynomial in U and its total derivatives and $' = \frac{d}{d\xi}$.

Let us make a transformation in which the unknown function $U(\xi)$ is considered as a functional variable in the form

$$U_\xi = F(U) \tag{8}$$

and some successively derivatives of U are

$$\begin{aligned} U_{\xi\xi} &= \tfrac{1}{2}\left(F^2\right)'\,,\\ U_{\xi\xi\xi} &= \tfrac{1}{2}\left(F^2\right)''\sqrt{F^2}\,,\\ U_{\xi\xi\xi\xi} &= \tfrac{1}{2}\left[\left(F^2\right)'''F^2 + \left(F^2\right)''\left(F^2\right)'\right]\,, \end{aligned} \tag{9}$$

where $' = \frac{d}{dU}$.

The ODE (7) can be reduced in terms of U, F and its derivatives upon using the expressions of Eq. (9) into Eq. (7) gives

$$R\left(U, F, F', F'', F''', \dots\right) = 0\,. \tag{10}$$

The key idea of this particular form Eq. (10) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, the Eq. (10) provides the expression of F, and this, in turn, together with Eq. (8) give the relevant solutions to the original problem.

3.2. Application to generalized CNLSE

We apply the functional variable method to construct the nontopological 1-soliton solution of the generalized CNLSE (4). We use the wave transformation

$$q(x,t) = U(\xi)e^{i(-\kappa x+\omega t+\theta)}\,, \qquad \xi = x - vt\,, \tag{11}$$

where κ represents the wave number of the soliton, while ω is the frequency of the soliton and θ is the phase constant, and finally v is the velocity of the soliton, all of them are to be determined later.

Thus, from Eq. (11), we have

$$\begin{aligned}(q^m)_t &= \left\{-v\left(U^m(\xi)\right)' + im\omega U^m(\xi)\right\} e^{im(-\kappa x+\omega t+\theta)}\,, &(12)\\ (q^m)_{xx} &= \left\{\left(U^m(\xi)\right)'' - 2imk\left(U^m(\xi)\right)' - m^2\kappa^2 U^m(\xi)\right\} e^{im(-\kappa x+\omega t+\theta)}\,, &(13)\end{aligned}$$

and

$$qq_x^* - q^*q_x = 2i\kappa U^2(\xi)\,. \tag{14}$$

Substituting Eqs. (11)–(14) into Eq. (4), and then decomposing into real and imaginary parts respectively yields

$$v = -2am\kappa\,, \tag{15}$$

and

$$a\left(U^m(\xi)\right)'' - \left(m\omega + am^2\kappa^2\right)U^m(\xi) - 2\kappa b U^{m+2}(\xi) = 0\,. \tag{16}$$

Now, we use the transformation

$$U(\xi) = V^{\frac{1}{m}}(\xi) \tag{17}$$

that will reduce Eq. (16) into the ODE

$$aV'' - \left(m\omega + am^2\kappa^2\right)V - 2\kappa b V^{1+\frac{2}{m}} = 0\,. \tag{18}$$

According to Eq. (9), we get from Eq. (18) the expression for the function $F(V)$ that reads

$$F(V) = \sqrt{\frac{m\left(\omega + am\kappa^2\right)}{a}}\,V\sqrt{1 + \frac{2\kappa b}{(m+1)\left(\omega + am\kappa^2\right)}V^{\frac{2}{m}}}\,. \tag{19}$$

After making the change of variables

$$Z=-\frac{2\kappa b}{(m+1)\left(\omega+am\kappa^2\right)}V^{\frac{2}{m}} \tag{20}$$

and using the relation $V_\xi = F(V)$, the solution of the Eq. (18) is in the following form

$$V(\xi)=\left\{\frac{(m+1)\left(\omega+am\kappa^2\right)}{2\kappa b}\operatorname{csch}^2\left(\sqrt{\frac{\omega+am\kappa^2}{ma}}\xi\right)\right\}^{\frac{m}{2}}. \tag{21}$$

Using the transformation (17), we can obtain the following exact 1-soliton solutions of Eq. (4):

Soliton solution:

$$q_1(x,t) = \sqrt{-\frac{(m+1)\left(\omega+am\kappa^2\right)}{2\kappa b}}\operatorname{sech}\left(\sqrt{\frac{\omega+am\kappa^2}{ma}}(x+2am\kappa t)\right) \times e^{i(-\kappa x+\omega t+\theta)}; \tag{22}$$

Singular soliton solution:

$$q_2(x,t) = \sqrt{\frac{(m+1)\left(\omega+am\kappa^2\right)}{2\kappa b}}\operatorname{csch}\left(\sqrt{\frac{\omega+am\kappa^2}{ma}}(x+2am\kappa t)\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{23}$$

for $\frac{\omega+am\kappa^2}{a}>0$; it is easy to see that solutions (22) and (23) can reduce to periodic solutions as follows

$$q_3(x,t) = \sqrt{-\frac{(m+1)\left(\omega+am\kappa^2\right)}{2\kappa b}}\sec\left(\sqrt{-\frac{\omega+am\kappa^2}{ma}}(x+2am\kappa t)\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{24}$$

and

$$q_4(x,t) = \sqrt{-\frac{(m+1)\left(\omega+am\kappa^2\right)}{2\kappa b}}\csc\left(\sqrt{-\frac{\omega+am\kappa^2}{ma}}(x+2am\kappa t)\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{25}$$

for $\frac{\omega+am\kappa^2}{a}<0$.

Remark I: Bright solitons are also known as bell-shaped solitons. These kinds of solitons are modeled by the sech function.

4. First integral method

In this section, we introduce a simple description of the first integral method [8–15] and then apply it to construct the topological soliton solution of the CNLSE with time-dependent coefficients [2] in the following form

$$iq_t + a(t)q_{xx} + ib(t)\left(qq_x^* - q^*q_x\right)q = 0\,. \tag{26}$$

This method is much more advanced but less standard. The basic idea of this method starts with the assumption that a second order nonlinear ODE has a first integral in the form of a polynomial of dependent variable and its first derivative. Then, treating the original second order equation as the first prolongation of the first integral, it is assumed that the first prolongation is another polynomial proportional to the first one, such that it is automatically zero for the solution. This assumption is not true in a general scenario. However, for specific problems, this can be valid. The first integral can be then found explicitly by comparing coefficients, namely by purely algebraic means. Besides, by considering higher degree polynomials, one can find solutions of increasing generality.

Recently, this useful method is widely used in many papers *e.g.* in [9–15] and the references therein. Lu [9] proposed the first integral method to solve some nonlinear fractional PDEs such as nonlinear fractional Klein–Gordon equation, Generalized Hirota–Satsuma coupled KdV system of time fractional order and nonlinear fractional Sharma–Tasso–Olever equation. Bekir and Unsal [10] solved combined KdV–mKdV equation, Pochhammer–Chree equation, and coupled nonlinear evolution equations using the first integral method. Tascan and Bekir [11] used the first integral method to obtain exact solutions of the modified Zakharov–Kuznetsov equation and ZK–MEW equation. Aslan and Mirzazadeh [12–16] proposed the first integral method to obtain exact solutions of some complex nonlinear PDEs.

Remark II: Biswas *et al.* [1] showed that for the generalized CNLSE, the topological solitons would exist only for $m = 1$ and no other value of m can be permitted for the topological soliton solutions to be valid.

4.1. Details of the method

Tascan *et al.* [11] summarized the main steps for using the first integral method, as follows:

Step I: Suppose a NLEE with independent variables x, t and dependent variable u as

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0 \tag{27}$$

can be converted to an ODE

$$Q\left(U(\xi), \frac{dU(\xi)}{d\xi}, \frac{d^2U(\xi)}{d\xi^2}, \ldots\right) = 0\,, \tag{28}$$

using a traveling wave variable $u(x,t) = U(\xi), \quad \xi = x - ct$, where the prime denotes the derivation with respect to ξ. If all terms contain derivatives, then Eq. (28) is integrated where integration constants are considered zeros.

Step II: Suppose that the solution of ODE (28) can be written as follows:

$$u(x,t) = U(\xi)\,. \tag{29}$$

Step III: We introduce a new independent variable

$$X(\xi) = U(\xi)\,, \qquad Y(\xi) = \frac{dU(\xi)}{d\xi}\,, \tag{30}$$

which leads to a following system

$$\begin{aligned} \frac{dX(\xi)}{d\xi} &= Y(\xi)\,, \\ \frac{dY(\xi)}{d\xi} &= F(X(\xi), Y(\xi))\,. \end{aligned} \tag{31}$$

Step IV: According to the qualitative theory of ordinary differential equations [17], if we can find the integrals to (31) under the same conditions, then the general solutions to (31) can be solved directly. However, in general, it is really difficult for us to realize this even for one first integral, because for a given plane autonomous system, there is neither a systematic theory that can tell us how to find its first integrals, nor is there a logical way for telling us what these first integrals are. We shall apply the Division Theorem to obtain one first integral to (31) which reduces (28) to a first-order integrable ordinary differential equation. An exact solution to (27) is then obtained by solving this equation. Now, let us recall the Division Theorem:

Division Theorem: Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $C[w,z]$; and $P(w,z)$ is irreducible in $C[w,z]$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $C[w,z]$ such that

$$Q(w,z) = P(w,z)G(w,z)\,.$$

4.2. Application to CNLSE with time-dependent coefficients

In this subsection, we study the CNLSE with time-dependent coefficients

$$iq_t + a(t)q_{xx} + ib(t)\left(qq_x^* - q^*q_x\right)q = 0\,. \tag{32}$$

We use the wave transformation

$$q(x,t) = U(\xi)e^{i(-\kappa x+\omega(t)t+\theta)}\,, \qquad \xi = x - v(t)t\,, \tag{33}$$

where $v(t)$ is the soliton velocity, κ is the wave number of the soliton, while $\omega(t)$ is the frequency of the soliton velocity and finally θ is the phase constant, all of them are to be determined.

Thus, from Eq. (33), we obtain

$$q_t = \left\{ i\left(t\frac{d\omega(t)}{dt}+\omega(t)\right)U(\xi)-\left(t\frac{dv(t)}{dt}+v(t)\right)\frac{dU(\xi)}{d\xi}\right\}e^{i(-\kappa x+\omega(t)t+\theta)}\,, \tag{34}$$

$$q_{xx} = \left\{\frac{d^2U(\xi)}{d\xi^2} - 2i\kappa\frac{dU(\xi)}{d\xi} - \kappa^2 U(\xi)\right\}e^{i(-\kappa x+\omega(t)t+\theta)}\,, \tag{35}$$

and

$$qq_x^* - q^*q_x = 2i\kappa U^2(\xi)\,. \tag{36}$$

Substituting Eqs. (33)–(36) into Eq. (32), and equating the real and imaginary parts, yields the following pair of relations

$$t\frac{dv(t)}{dt} + v(t) + 2\kappa a(t) = 0\,, \tag{37}$$

and

$$a(t)\frac{d^2U(\xi)}{d\xi^2} - \left(\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)\right)U(\xi) - 2\kappa b(t)U^3(\xi) = 0\,. \tag{38}$$

Then, integrating Eq. (37), we obtain

$$v(t) = -\frac{2\kappa}{t}\int a(t)dt\,. \tag{39}$$

Using (30), we get

$$\frac{dX(\xi)}{d\xi} = Y(\xi)\,, \tag{40}$$

$$\frac{dY(\xi)}{d\xi} = \left(\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{a(t)}\right)X(\xi) + \frac{2\kappa b(t)}{a(t)}X^3(\xi)\,. \tag{41}$$

According to the first integral method, we suppose that $X(\xi)$ and $Y(\xi)$ are nontrivial solutions of (40), (41), and $q(X,Y) = \sum_{i=0}^{m} a_i(X)Y^i$ is an irreducible polynomial in the complex domain $C[X,Y]$ such that

$$q\left(X(\xi), Y(\xi)\right) = \sum_{i=0}^{m} a_i(X(\xi))Y^i(\xi) = 0\,, \tag{42}$$

where $a_i(X)$, $i = 0, 1, ..., m$, are polynomials of X and $a_m(X) \neq 0$. Equation (42) is called the first integral to (40)–(41). According to the division theorem, there exists a polynomial $T(X, Y) = g(X) + h(X)Y$ in the complex domain $C[X, Y]$ such that

$$\frac{dq}{d\xi} = \frac{dq}{dX}\frac{dX}{d\xi} + \frac{dq}{dY}\frac{dY}{d\xi} = (g(X) + h(X)Y)\sum_{i=0}^{m} a_i(X)Y^i \,. \tag{43}$$

In this example, we take two different cases, by assuming $m = 1$ and $m = 2$ in Eq. (42).

Case I: Suppose $m = 1$. By equating the coefficients of Y^i, $i = 0, 1, 2$, on both sides of Eq. (43), we have

$$\dot{a}_1(X) = h(X)a_1(X) \,, \tag{44}$$
$$\dot{a}_0(X) = g(X)a_1(X) + h(X)a_0(X) \,, \tag{45}$$

$$a_1(X)\dot{Y} = a_1(X)\left(\left(\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{a(t)}\right)X + \frac{2\kappa b(t)}{a(t)}X^3\right) = g(X)a_0(X). \tag{46}$$

From Eq. (44), we obtain $a_1(X) = c_0 e^{\int h(X)dX}$, where c_0 is an integration constant. As $a_1(X)$ and $h(X)$ are polynomials, we deduce that $h(X) = 0$ and $a_1(X)$ must be a constant. For simplicity, we can take $a_1(X) = 1$. Balancing the degrees of $g(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, and $A_1 \neq 0$, then we find

$$a_0(X) = \frac{A_1}{2}X^2 + B_0X + A_0 \,. \tag{47}$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in Eq. (46), and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$A_1 = 2\sqrt{\frac{\kappa b(t)}{a(t)}} \,, \quad B_0 = 0 \,, \quad \omega(t) = -\frac{1}{t}\int\left\{\kappa^2 a(t) - 2A_0\sqrt{ka(t)b(t)}\right\}dt \,, \tag{48}$$

$$A_1 = -2\sqrt{\frac{\kappa b(t)}{a(t)}} \,, \quad B_0 = 0 \,, \quad \omega(t) = -\frac{1}{t}\int\left\{\kappa^2 a(t) + 2A_0\sqrt{ka(t)b(t)}\right\}dt \,, \tag{49}$$

where A_0 and κ are arbitrary constants.

Substituting (48) into Eq. (42), we obtain

$$Y(\xi) = -\sqrt{\frac{\kappa b(t)}{a(t)}}X^2(\xi) - A_0 \,. \tag{50}$$

Combining (50) with (40), we obtain the exact solution to Eq. (38) and then exact solutions for the CNLSE with time-dependent coefficients (32) can be written as:
When $A_0\sqrt{\frac{\kappa b(t)}{a(t)}} < 0$, we have:

Topological soliton solution:

$$\begin{aligned} q_1(x,t) &= -\sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\ &\quad \times \tanh\left[\sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) - 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)}\,; \end{aligned} \tag{51}$$

Singular soliton solution:

$$\begin{aligned} q_2(x,t) &= -\sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\ &\quad \times \coth\left[\sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) - 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)}\,. \end{aligned} \tag{52}$$

When $A_0\sqrt{\frac{\kappa b(t)}{a(t)}} > 0$, we can obtain the following periodic solutions

$$\begin{aligned} q_3(x,t) &= \sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\ &\quad \times \tan\left[\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) - 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)}\,, \end{aligned} \tag{53}$$

$$
\begin{aligned}
q_4(x,t) &= -\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\
&\quad \times \cot\left[\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\
&\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) - 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)} . \qquad (54)
\end{aligned}
$$

Remark III: The topological solitons are also known as dark solitons (in the context of nonlinear optics) or simply topological defects. These kinds of solitons are modeled by the tanh functions.

Similarly, in the case of (49), from Eq. (42), we obtain

$$
Y(\xi) = \sqrt{\frac{\kappa b(t)}{a(t)}}X^2(\xi) - A_0\,, \qquad (55)
$$

and then the exact solutions to the CNLSE with time-dependent coefficients (32) can be written as:

When $A_0\sqrt{\frac{\kappa b(t)}{a(t)}} > 0$, we have:

Topological soliton solution:

$$
\begin{aligned}
q_5(x,t) &= -\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\
&\quad \times \tanh\left[\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\
&\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) + 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)} ; \qquad (56)
\end{aligned}
$$

Singular soliton solution:

$$
\begin{aligned}
q_6(x,t) &= -\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\
&\quad \times \coth\left[\sqrt{\frac{A_0\sqrt{\kappa a(t)b(t)}}{a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\
&\quad \times e^{i\left(-\kappa x - \int\left\{\kappa^2 a(t) + 2A_0\sqrt{\kappa a(t)b(t)}\right\}dt + \theta\right)} . \qquad (57)
\end{aligned}
$$

When $A_0\sqrt{\frac{\kappa b(t)}{a(t)}} < 0$, we can obtain the following periodic solutions

$$\begin{aligned} q_7(x,t) &= \sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{\kappa b(t)}} \\ &\quad \times \tan\left[\sqrt{-\frac{A_0\sqrt{ka(t)b(t)}}{a(t)}}\left(x+2\kappa\int a(t)dt+C_1\right)\right] \\ &\quad \times e^{i\left(-\kappa x-\int\left\{\kappa^2 a(t)+2A_0\sqrt{\kappa a(t)b(t)}\right\}dt+\theta\right)}, \end{aligned} \tag{58}$$

$$\begin{aligned} q_8(x,t) &= -\sqrt{-\frac{A_0\sqrt{\kappa a(t)b(t)}}{kb(t)}} \\ &\quad \times \cot\left[\sqrt{-\frac{A_0\sqrt{ka(t)b(t)}}{a(t)}}\left(x+2\kappa\int a(t)dt+C_1\right)\right] \\ &\quad \times e^{i\left(-\kappa x-\int\left\{\kappa^2 a(t)+2A_0\sqrt{\kappa a(t)b(t)}\right\}dt+\theta\right)}. \end{aligned} \tag{59}$$

Remark IV: The topological soliton solution (56) that we obtained for Eq. (32) is exactly the same as the one that was obtained by Biswas [2] who used the solitary wave ansatz method.

Case II: Suppose $m = 2$. By equating the coefficients of Y^i, $i = 0,1,2,3$ on both sides of Eq. (43), we have

$$\dot{a_2}(X) = h(X)a_2(X)\,, \tag{60}$$

$$\dot{a_1}(X) = g(X)a_2(X) + h(X)a_1(X)\,, \tag{61}$$

$$\begin{aligned} \dot{a_0}(X) &= -2a_2(X)\left(\left(\frac{\kappa^2 a(t)+t\frac{d\omega(t)}{dt}+\omega(t)}{a(t)}\right)X+\frac{2\kappa b(t)}{a(t)}X^3\right) \\ &\quad +g(X)a_1(X)+h(X)a_0(X)\,, \end{aligned} \tag{62}$$

$$a_1(X)\dot{Y} = a_1(X)\left(\left(\frac{\kappa^2 a(t)+t\frac{d\omega(t)}{dt}+\omega(t)}{a(t)}\right)X+\frac{2\kappa b(t)}{a(t)}X^3\right) = g(X)a_0(X)\,. \tag{63}$$

Since $a_2(X)$ is a polynomial of X, then from Eq. (60) we deduce that $a_2(X)$ is a constant and $h(X) = 0$. For simplicity, we take $a_2(X) = 1$. Balancing the degrees of $g(X)$, $a_0(X)$ and $a_0(X)$, we conclude that $\deg(g(X)) = 1$ only. Suppose that $g(X) = A_1X + B_0$, then we find $a_1(X)$ and $a_0(X)$ as follows

$$a_1(X) = \tfrac{1}{2}A_1X^2 + B_0X + A_0\,, \tag{64}$$

$$a_0(X) = \left(\frac{A_1^2}{8} - \frac{\kappa b(t)}{a(t)}\right) X^4 + \frac{B_0 A_1}{2} X^3 + \left(-\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{a(t)} + \frac{B_0^2}{2} + \frac{A_0 A_1}{2}\right) X^2 + A_0 B_0 X + d\,. \quad (65)$$

Substituting $a_0(X)$, $a_1(X)$ and $g(X)$ in Eq. (63) and setting all the coefficients of powers X to be zero, we obtain a system of nonlinear algebraic equations and by solving it with aid Maple, we obtain

$$A_1 = 4\sqrt{\frac{\kappa b(t)}{a(t)}}, \qquad A_0 = \frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{\sqrt{\kappa a(t) b(t)}}, \qquad B_0 = 0,$$
$$d = \frac{\left(\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)\right)^2}{4\kappa a(t) b(t)}, \quad (66)$$
$$A_1 = -4\sqrt{\frac{\kappa b(t)}{a(t)}}, \qquad A_0 = -\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{\sqrt{\kappa a(t) b(t)}}, \qquad B_0 = 0,$$
$$d = \frac{\left(\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)\right)^2}{4\kappa a(t) b(t)}. \quad (67)$$

Now, taking the Eqs. (66)–(67) into account, Eq. (42) becomes

$$\frac{\left(\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)\right)^2}{4\kappa a(t) b(t)} + \frac{\left(\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)\right)^2}{a(t)} X^2 + \frac{\kappa b(t)}{a(t)} X^4 \pm \left(\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{\sqrt{\kappa a(t) b(t)}} + 2\sqrt{\frac{\kappa b(t)}{a(t)}} X^2\right) Y + Y^2 = 0 \quad (68)$$

which is a first integral of Eqs. (40)–(41). Solving Eq. (68), we get

$$Y(\xi) = \pm \frac{1}{2\sqrt{\kappa a(t) b(t)}} \left[\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t) + 2\kappa b(t) X^2(\xi)\right]. \quad (69)$$

Combining (69) with (40), we obtain the exact solution to Eq. (38) and then exact solutions to the CNLSE with time-dependent coefficients (32) can be written as:

When $\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{a(t)} < 0$, we have:

Topological soliton solutions:

$$\begin{aligned} q_{9,10}(x,t) &= \pm\sqrt{-\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2\kappa b(t)}} \\ &\quad \times \tanh\left[\sqrt{-\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i(-\kappa x + \omega(t)t + \theta)}; \end{aligned} \tag{70}$$

Singular soliton solutions:

$$\begin{aligned} q_{11,12}(x,t) &= \pm\sqrt{-\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2\kappa b(t)}} \\ &\quad \times \coth\left[\sqrt{-\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i(-\kappa x + \omega(t)t + \theta)}. \end{aligned} \tag{71}$$

When $\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{a(t)} > 0$, we can obtain the following periodic solutions

$$\begin{aligned} q_{13,14}(x,t) &= \pm\sqrt{\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2\kappa b(t)}} \\ &\quad \times \tan\left[\sqrt{\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i(-\kappa x + \omega(t)t + \theta)}, \end{aligned} \tag{72}$$

$$\begin{aligned} q_{15,16}(x,t) &= \pm\sqrt{\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2\kappa b(t)}} \\ &\quad \times \cot\left[\sqrt{\frac{\kappa^2 a(t) + t\frac{d\omega(t)}{dt} + \omega(t)}{2a(t)}}\left(x + 2\kappa\int a(t)dt + C_1\right)\right] \\ &\quad \times e^{i(-\kappa x + \omega(t)t + \theta)}. \end{aligned} \tag{73}$$

Remark V: The derivation of the soliton solution for the CNLSE with constant coefficients was carried out by Nishino *et al.* [18]. It is important to consider the time-dependent coefficients to the CNLSE as this is more close to reality. It will be seen that the only criterion for the chiral solitons to exist is that the dispersion coefficient must be Riemann integrable.

By using the Eqs. (51), (56) and (70), the topological soliton solutions of the CNLSE

$$iq_t + aq_{xx} + ib\left(qq_x^* - q^*q_x\right)q = 0 \tag{74}$$

are in the following forms:

$$\begin{aligned} q_1(x,t) &= -\sqrt{-\frac{A_0\sqrt{\kappa ab}}{\kappa b}}\tanh\left[\sqrt{-\frac{A_0\sqrt{\kappa ab}}{a}}(x+2\kappa at+C_1)\right] \\ &\quad \times e^{i\left(-\kappa x-\left\{\kappa^2 a-2A_0\sqrt{\kappa ab}\right\}t+\theta\right)}, \end{aligned} \tag{75}$$

$$\begin{aligned} q_2(x,t) &= -\sqrt{\frac{A_0\sqrt{\kappa ab}}{kb}}\tanh\left[\sqrt{\frac{A_0\sqrt{\kappa ab}}{a}}(x+2\kappa at+C_1)\right] \\ &\quad \times e^{i\left(-\kappa x-\left\{\kappa^2 a+2A_0\sqrt{\kappa ab}\right\}t+\theta\right)}, \end{aligned} \tag{76}$$

and

$$\begin{aligned} q_{3,4}(x,t) &= \pm\sqrt{-\frac{\kappa^2 a+\omega}{2\kappa b}}\tanh\left[\sqrt{-\frac{\kappa^2 a+\omega}{2a}}(x+2\kappa at+C_1)\right] \\ &\quad \times e^{i(-\kappa x+\omega t+\theta)}. \end{aligned} \tag{77}$$

The dark soliton solution (76) that we obtained for Eq. (74) is the same with the solution that was obtained by Biswas [1] who used the solitary wave ansatz method.

5. Ansatz approach

This section will utilize the ansatz method to solve the chiral NLSE with constant coefficients. This method consist of guessing the correct functional dependence such that only constraints remain to be determined. This method is not general since one knows the solution structure, more or less, in advance. The focus will be on equation (3) with $m = 1$. Therefore, the governing equation of study here is

$$iq_t + aq_{xx} + ib\left(qq_x^* - q^*q_x\right) = 0\,. \tag{78}$$

The singular soliton solutions to (78) will be obtained by the aid of ansatz method. The starting point is the assumption

$$q(x,t) = P(x,t)e^{i\phi(x,t)}\,, \tag{79}$$

where $P(x,t)$ is the amplitude part and the phase component $\phi(x,t)$ is given by

$$\phi(x,t) = -\kappa x + \omega t + \theta \,. \tag{80}$$

In (80), κ represents the soliton wave number, while ω is the frequency and θ is the phase constant. Substituting (79) into (78), and decomposing into real and imaginary parts yield

$$v = -2a\kappa \tag{81}$$

and

$$\left(\omega + a\kappa^2\right) P + 2b\kappa P^3 - a\frac{\partial^2 P}{\partial x^2} = 0 \,. \tag{82}$$

For singular soliton, the hypothesis is

$$P(x,t) = A \operatorname{csch}^p \tau \,, \tag{83}$$

where

$$\tau = B(x - vt) \,. \tag{84}$$

The value of the unknown exponent p will fall out during the course of derivation of the soliton solutions. Also A and B are free parameters, while v is the speed of the soliton. Substution of (83) into the real part equation given by (82) leads to

$$\left(\omega + a\kappa^2 - ap^2B^2\right) \operatorname{csch}^p\tau + 2b\kappa^2 \operatorname{csch}^{3p}\tau - ap(p+1)B^2 \operatorname{csch}^{p+2}\tau = 0 \,. \tag{85}$$

From (85), the balancing principle yields

$$p = 1 \,. \tag{86}$$

Next, from (85) setting the coefficients of the linearly independent functions to zero implies

$$A = \sqrt{\frac{\omega + a\kappa^2}{b\kappa}} \,, \tag{87}$$

and

$$B = \sqrt{\frac{\omega + a\kappa^2}{a}} \,. \tag{88}$$

Equations (87) and (88) prompts the constraints

$$b\kappa\left(\omega + a\kappa^2\right) > 0 \tag{89}$$

and

$$a\left(\omega + a\kappa^2\right) > 0 \,, \tag{90}$$

respectively. Thus, the 1-soliton solution to (78) is given by

$$q(x,t) = A \operatorname{csch}[B(x - vt)] e^{i(-\kappa x + \omega t + \theta)} \,, \tag{91}$$

where the free parameters A and B are respectively given by (87) and (88) with the constraints (89) and (90). The velocity of the soliton is seen in (81).

6. Results and discussion

The results of this paper on chiral solitons are interesting. There are three independent integration tools that are applied to extract the solutions of the model equation that was studied in this paper. These integration architectures revealed several solutions that are generalized forms of results reported earlier in the literature. The functional variable method, first integral method and the ansatz approach are all indeed powerful tools that reveal several solutions to the CNLSE. The results of this paper thus supplement the existing material in the literature of chiral solitons in nuclear physics.

7. Conclusions

This paper studied the CNLSE as well as the generalized CNLSE with time-dependent coefficients by three methods of integration. There are the soliton as well as singular periodic solutions that are obtained. The constraint conditions fell out as necessary conditions of integrability as well as existence of soliton solutions. These interesting results are going to be profoundly useful and will thus set a foundation stone for further research activities in this avenue. There are several additional aspects that will be considered in future. The perturbation term will be considered. Instead of time-dependent coefficients, the stochastic coefficients will be taken into consideration. Additionally, stochastic perturbation terms will be studied and these results will be reported in future publications.

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